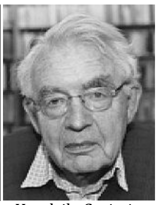
M2/International Centre for Fundamental Physics
Parcours of Physique Théorique

Invariances in Physics and Group Theory

Jean-Bernard Zuber

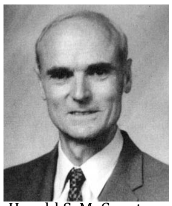

Niels Henrik Abel
1802 – 1829


Elie Cartan
1869 – 1951


Hendrik Casimir
1909-2000


Claude Chevalley
1909 – 1984


Rudolf F. A. Clebsch
1833 – 1872


Harold S. M. Coxeter
1907 – 2003


Eugene B. Dynkin
1924 -


Hans Freudenthal
1905 - 1990


Ferdinand Frobenius
1849 – 1917


Paul Albert Gordan
1837 – 1912


Alfréd Haar
1885 - 1933


Sir William R. Hamilton
1805 - 1865


Wilhelm K. J. Killing
1847 – 1923


Sophus Lie
1842 – 1899

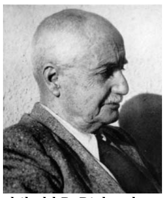

Dudley E. Littlewood
1903 - 1979


Hendrik A. Lorentz
1853 - 1928


Hermann Minkowski
1864 – 1909


Emmy A. Noether
1882 - 1935


Henri Poincaré
1854 – 1912

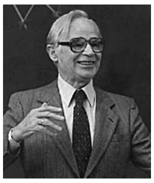

Archibald R. Richardson
1881 – 1954


Olinde Rodrigues
1795–1851


Issai Schur
1875 – 1941


Jean-Pierre Serre
1926 -


Miguel Virasoro
1940-

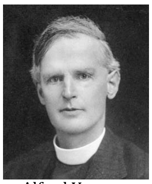

Bartel van der Waerden
1903 – 1996


André Weil
1906 – 1998


Hermann Weyl
1885 – 1955


Eugene P. Wigner
1902 – 1995


Ernst Witt
1911 – 1991


Alfred Young
1873 – 1940

Some of the major actors of group theory mentionned in the first part of these notes

Foreword

The following notes cover the content of the course "Invariances in Physique and Group Theory" given in the fall 2013. Additional lectures were given during the week of "prérentrée" on the SO(3), SU(2), SL(2,$\mathbb{C}$) groups, see below Chap. 0.

Chapters 1 to 5 also contain, in sections in smaller characters and Appendices, additional details that are not treated in the oral course.

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Contents

Chapter 0

Some basic elements on the groups SO(3), SU(2) and SL(2,$\mathbb{C}$)

0.1 Rotations of $\mathbb{R}^3$, the groups SO(3) and SU(2)

0.1.1 The group SO(3), a 3-parameter group

Let us consider the rotation group in three-dimensional Euclidean space. These rotations leave invariant the squared norm of any vector $\mathbf{OM}$, $\mathbf{OM}^2 = x_1^2 + x_2^2 + x_3^2 = x^2 + y^2 + z^2$ [1] and preserve orientation. They are represented in an orthonormal bases by 3×3 orthogonal real matrices, of determinant 1 : they form the "special orthogonal" group SO(3).

Olinde Rodrigues formula

Any rotation of SO(3) is a rotation by some angle ψ around an axis colinear to a unit vector $\mathbf{n}$, and the rotations associated with $(\mathbf{n}, \psi)$ and $(-\mathbf{n}, -\psi)$ are identical. We denote $R_{\mathbf{n}}(\psi)$ this rotation. In a very explicit way, one writes $\mathbf{x} = \mathbf{x}_\parallel + \mathbf{x}_\perp = (\mathbf{x}.\mathbf{n})\mathbf{n} + (\mathbf{x} - (\mathbf{x}.\mathbf{n})\mathbf{n})$ and $\mathbf{x}' = \mathbf{x}_\parallel + \cos\psi\, \mathbf{x}_\perp + \sin\psi\, \mathbf{n} \times \mathbf{x}_\perp$, whence Rodrigues formula

$$\mathbf{x}' = R_{\mathbf{n}}(\psi)\mathbf{x} = \cos\psi\, \mathbf{x} + (1 - \cos\psi)(\mathbf{x}.\mathbf{n})\, \mathbf{n} + \sin\psi\, (\mathbf{n} \times \mathbf{x})\ . \tag{0.1}$$

As any unit vector $\mathbf{n}$ in $\mathbb{R}^3$ depends on two parameters, for example the angle θ it makes with the Oz axis and the angle ϕ of its projection in the Ox, Oy plane with the Ox axis (see Fig. 1) an element of SO(3) is parametrized by 3 continuous variables. One takes

$$0 \le \theta \le \pi, \quad 0 \le \phi < 2\pi, \quad 0 \le \psi \le \pi\ . \tag{0.2}$$

But there remains an innocent-looking redundancy, $R_{\mathbf{n}}(\pi) = R_{-\mathbf{n}}(\pi)$, the consequences of which we see later ...

[1] In this chapter, we use alternately the notations (x, y, z) or (x_1, x_2, x_3) to denote coordinates in an orthonormal frame.

SO(3) is thus a dimension 3 manifold. For the rotation of axis $\mathbf{n}$ colinear to the Oz axis, we have the matrix

$$\mathcal{R}_z(\psi) = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{0.3}$$

whereas around the Ox and Oy axes

$$\mathcal{R}_x(\psi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{pmatrix} \qquad \mathcal{R}_y(\psi) = \begin{pmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{pmatrix}. \tag{0.4}$$

Conjugation of $R_{\mathbf{n}}(\psi)$ by another rotation
A relation that we are going to use frequently reads

$$\boxed{R R_{\mathbf{n}}(\psi) R^{-1} = R_{\mathbf{n}'}(\psi)} \tag{0.5}$$

where $\mathbf{n}'$ is the transform of $\mathbf{n}$ by rotation R, $\mathbf{n}' = R\mathbf{n}$ (check it!). Conversely any rotation of angle ψ around a vector $\mathbf{n}'$ can be cast under the form (0.5) : we'll say later that the "conjugation classes" of the group SO(3) are characterized by the angle ψ.

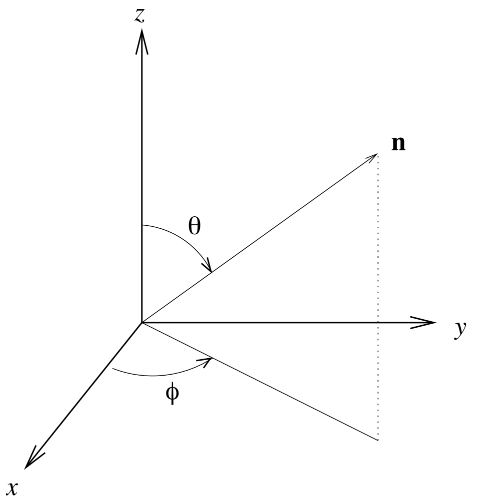


Fig. 1

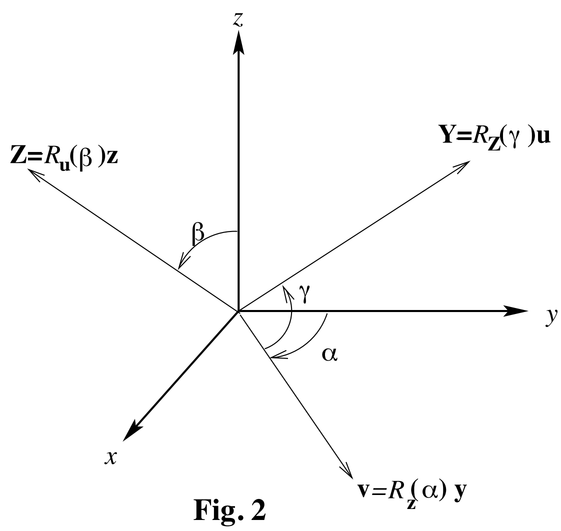


Fig. 2

Euler angles
Another description makes use of *Euler angles* : given an orthonormal frame (Ox, Oy, Oz), any rotation around O that maps it onto another frame (OX, OY, OZ) may be regarded as resulting from the composition of a rotation of angle α around Oz, which brings the frame onto (Ou, Ov, Oz), followed by a rotation of angle β around Ov bringing it on (Ou', Ov, OZ), and lastly, by a rotation of angle γ around OZ bringing the frame onto (OX, OY, OZ), (see Fig. 2). One thus takes $0 \le \alpha < 2\pi,\ 0 \le \beta \le \pi,\ 0 \le \gamma < 2\pi$ and one writes

$$R(\alpha, \beta, \gamma) = R_Z(\gamma) R_v(\beta) R_z(\alpha) \tag{0.6}$$

but according to (0.5)

$$R_Z(\gamma) = R_v(\beta) R_z(\gamma) R_v^{-1}(\beta) \qquad\qquad R_v(\beta) = R_z(\alpha) R_y(\beta) R_z^{-1}(\alpha)$$

thus, by inserting into (0.6)

$$R(\alpha,\beta,\gamma) = R_z(\alpha)R_y(\beta)R_z(\gamma)\ . \tag{0.7}$$

where one used the fact that $R_z(\alpha)R_z(\gamma)R_z^{-1}(\alpha) = R_z(\gamma)$ since rotations around a given axis commute (they form an abelian subgroup, isomorphic to $SO(2)$).

Exercise : using (0.5), write the expression of a matrix R which maps the unit vector $\mathbf{z}$ colinear to Oz to the unit vector $\mathbf{n}$, in terms of $R_z(\phi)$ and $R_y(\theta)$; then write the expression of $R_{\mathbf{n}}(\psi)$ in terms of R_y and R_z. Write the explicit expression of that matrix and of (0.7) and deduce the relations between θ,ϕ,ψ and Euler angles. (See also below, equ. (0.66).)

0.1.2 From SO(3) to SU(2)

Consider another parametrization of rotations. To the rotation $R_{\mathbf{n}}(\psi)$, we associate the unitary 4-vector u : $(u_0 = \cos\frac{\psi}{2}, \mathbf{u} = \mathbf{n}\sin\frac{\psi}{2})$; we have $u^2 = u_0^2 + \mathbf{u}^2 = 1$, and u belongs to the unit sphere S^3 in the space $\mathbb{R}^4$. Changing the determination of ψ by an odd multiple of 2π changes u into $-u$. There is thus a bijection between $R_{\mathbf{n}}(\psi)$ and the pair $(u,-u)$, *i.e.* between SO(3) and $S^3/\mathbb{Z}_2$, the sphere in which diametrically opposed points are identified. We shall say that the sphere S^3 is a "covering group" of SO(3). In which sense is this sphere a group? To answer that question, introduce Pauli matrices σ_i, $i = 1,2,3$.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ . \tag{0.8}$$

Together with the identity matrix $\mathbf{I}$, they form a basis of the vector space of 2×2 Hermitian matrices. They satisfy the identity

$$\sigma_i\sigma_j = \delta_{ij}\mathbf{I} + i\epsilon_{ijk}\sigma_k\ , \tag{0.9}$$

with ϵ_{ijk} the completely antisymmetric tensor, $\epsilon_{123} = +1$, $\epsilon_{ijk} =$ the signature of permutation (ijk).

From u a real unit 4-vector unitary (*i.e.* a point of S^3), we form the matrix

$$U = u_0\mathbf{I} - i\mathbf{u}.\boldsymbol{\sigma} \tag{0.10}$$

which is unitary and of determinant 1 (check it and also show the converse: any unimodular (= of determinant 1) unitary 2×2 matrix is of the form (0.10), with $u^2 = 1$). These matrices form the *special unitary* group SU(2) which is thus isomorphic to S^3. By a power expansion of the exponential and making use of $(\mathbf{n}.\boldsymbol{\sigma})^2 = \mathbf{I}$, a consequence of (0.9), one may verify that

$$e^{-i\frac{\psi}{2}\mathbf{n}.\boldsymbol{\sigma}} = \cos\frac{\psi}{2} - i\sin\frac{\psi}{2}\mathbf{n}.\boldsymbol{\sigma}\ . \tag{0.11}$$

It is then suggested that the multiplication of matrices

$$U_n(\psi) = e^{-i\frac{\psi}{2}\mathbf{n}.\boldsymbol{\sigma}} = \cos\frac{\psi}{2} - i\sin\frac{\psi}{2}\mathbf{n}.\boldsymbol{\sigma}, \qquad 0\le\psi\le 2\pi, \quad \mathbf{n}\in S^2 \tag{0.12}$$

gives the desired group law in S^3. Let us show indeed that to a matrix of SU(2) one may associate a rotation of SO(3) and that to the product of two matrices of SU(2) corresponds the product of the SO(3) rotations (this is the *homomorphism* property). To the point x of $\mathbb{R}^3$ of coordinates x_1, x_2, x_3, we associate the Hermitian matrix

$$X = \mathbf{x}.\boldsymbol{\sigma} = \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix} , \tag{0.13}$$

with conversely $x_i = \frac{1}{2}\mathrm{tr}\,(X\sigma_i)$, and we let SU(2) act on that matrix according to

$$X \mapsto X' = UXU^\dagger , \tag{0.14}$$

which defines a linear transform $x \mapsto x' = \mathcal{T}x$. One readily computes that

$$\det X = -(x_1^2 + x_2^2 + x_3^2) \tag{0.15}$$

and as $\det X = \det X'$, the linear transform $x \mapsto x' = \mathcal{T}x$ is an isometry, hence $\det \mathcal{T} = 1$ or -1. To convince oneself that this is indeed a rotation, *i.e.* that the transformation has a determinant 1, it suffices to compute that determinant for $U = \mathbf{I}$ where $\mathcal{T} =$ the identity, hence $\det \mathcal{T} = 1$, and then to invoke the connexity of the manifold $SU(2)(\cong S^3)$ to conclude that the continuous function $\det \mathcal{T}(U)$ cannot jump to the value -1. In fact, using identity (0.9), the explicit calculation of X' leads, after some algebra, to

$$\begin{aligned} X' &= (\cos\frac{\psi}{2} - i\mathbf{n}.\boldsymbol{\sigma}\sin\frac{\psi}{2})X(\cos\frac{\psi}{2} + i\mathbf{n}.\boldsymbol{\sigma}\sin\frac{\psi}{2}) \\ &= \big(\cos\psi\,\mathbf{x} + (1-\cos\psi)(\mathbf{x}.\mathbf{n})\,\mathbf{n} + \sin\psi\,(\mathbf{n}\times\mathbf{x})\big).\boldsymbol{\sigma} \end{aligned} \tag{0.16}$$

which is nothing else than the Rodrigues formula (0.1). We thus conclude that the transformation $x \mapsto x'$ performed by the matrices of SU(2) in (0.14) is indeed the rotation of angle ψ around $\mathbf{n}$. To the product $U_{n'}(\psi')U_n(\psi)$ in SU(2) corresponds in SO(3) the composition of the two rotations $R_{\mathbf{n}'}(\psi')R_{\mathbf{n}}(\psi)$ of SO(3). There is thus a "homomorphism" of the group SU(2) into SO(3). This homomorphism maps the two matrices U and $-U$ onto one and the same rotation of SO(3).

Let us summarize what we have learnt in this section. The group SU(2) is a covering group (of order 2) of the group SO(3) (the precise topological meaning of which will be given in Chap. 1), and the 2-to-1 homomorphism from SU(2) to SO(3) is given by equations (0.12)-(0.14).

Exercise : prove that any matrix of SU(2) may be written as $\begin{pmatrix} a & -b \\ b^* & a^* \end{pmatrix}$ with $|a|^2 + |b|^2 = 1$. What is the connection with (0.10) ?

0.2 Infinitesimal generators. The $su(2)$ Lie algebra

0.2.1 Infinitesimal generators of SO(3)

Rotations $R_{\mathbf{n}}(\psi)$ around a given axis $\mathbf{n}$ form a *one-parameter subgroup*, isomorphic to $SO(2)$. In this chapter, we follow the common use (among physicists) and write the infinitesimal generators of rotations as *Hermitian* operators $J = J^\dagger$. Thus

$$R_{\mathbf{n}}(\mathrm{d}\psi) = (I - i\mathrm{d}\psi J_{\mathbf{n}}) \tag{0.17}$$

where $J_{\mathbf{n}}$ is the "generator" of these rotations, a Hermitian 3×3 matrix. Let us first show that we may reconstruct the finite rotations from these infinitesimal generators. By the group property,

$$R_{\mathbf{n}}(\psi + \mathrm{d}\psi) = R_{\mathbf{n}}(\mathrm{d}\psi)R_{\mathbf{n}}(\psi) = (I - i\mathrm{d}\psi J_{\mathbf{n}})R_{\mathbf{n}}(\psi)\ , \tag{0.18}$$

or equivalently

$$\frac{\partial R_{\mathbf{n}}(\psi)}{\partial \psi} = -iJ_{\mathbf{n}}R_{\mathbf{n}}(\psi) \tag{0.19}$$

which, on account of $R_{\mathbf{n}}(0) = I$, may be integrated into

$$\boxed{R_{\mathbf{n}}(\psi) = e^{-i\psi J_{\mathbf{n}}}\ .} \tag{0.20}$$

To be more explicit, introduce the three basic J_1, J_2 and J_3 describing the infinitesimal rotations around the corresponding axes[2]. From the infinitesimal version of (0.3) it follows that

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad J_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \qquad J_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{0.21}$$

which may be expressed by a unique formula

$$\boxed{(J_k)_{ij} = -i\epsilon_{ijk}} \tag{0.22}$$

with the completely antisymmetric tensor ϵ_{ijk}.

We now show that matrices (0.21) form a basis of infinitesimal generators and that $J_{\mathbf{n}}$ is simply expressed as

$$J_{\mathbf{n}} = \sum_k J_k n_k \tag{0.23}$$

which allows us to rewrite (0.20) in the form

$$\boxed{R_{\mathbf{n}}(\psi) = e^{-i\psi\sum_k n_k J_k}\ .} \tag{0.24}$$

The expression (0.23) follows simply from the infinitesimal form of Rodrigues formula, $R_{\mathbf{n}}(\mathrm{d}\psi) = (\mathbf{I} + \mathrm{d}\psi\,\mathbf{n}\times)$ hence $-iJ_{\mathbf{n}} = \mathbf{n}\times$ or alternatively $-i(J_{\mathbf{n}})_{ij} = \epsilon_{ikj}n_k = n_k(-iJ_k)_{ij}$, q.e.d. (Here and in the following, we make use of the convention of summation over repeated indices: $\epsilon_{ikj}n_k \equiv \sum_k \epsilon_{ikj}n_k$, etc.)

A comment about (0.24): it is obviously wrong to write in general $R_{\mathbf{n}}(\psi) = e^{-i\psi\sum_k n_k J_k} \stackrel{?}{=} \prod_{k=1}^3 e^{-i\psi n_k J_k}$ because of the non commutativity of the J's. On the other hand, formula (0.7) shows that any rotation of SO(3) may be written under the form

$$R(\alpha, \beta, \gamma) = e^{-i\alpha J_3}e^{-i\beta J_2}e^{-i\gamma J_3}\ . \tag{0.25}$$

[2] Do not confuse $J_{\mathbf{n}}$ labelled the unit vector $\mathbf{n}$ with J_k, k-th component of $\mathbf{J}$. The relation between the two will be explained shortly.

The three matrices J_i, $i = 1, 2, 3$ satisfy the very important commutation relations

$$\boxed{[J_i, J_j] = i\epsilon_{ijk} J_k} \tag{0.26}$$

which follow from the identity (Jacobi) verified by the tensor ϵ

$$\epsilon_{iab}\epsilon_{bjc} + \epsilon_{icb}\epsilon_{baj} + \epsilon_{ijb}\epsilon_{bca} = 0 \ . \tag{0.27}$$

Exercise: note the structure of this identity (i is fixed, b summed over, cyclic permutation over the three others) and check that it implies (0.26).

In view of the importance of relations (0.23–0.26), it may be useful to recover them by another route. Note first that equation (0.5) implies that for any R

$$Re^{-i\psi J_{\mathbf{n}}} R^{-1} = e^{-i\psi R J_{\mathbf{n}} R^{-1}} = e^{-i\psi J_{\mathbf{n}'}} \tag{0.28}$$

with $\mathbf{n}' = R\mathbf{n}$, whence

$$R J_{\mathbf{n}} R^{-1} = J_{\mathbf{n}'} \ . \tag{0.29}$$

The tensor ϵ_{ijk} is invariant under rotations

$$\epsilon_{lmn}\mathcal{R}_{il}\mathcal{R}_{jm}\mathcal{R}_{kn} = \epsilon_{ijk} \det \mathcal{R} = \epsilon_{ijk} \tag{0.30}$$

since the matrix $\mathcal{R}$ is of determinant 1. That matrix being also orthogonal, one may push one $\mathcal{R}$ to the right-hand side

$$\epsilon_{lmn}\mathcal{R}_{jm}\mathcal{R}_{kn} = \epsilon_{ijk}\mathcal{R}_{il} \tag{0.31}$$

which thanks to (0.22) expresses that

$$\mathcal{R}_{jm}(J_l)_{mn}\mathcal{R}^{-1}_{nk} = (J_i)_{jk}\mathcal{R}_{il} \tag{0.32}$$

i.e. for any R and its matrix $\mathcal{R}$,

$$R J_l R^{-1} = J_i \mathcal{R}_{il} \ . \tag{0.33}$$

Let R be a rotation which maps the unit vector $\mathbf{z}$ colinear to Oz on the vector $\mathbf{n}$, thus $n_k = \mathcal{R}_{k3}$ and

$$J_{\mathbf{n}} \stackrel{(0.29)}{=} R J_3 R^{-1} \stackrel{(0.33)}{=} J_k \mathcal{R}_{k3} = J_k n_k \ , \tag{0.34}$$

which is just (0.23). Note that equations (0.33) and (0.34) are compatible with (0.29)

$$J_{\mathbf{n}'} \stackrel{(0.29)}{=} R J_{\mathbf{n}} R^{-1} \stackrel{(0.34)}{=} R J_k n_k R^{-1} \stackrel{(0.33)}{=} J_l \mathcal{R}_{lk} n_k = J_l n'_l \ .$$

As we shall see later in a more systematic way, the commutation relation (0.26) of infinitesimal generators J encodes an infinitesimal version of the group law. Consider for example a rotation of infinitesimal angle $d\psi$ around Oy acting on J_1

$$R_2(d\psi) J_1 R_2^{-1}(d\psi) \stackrel{(0.33)}{=} J_k [\mathcal{R}_2(d\psi)]_{k1} \tag{0.35}$$

but to first order, $R_2(d\psi) = \mathbf{I} - id\psi J_2$, and thus the left hand side of (0.35) equals $J_1 - id\psi[J_2, J_1]$ while on the right hand side, $[\mathcal{R}_2(d\psi)]_{k1} = \delta_{k1} - id\psi(J_2)_{k1} = \delta_{k1} - d\psi\delta_{k3}$ by (0.22), whence $i[J_1, J_2] = -J_3$, which is one of the relations (0.26).

0.2.2 Infinitesimal generators of SU(2)

Let us examine now things from the point of view of SU(2). Any unitary matrix U (here 2×2) may be diagonalized by a unitary change of basis $U = V\exp\{i\,\mathrm{diag}\,(\lambda_k)\}V^\dagger$, V unitary, and hence written as

$$U = \exp iH = \sum_0^\infty \frac{(iH)^n}{n!} \tag{0.36}$$

with H Hermitian, $H = V\,\mathrm{diag}\,(\lambda_k)\,V^\dagger$. The sum converges (for the norm $||M||^2 = \mathrm{tr}\,MM^\dagger$). The unimodularity condition $1 = \det U = \exp i\mathrm{tr}\,H$ is ensured if $\mathrm{tr}\,H = 0$. The set of such Hermitian traceless matrices forms a vector space $\mathcal{V}$ of dimension 3 over $\mathbb{R}$, with a basis given by the three Pauli matrices

$$H = \sum_{k=1}^3 \eta_k \frac{\sigma_k}{2}\ , \tag{0.37}$$

which may be inserted back into (0.36). (In fact we already observed that any unitary 2×2 matrix may be written in the form (0.11)). Comparing that form with (0.24), or else comparing its infinitesimal version $U_n(\mathrm{d}\psi) = (I - i\,\mathrm{d}\psi\mathbf{n}.\frac{\boldsymbol{\sigma}}{2})$ with (0.17), we see that matrices $\frac12\sigma_j$ play in SU(2) the role played by infinitesimal generators J_j in SO(3). But these matrices $\frac12\sigma_.$ verify the same commutation relations

$$\left[\frac{\sigma_i}{2},\frac{\sigma_j}{2}\right] = i\epsilon_{ijk}\frac{\sigma_k}{2} \tag{0.38}$$

with the same *structure constants* ϵ_{ijk} as in (0.26). In other words, we have just discovered that infinitesimal generators J_i (eq. (0.21) of SO(3) and $\frac12\sigma_i$ of SU(2) satisfy the same commutation relations (we shall say later that they are the bases of two different representations of the same Lie algebra $su(2) = so(3)$). This has the consequence that calculations carried out with the $\frac12\vec\sigma$ and *making only use of commutation relations* are also valid with the $\vec J$, and vice versa. For instance, from (0.33), for example $R_2(\beta)J_kR_2^{-1}(\beta) = J_l\mathcal{R}_y(\beta)_{lk}$, it follows immediately, with no further calculation, that for Pauli matrices, we have

$$e^{-i\frac{\beta}{2}\sigma_2}\sigma_k e^{i\frac{\beta}{2}\sigma_2} = \sigma_l\mathcal{R}_y(\beta)_{lk} \tag{0.39}$$

where the matrix elements $\mathcal{R}_y$ are read off (0.4). Indeed there is a general identity stating that $e^ABe^{-A} = B + \sum_{n=1}^\infty \frac{1}{n!}\underbrace{[A[A,[\cdots,[A,B]\cdots]]]}_{n \text{ commutators}}$, see Chap. 1, eq. (1.29), and that computation thus involves only commutators. On the other hand, the relation

$$\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$$

(which does *not* involve only commutators) is specific to the dimension 2 representation of the su(2) algebra.

0.2.3 Lie algebra $su(2)$

Let us recapitulate: we have just introduced the commutation algebra (or Lie algebra) of infinitesimal generators of the group SU(2) (or SO(3)), denoted $su(2)$ or $so(3)$. It is defined by relations (0.26), that we write once again

$$[J_i, J_j] = i\epsilon_{ijk} J_k \ . \tag{0.26}$$

We shall also make frequent use of the three combinations

$$J_z \equiv J_3, \quad J_+ = J_1 + iJ_2, \quad J_- = J_1 - iJ_2 \ . \tag{0.40}$$

It is then immediate to compute

$$\begin{aligned} [J_3, J_+] &= J_+ \\ [J_3, J_-] &= -J_- \\ [J_+, J_-] &= 2J_3 \, . \end{aligned} \tag{0.41}$$

One also verifies that the *Casimir operator* defined as

$$\mathbf{J}^2 = J_1^2 + J_2^2 + J_3^2 = J_3^2 + J_3 + J_- J_+ \tag{0.42}$$

commutes with all the J's

$$[\mathbf{J}^2, J_.] = 0 \ , \tag{0.43}$$

which means that it is invariant under rotations.

Anticipating a little on the following, we shall be mostly interested in "unitary representations", where the generators J_i, $i = 1, 2, 3$ are Hermitian, hence

$$J_i^\dagger = J_i, \quad i = 1, 2, 3 \qquad J_\pm^\dagger = J_\mp \ . \tag{0.44}$$

Let us finally mention an interpretation of the J_i as differential operators acting on differentiable functions of coordinates in the space $\mathbb{R}^3$. In that space $\mathbb{R}^3$, an infinitesimal rotation acting on the vector $\mathbf{x}$ changes it into

$$\mathbf{x}' = R\mathbf{x} = \mathbf{x} + \delta\psi \mathbf{n} \times \mathbf{x}$$

hence a *scalar* function of $\mathbf{x}$, $f(\mathbf{x})$, is changed into $f'(\mathbf{x}') = f(\mathbf{x})$ or

$$\begin{aligned} f'(\mathbf{x}) &= f\left(R^{-1}\mathbf{x}\right) = f(\mathbf{x} - \delta\psi \mathbf{n} \times \mathbf{x}) \\ &= \left(1 - \delta\psi \mathbf{n}.\mathbf{x} \times \nabla\right) f(\mathbf{x}) \\ &= (1 - i\delta\psi \mathbf{n}.\mathbf{J}) f(\mathbf{x}) \ . \end{aligned} \tag{0.45}$$

We thus identify

$$\mathbf{J} = -i\mathbf{x} \times \nabla, \qquad J_i = -i\epsilon_{ijk} x_j \frac{\partial}{\partial x_k} \tag{0.46}$$

which allows us to compute it in arbitrary coordinates, for example spherical, see Appendix 0. (Compare also (0.46) with the expression of (orbital) angular momentum in Quantum Mechanics $L_i = \frac{\hbar}{i}\epsilon_{ijk} x_j \frac{\partial}{\partial x_k}$). Exercise: check that these differential operators do satisfy the commutation relations (0.26).

Among the combinations of J that one may construct, there is one that must play a particular role, namely the Laplacian on the sphere S^2, a second order differential operator which is invariant under changes of coordinates (see Appendix 0). It is in particular rotation invariant, of degree 2 in the $J_.$, this may only be the Casimir operator $\mathbf{J}^2$ (up to a factor). In fact the Laplacian in $\mathbb{R}^3$ reads in spherical coordinates

$$\begin{aligned} \Delta_3 &= \frac{1}{r}\frac{\partial^2}{\partial r^2} r - \frac{\mathbf{J}^2}{r^2} \\ &= \frac{1}{r}\frac{\partial^2}{\partial r^2} r + \frac{\Delta_{\text{sphere } S^2}}{r^2} \ . \end{aligned} \tag{0.47}$$

For the sake of simplicity we have restricted this discussion to scalar functions, but one might more generally consider the transformation of a collection of functions "forming a representation" of SO(3), *i.e.* transforming linearly among themselves under the action of that group

$$A'(\mathbf{x}') = D(R)A(\mathbf{x})$$

or else

$$A'(\mathbf{x}) = D(R)A\left(R^{-1}\mathbf{x}\right) \ ,$$

for example a vector field transforming as

$$\mathbf{A}'(\mathbf{x}) = R\mathbf{A}(R^{-1}\mathbf{x}) \ .$$

What are now the infinitesimal generators for such objects ? Show that they now have two contributions, one given by (0.46) and the other coming from the infinitesimal form of R; in physical terms, these two contributions correspond to the orbital and to the intrinsic (spin) angular momenta.

0.3 Representations of SU(2)

0.3.1 Representations of the groups SO(3) and SU(2)

We are familiar with the notions of vectors or tensors in the geometry of the space $\mathbb{R}^3$. They are objects that transform linearly under rotations

$$V_i \mapsto \mathcal{R}_{ii'}V_{i'} \qquad (V\otimes W)_{ij} = V_iW_j \mapsto \mathcal{R}_{ii'}\mathcal{R}_{jj'}(V\otimes W)_{i'j'} = \mathcal{R}_{ii'}\mathcal{R}_{jj'}V_{i'}W_{j'} \quad \text{etc.}$$

More generally we call *representation* of a group G in a vector space E a homomorphism of G into the group $\mathrm{GL}(E)$ of linear transformations of E (see Chap. 2). Thus, as we just saw, the group SO(3) admits a representation in the space $\mathbb{R}^3$ (the vectors V of the above example), another representation in the space of rank 2 tensors, etc. We now want to build the general representations of SO(3) and SU(2). For the needs of physics, in particular of quantum mechanics, we are mostly interested in unitary representations, in which the representation matrices are unitary. In fact, as we'll see, it is enough to study the representations of SU(2) to also get those of SO(3), and even better, it is enough to study the way the group elements close to the identity are represented, *i.e.* to find the representations of the infinitesimal generators of SU(2) (and SO(3)).

To summarize : to find all the unitary representations of the group SU(2), it is thus sufficient to find the representations by Hermitian matrices of its Lie algebra $su(2)$, that is, Hermitian operators satisfying the commutation relations (0.26).

0.3.2 Representations of the algebra $su(2)$

We now proceed to the classical construction of representations of the algebra $su(2)$. As above, $J_\pm$ and J_z denote the representatives of infinitesimal generators in a certain representation. They thus satisfy the commutation relations (0.41) and hermiticity (0.44). Commutation of operators J_z and $\mathbf{J}^2$ ensures that one may find common eigenvectors. The eigenvalues of these

Hermitian operators are real, and moreover, $\mathbf{J}^2$ being semi-definite positive, one may always write its eigenvalues in the form $j(j+1)$, j real non negative (*i.e.* $j \geq 0$), and one thus considers a common eigenvector $|j\,m\,\rangle$

$$\begin{aligned} \mathbf{J}^2|j\,m\,\rangle &= j(j+1)|j\,m\,\rangle \\ J_z|j\,m\,\rangle &= m|j\,m\,\rangle\,, \end{aligned} \tag{0.48}$$

with m a real number, a priori arbitrary at this stage. By a small abuse of language, we call $|jm\,\rangle$ an "eigenvector of eigenvalues (j, m)".
(i) Act with J_+ and $J_- = J_+^\dagger$ on $|j\,m\,\rangle$. Using the relation $J_\pm J_\mp = \mathbf{J}^2 - J_z^2 \pm J_z$ (a consequence of (0.41)), the squared norm of $J_\pm|j\,m\,\rangle$ is computed to be:

$$\begin{aligned} \langle\,j\,m|J_-J_+|j\,m\,\rangle &= (j(j+1) - m(m+1))\,\langle\,j\,m|j\,m\,\rangle \\ &= (j-m)(j+m+1)\langle\,j\,m|j\,m\,\rangle \\ \langle\,j\,m|J_+J_-|j\,m\,\rangle &= (j(j+1) - m(m-1))\,\langle\,j\,m|j\,m\,\rangle \\ &= (j+m)(j-m+1)\langle\,j\,m|j\,m\,\rangle\,. \end{aligned} \tag{0.49}$$

These squared norms cannot be negative and thus

$$\begin{aligned} (j-m)(j+m+1) \geq 0 \qquad &: \qquad -j-1 \leq m \leq j \\ (j+m)(j-m+1) \geq 0 \qquad &: \qquad -j \leq m \leq j+1 \end{aligned} \tag{0.50}$$

which implies

$$-\,j \leq m \leq j\,. \tag{0.51}$$

Moreover $J_+|j\,m\,\rangle = 0$ iff $m = j$ and $J_-|j\,m\,\rangle = 0$ iff $m = -j$

$$J_+|j\,j\,\rangle = 0 \qquad\qquad J_-|j\,\,-j\,\rangle = 0\,. \tag{0.52}$$

(ii) If $m \neq j$, $J_+|j\,m\,\rangle$ is non vanishing, hence is an eigenvector of eigenvalues $(j, m+1)$. Indeed

$$\begin{aligned} \mathbf{J}^2 J_+|j\,m\,\rangle &= J_+\mathbf{J}^2|j\,m\,\rangle = j(j+1)J_+|j\,m\,\rangle \\ J_zJ_+|j\,m\,\rangle &= J_+(J_z+1)|j\,m\,\rangle = (m+1)J_+|j\,m\,\rangle\,. \end{aligned} \tag{0.53}$$

Likewise if $m \neq -j$, $J_-|j\,m\,\rangle$ is a (non vanishing) eigenvector of eigenvalues $(j, m-1)$.
(iii) Consider now the sequence of vectors

$$|j\,m\,\rangle,\; J_-|j\,m\,\rangle,\; J_-^2|j\,m\,\rangle,\; \cdots, J_-^p|j\,m\,\rangle\cdots$$

If non vanishing, they are eigenvectors of J_z of eigenvalues $m, m-1, m-2, \cdots, m-p\cdots$. As the allowed eigenvalues of J_z are bound by (0.51), this sequence must stop after a finite number of steps. Let p be the integer such that $J_-^p|j\,m\,\rangle \neq 0$, $J_-^{p+1}|j\,m\,\rangle = 0$. By (0.52), $J_-^p|j\,m\,\rangle$ is an eigenvector of eigenvalues $(j, -j)$ hence $m - p = -j$, *i.e.*

$$(j+m) \text{ is a non negative integer}\,. \tag{0.54}$$

Acting likewise with $J_+, J_+^2, \cdots$ sur $|j\,m\,\rangle$, we are led to the conclusion that

$$(j-m) \text{ is a non negative integer}\,. \tag{0.55}$$

and thus j *and* m *are simultaneously integers or half-integers*. For each value of j

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \cdots$$

m may take the $2j+1$ values [3]

$$m = -j, -j+1, \cdots, j-1, j\ . \tag{0.56}$$

Starting from the vector $|j\,m=j\,\rangle$, ("highest weight vector"), now chosen of norm 1, we construct the orthonormal basis $|j\,m\,\rangle$ by iterated application of J_- and we have

$$\begin{aligned} J_+|j\,m\,\rangle &= \sqrt{j(j+1)-m(m+1)}|j\,m+1\,\rangle \\ J_-|j\,m\,\rangle &= \sqrt{j(j+1)-m(m-1)}|j\,m-1\,\rangle \\ J_z|j\,m\,\rangle &= m|j\,m\,\rangle\ . \end{aligned} \tag{0.57}$$

These $2j+1$ vectors form a basis of the "spin j representation" of the $su(2)$ algebra.

In fact this representation of the *algebra* $su(2)$ extends to a representation of the *group* SU(2), as we now show.

Remark. The previous discussion has given a central role to the unitarity of the representation and hence to the hermiticity of infinitesimal generators, hence to positivity: $||J_\pm|j\,m\,\rangle||^2 \geq 0 \Longrightarrow -j \leq m \leq j$, etc, which allowed us to conclude that the representation is necessarily of finite dimension. Conversely one may insist on the latter condition, and show that it suffices to ensure the previous conditions on j and m. Starting from an eigenvector $|\psi\,\rangle$ of J_z, the sequence $J_+^p|\psi\,\rangle$ yields eigenvectors of J_z of increasing eigenvalue, hence linearly independent, as long as they do not vanish. If by hypothesis the representation is of finite dimension, this sequence is finite, and there exists a vector denoted $|j\,\rangle$ such that $J_+|j\,\rangle = 0$, $J_z|j\,\rangle = j|j\,\rangle$. By the relation $\mathbf{J}^2 = J_-J_+ + J_z(J_z+1)$, it is also an eigenvector of eigenvalue $j(j+1)$ of $\mathbf{J}^2$. It thus identifies with the highest weight vector denoted previously $|j\,j\,\rangle$, a notation that we thus adopt in the rest of this discussion. Starting from this vector, the $J_-^p|j\,j\,\rangle$ form a sequence that must also be finite

$$\exists q \quad J_-^{q-1}|j\,j\,\rangle \neq 0 \qquad J_-^q|j\,j\,\rangle = 0\ . \tag{0.58}$$

One easily shows by induction that

$$J_+J_-^q|j\,j\,\rangle = [J_+, J_-^q]|j\,j\,\rangle = q(2j+1-q)J_-^{q-1}|j\,j\,\rangle = 0 \tag{0.59}$$

hence $q = 2j+1$. The number j is thus integer or half-integer, the vectors of the representation built in that way are eigenvectors of $\mathbf{J}^2$ of eigenvalue $j(j+1)$ and of J_z of eigenvalue m satisfying (0.56). We have recovered all the previous results. In this form, the construction of these "highest weight representations" generalizes to other Lie algebras, (even of infinite dimension, such as the Virasoro algebra, see Chap. 1, § 1.3.6).

The matrices $\mathcal{D}^j$ of the spin j representation are such that under the action of the rotation $U \in SU(2)$

$$|j\,m\,\rangle \mapsto D^j(U)|j\,m\,\rangle = |j\,m'\,\rangle \mathcal{D}^j_{m'm}(U)\ . \tag{0.60}$$

[3]In fact, we have just found a necessary condition on the j, m. That all these j give indeed rise to representations will be verified in the next subsection.

Depending on the parametrization ($(\mathbf{n},\psi)$, angles d'Euler, ...), we write $\mathcal{D}^j_{m'm}(\mathbf{n},\psi)$, $\mathcal{D}^j_{m'm}(\alpha,\beta,\gamma)$, etc. By (0.7), we thus have

$$\begin{aligned}\mathcal{D}^j_{m'm}(\alpha,\beta,\gamma) &= \langle\, j\,m'|D(\alpha,\beta,\gamma)|j\,m\,\rangle \\ &= \langle\, j\,m'|e^{-i\alpha J_z}e^{-i\beta J_y}e^{-i\gamma J_z}|j\,m\,\rangle \\ &= e^{-i\alpha m'}d^j_{m'm}(\beta)e^{-i\gamma m}\end{aligned} \tag{0.61}$$

where the Wigner matrix d^j is defined by

$$d^j_{m'm}(\beta) = \langle\, j\,m'|e^{-i\beta J_y}|j\,m\,\rangle\ . \tag{0.62}$$

An explicit formula for d^j will be given in the next subsection. We also have

$$\begin{aligned}\mathcal{D}^j_{m'm}(\mathbf{z},\psi) &= e^{-i\psi m}\delta_{mm'} \\ \mathcal{D}^j_{m'm}(\mathbf{y},\psi) &= d^j_{m'm}(\psi)\ .\end{aligned} \tag{0.63}$$

Exercise : Compute $\mathcal{D}^j(\mathbf{x},\psi)$. (Hint : use (0.5).)

One notices that $\mathcal{D}^j(\mathbf{z},2\pi) = (-1)^{2j}I$, since $(-1)^{2m} = (-1)^{2j}$ using (0.55), and this holds true for any axis $\mathbf{n}$ by the conjugation (0.5)

$$\mathcal{D}^j(\mathbf{n},2\pi) = (-1)^{2j}I\ . \tag{0.64}$$

This shows that a 2π rotation in SO(3) is represented by $-I$ in a half-integer-spin representation of SU(2). Half-integer-spin representations of SU(2) are said to be "projective", (*i.e.* here, up to a sign), representations of SO(3); we return in Chap. 2 to this notion of projective representation.

We also verify the *unimodularity* of matrices $\mathcal{D}^j$ (or equivalently, the fact that representatives of infinitesimal generators are traceless). If $\mathbf{n} = R\mathbf{z}$, $\mathcal{D}(\mathbf{n},\psi) = \mathcal{D}(R)\mathcal{D}(\mathbf{z},\psi)\mathcal{D}^{-1}(R)$, hence

$$\det\mathcal{D}(\mathbf{n},\psi) = \det\mathcal{D}(\mathbf{z},\psi) = \det e^{-i\psi J_z} = \prod_{m=-j}^{j} e^{-im\psi} = 1\ . \tag{0.65}$$

It may be useful to write explicitly these matrices in the cases $j=\frac12$ and $j=1$. The case of $j=\frac12$ is very simple, since

$$\begin{aligned}\mathcal{D}^{\frac12}(U) &= U = e^{-i\frac12\psi\mathbf{n}.\boldsymbol{\sigma}} = \begin{pmatrix}\cos\frac{\psi}{2} - i\cos\theta\sin\frac{\psi}{2} & -i\sin\frac{\psi}{2}\sin\theta\,e^{-i\phi} \\ -i\sin\frac{\psi}{2}\sin\theta\,e^{i\phi} & \cos\frac{\psi}{2} + i\cos\theta\sin\frac{\psi}{2}\end{pmatrix} \\ &= e^{-i\frac{\alpha}{2}\sigma_3}e^{-i\frac{\beta}{2}\sigma_2}e^{-i\frac{\gamma}{2}\sigma_3} = \begin{pmatrix}\cos\frac{\beta}{2}e^{-\frac{i}{2}(\alpha+\gamma)} & -\sin\frac{\beta}{2}e^{-\frac{i}{2}(\alpha-\gamma)} \\ \sin\frac{\beta}{2}e^{\frac{i}{2}(\alpha-\gamma)} & \cos\frac{\beta}{2}e^{\frac{i}{2}(\alpha+\gamma)}\end{pmatrix}\end{aligned} \tag{0.66}$$

an expected result since the matrices U of the group form obviously a representation. (As a by-product, we have derived relations between the two parametrizations, $(\mathbf{n},\psi) = (\theta,\phi,\psi)$ and Euler angles (α,β,γ).) For $j=1$, in the basis $|1,1\,\rangle$, $|1,0\,\rangle$ and $|1,-1\,\rangle$ where J_z is diagonal (which *is not* the basis (0.21) !)

$$J_z = \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix} \qquad J_+ = \sqrt{2}\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix} \qquad J_- = \sqrt{2}\begin{pmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix} \tag{0.67}$$

whence

$$d^1(\beta) = e^{-i\beta J_y} = \begin{pmatrix} \frac{1+\cos\beta}{2} & -\frac{\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & -\frac{\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2} \end{pmatrix} \tag{0.68}$$

as the reader may check.

In the following subsection, we write more explicitly these representation matrices of the group SU(2), and in Appendix E of Chap. 2, give more details on the differential equations they satisfy and on their relations with "special functions", orthogonal polynomials and spherical harmonics...

Irreductibility

A central notion in the study of representations is that of irreducibility. A representation is irreducible if it has no invariant subspace. Let us show that the spin j representation of SU(2) that we have just built is irreducible. We show below in Chap. 2 that, as the representation is unitary, it is either irreducible or "completely reducible" (there exists an invariant subspace and its supplementary space is also invariant) ; in the latter case, there would exist block-diagonal operators, different from the identity and commuting the matrices of the representation, in particulier with the generators J_i. But in the basis (0.5) any matrix M that commutes with J_z is diagonal, $M_{mm'} = \mu_m \delta_{mm'}$, (check it !), and commutation with J_+ forces all μ_m to be equal: the matrix M is a multiple of the identity and the representation is indeed irreducible.

One may also wonder why the study of finite dimensional representations that we just carried out suffices to the physicist's needs, for instance in quantum mechanics, where the scene usually takes place in an infinite dimensional Hilbert space. We show below (Chap. 2) that *Any representation of SU(2) or SO(3) in a Hilbert space is equivalent to a unitary representation, and is completely reducible to a (finite or infinite) sum of finite dimensional irreducible representations.*

0.3.3 Explicit construction

Let ξ and η be two complex variables on which matrices $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of SU(2) act according to $\xi' = a\xi + c\eta$, $\eta' = b\xi + d\eta$. In other terms, ξ and η are the basis vectors of the representation of dimension 2 (representation of spin $\frac{1}{2}$) of SU(2). An explicit construction of the previous representations is then obtained by considering homogenous polynomials of degree $2j$ in the two variables ξ and η, a basis of which is given by the $2j+1$ polynomials

$$P_{jm} = \frac{\xi^{j+m}\eta^{j-m}}{\sqrt{(j+m)!(j-m)!}} \qquad m = -j, \cdots j\,. \tag{0.69}$$

(In fact, the following considerations also apply if U is an arbitrary matrix of the group GL(2,$\mathbb{C}$) and provide a representation of that group.) Under the action of U on ξ and η, the $P_{jm}(\xi,\eta)$ transform into $P_{jm}(\xi',\eta')$, also homogenous of degree $2j$ in ξ and η, which may thus be expanded on the $P_{jm}(\xi,\eta)$. The latter thus span a dimension $2j+1$ representation of SU(2) (or of

GL(2,ℂ)), which is nothing else than the previous spin j representation. This enables us to write quite explicit formulae for the $\mathcal{D}^j$

$$P_{jm}(\xi',\eta') = \sum_{m'} P_{jm'}(\xi,\eta)\mathcal{D}^j_{m'm}(U) \ . \tag{0.70}$$

We find explicitly

$$\mathcal{D}^j_{m'm}(U) = \big((j+m)!(j-m)!(j+m')!(j-m')!\big)^{\frac12} \sum_{\substack{n_1,n_2,n_3,n_4\geq 0\\ n_1+n_2=j+m';\ n_3+n_4=j-m'\\ n_1+n_3=j+m;\ n_2+n_4=j-m}} \frac{a^{n_1}b^{n_2}c^{n_3}d^{n_4}}{n_1!n_2!n_3!n_4!} \ . \tag{0.71}$$

For $U=-\mathbf{I}$, one may check once again that $\mathcal{D}^j(-\mathbf{I}) = (-1)^{2j}I$. In the particular case of $U = e^{-i\psi\frac{\sigma_2}{2}} = \cos\frac{\psi}{2}\mathbf{I} - i\sin\frac{\psi}{2}\sigma_2$, we thus have

$$d^j_{m'm}(\psi) = \big((j+m)!(j-m)!(j+m')!(j-m')!\big)^{\frac12} \sum_{k\geq 0} \frac{(-1)^{k+j-m}\cos\frac{\psi}{2}^{2k+m+m'}\sin\frac{\psi}{2}^{2j-2k-m-m'}}{(m+m'+k)!(j-m-k)!(j-m'-k)!k!} \tag{0.72}$$

where the sum runs over $k\in[\inf(0,-m-m'),\sup(j-m,j-m')]$. The expression of the infinitesimal generators acting on polynomials P_{jm} is obtained by considering U close to the identity. One finds

$$J_+ = \xi\frac{\partial}{\partial\eta} \qquad J_- = \eta\frac{\partial}{\partial\xi} \qquad J_z = \frac12\left(\xi\frac{\partial}{\partial\xi} - \eta\frac{\partial}{\partial\eta}\right) \tag{0.73}$$

on which it is easy to check commutation relations as well as the action on the P_{jm} in accordance with (0.57). This completes the identification of (0.69) with the spin j representation.

Remarks

1. Repeat the proof of irreducibility of the spin j representation in that new form.
2. Notice that the space of the homogenous polynomials of degree $2j$ in the variables ξ and η is nothing else than the symmetrized $2j$-th tensor power of the representation of dimension 2 (see the definition below).

0.4 Direct product of representations of SU(2)

0.4.1 Direct product of representations and the "addition of angular momenta"

Consider the direct (or tensor) product of two representations of spin j_1 and j_2 and their decomposition on vectors of given total spin ("decomposition into irreducible representations"). We start with the product representation spanned by the vectors

$$|j_1\,m_1\,\rangle\otimes|j_2\,m_2\,\rangle \equiv |j_1\,m_1;j_2\,m_2\,\rangle \quad \text{written in short as} \quad |m_1\,m_2\,\rangle \tag{0.74}$$

on which the infinitesimal generators act as

$$\mathbf{J} = \mathbf{J}^{(1)}\otimes\mathbf{I}^{(2)} + \mathbf{I}^{(1)}\otimes\mathbf{J}^{(2)} \ . \tag{0.75}$$

The upper index indicates on which space the operators act. By an abuse of notation, one frequently writes, instead of (0.75)

$$\mathbf{J} = \mathbf{J}^{(1)} + \mathbf{J}^{(2)} \tag{0.75$'$}$$

and (in Quantum Mechanics) one refers to the "addition of angular momenta" $\mathbf{J}^{(1)}$ and $\mathbf{J}^{(2)}$. The problem is thus to decompose the vectors (0.74) onto a basis of eigenvectors of $\mathbf{J}$ and J_z. As $\mathbf{J}^{(1)2}$ and $\mathbf{J}^{(2)2}$ commute with one another and with $\mathbf{J}^2$ and J_z, one may seek common eigenvectors that we denote

$$|(j_1\, j_2)\, J\, M\,\rangle \qquad \text{or more simply} \qquad |J\, M\,\rangle \tag{0.76}$$

where it is understood that the value of j_1 and j_2 is fixed. The question is thus twofold: which values can J and M take; and what is the matrix of the change of basis $|m_1\, m_2\,\rangle \to |J\, M\,\rangle$? In other words, what is the (Clebsch-Gordan) decomposition and what are the Clebsch-Gordan coefficients?

The possible values of M, eigenvalue of $J_z = J_z^{(1)} + J_z^{(2)}$, are readily found

$$\begin{aligned} \langle\, m_1\, m_2 | J_z | J\, M\,\rangle &= (m_1 + m_2)\langle\, m_1\, m_2 | J\, M\,\rangle \\ &= M \langle\, m_1\, m_2 | J\, M\,\rangle \end{aligned} \tag{0.77}$$

and the only value of M such that $\langle\, m_1\, m_2 | J\, M\,\rangle \neq 0$ is thus

$$M = m_1 + m_2\ . \tag{0.78}$$

For j_1, j_2 and M fixed, there are as many independent vectors with that eigenvalue of M as there are couples (m_1, m_2) satisfying (0.78), thus

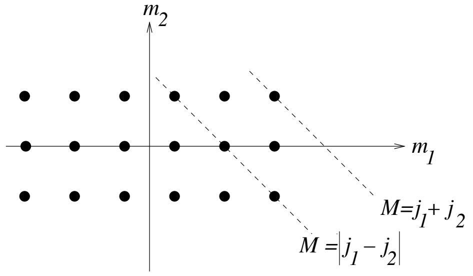
$$n(M) = \begin{cases} 0 & \text{if } |M| > j_1 + j_2 \\ j_1 + j_2 + 1 - |M| & \text{if } |j_1 - j_2| \leq |M| \leq j_1 + j_2 \\ 2\mathrm{inf}(j_1, j_2) + 1 & \text{if } 0 \leq |M| \leq |j_1 - j_2| \end{cases} \tag{0.79}$$

(see the left Fig. 3 in which $j_1 = 5/2$ and $j_2 = 1$). Let N_J be the number of times the representation of spin J appears in the decomposition of the representations of spin j_1 et j_2. The $n(M)$ vectors of eigenvalue M for J_z may also be regarded as coming from the N_J vectors $|J\, M\,\rangle$ for the different values of J compatible with that value of M

$$n(M) = \sum_{J \geq |M|} N_J \tag{0.80}$$

hence, by subtracting two such relations

$$\begin{aligned} N_J &= n(J) - n(J+1) \\ &= 1 \qquad \text{iff si} \quad |j_1 - j_2| \leq J \leq j_1 + j_2 \\ &= 0 \qquad \text{otherwise.} \end{aligned} \tag{0.81}$$



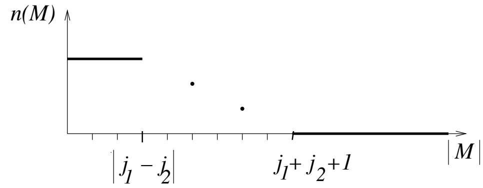


Fig. 3

To summarize, we have just shown that the $(2j_1+1)(2j_2+1)$ vectors (0.74) (with j_1 and j_2 fixed) may be reexpressed in terms of vectors $|J\,M\,\rangle$ with

$$\begin{aligned} J &= |j_1-j_2|, |j_1-j_2|+1, \cdots, j_1+j_2 \\ M &= -J, -J+1, \cdots, J\ . \end{aligned} \tag{0.82}$$

Note that multiplicities N_J take the value 0 or 1 ; it is a pecularity of SU(2) that multiplicities larger than 1 do not occur in the decomposition of the tensor product of irreducible representations, *i.e.* here of fixed spin.

0.4.2 Clebsch-Gordan coefficients, 3-j and 6-j symbols. . .

The change of orthonormal basis $|j_1\,m_1; j_2\,m_2\,\rangle \to |(j_1\,j_2)\,J\,M\,\rangle$ is carried out by the Clebsch-Gordan coefficients (C.G.) $\langle\,(j_1\,j_2); J\,M|j_1\,m_1; j_2\,m_2\,\rangle$ which form a unitary matrix

$$|j_1\,m_1; j_2\,m_2\,\rangle = \sum_{J=|j_1-j_2|}^{j_1+j_2} \sum_{M=-J}^{J} \langle\,(j_1\,j_2)\,J\,M|j_1\,m_1; j_2\,m_2\,\rangle |(j_1\,j_2)\,J\,M\,\rangle \tag{0.83}$$

$$|j_1\,j_2; J\,M\,\rangle = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \langle\,(j_1\,j_2)\,J\,M|j_1\,m_1; j_2\,m_2\,\rangle^* |j_1\,m_1; j_2\,m_2\,\rangle\ . \tag{0.84}$$

Note that in the first line, M is fixed in terms of m_1 and m_2; and that in the second one, m_2 is fixed in terms of m_1, for given M. Each relation thus implies only one summation. The value of these C.G. depends in fact on a choice of a relative phase between vectors (0.74) and (0.76); the usual convention is that for each value of J, one chooses

$$\langle\,J\,M = J\,|\,j_1\,m_1 = j_1; j_2\,m_2 = J - j_1\,\rangle \qquad \text{real}\,. \tag{0.85}$$

The other vectors are then unambiguously defined by (0.57) and we shall now show that all C.G. are real. C.G. satisfy recursion relations that are consequences of (0.57). Applying indeed $J_\pm$ to the two sides of (0.83), one gets

$$\begin{aligned} \sqrt{J(J+1)-M(M\pm1)} \qquad & \langle\,(j_1\,j_2)\,J\,M|j_1\,m_1; j_2\,m_2\,\rangle \\ = \ & \sqrt{j_1(j_1+1)-m_1(m_1\pm1)}\langle\,(j_1\,j_2)\,J\,M\pm1|j_1\,m_1\pm1; j_2\,m_2\,\rangle \\ + \ & \sqrt{j_2(j_2+1)-m_2(m_2\pm1)}\langle\,(j_1\,j_2)\,J\,M\pm1|j_1\,m_1; j_2\,m_2\pm1\,\rangle \end{aligned} \tag{0.86}$$

which, together with the normalization $\sum_{m_1,m_2} |\langle\,j_1\,m_1; j_2\,m_2|(j_1\,j_2)\,J\,M\,\rangle|^2 = 1$ and the convention (0.85), allows one to determine all the C.G. As stated before, they are clearly all real.

The C.G. of the group SU(2), which describe a change of orthonormal basis, form a unitary matrix and thus satisfy orthogonality and completeness properties

$$\begin{aligned}\sum_{m_1=-j_1}^{j_1} \langle\, j_1\, m_1; j_2\, m_2 | (j_1\, j_2)\, J\, M \,\rangle \langle\, j_1\, m_1; j_2\, m_2 | (j_1\, j_2)\, J'\, M' \,\rangle &= \delta_{JJ'}\delta_{MM'} \quad \text{if } |j_1 - j_2| \le J \le j_1 + j_2 \\ \sum_{J=|j_1-j_2|}^{j_1+j_2} \langle\, j_1\, m_1; j_2\, m_2 | (j_1\, j_2)\, J\, M \,\rangle \langle\, j_1\, m_1'; j_2\, m_2' | (j_1\, j_2)\, J\, M \,\rangle &= \delta_{m_1 m_1'}\delta_{m_2 m_2'} \quad \text{if } |m_1| \le j_1,\ |m_2| \le j_2\ .\end{aligned} \tag{0.87}$$

Once again, each relation implies only one non trivial summation.

Rather than the C.G. coefficients, one may consider another set of equivalent coefficients, called *3-j symbols*. They are defined through

$$\begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & -M \end{pmatrix} = \frac{(-1)^{j_1-j_2+M}}{\sqrt{2J+1}} \langle\, j_1\, m_1; j_2\, m_2 | (j_1\, j_2)\, J\, M \,\rangle \tag{0.88}$$

and they enjoy simple symmetry properties:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

is invariant under cyclic permutation of its three columns, and changes by the sign $(-1)^{j_1+j_2+j_3}$ when two columns are interchanged *or* when the signs of m_1, m_2 and m_3 are reversed. The reader will find a multitude of tables and explicit formulas of the C.G. and $3j$ coefficients in the literature.

Let us just give some values of C.G. for low spins

$$\frac{1}{2}\otimes\frac{1}{2}\ :\quad \begin{aligned} |(\tfrac12,\tfrac12)1,1\,\rangle &= |\tfrac12,\tfrac12;\tfrac12,\tfrac12\,\rangle \\ |(\tfrac12,\tfrac12)1,0\,\rangle &= \tfrac{1}{\sqrt2}\left(|\tfrac12,\tfrac12;\tfrac12,-\tfrac12\,\rangle + |\tfrac12,-\tfrac12;\tfrac12,\tfrac12\,\rangle\right) \\ |(\tfrac12,\tfrac12)0,0\,\rangle &= \tfrac{1}{\sqrt2}\left(|\tfrac12,\tfrac12;\tfrac12,-\tfrac12\,\rangle - |\tfrac12,-\tfrac12;\tfrac12,\tfrac12\,\rangle\right) \\ |(\tfrac12,\tfrac12)1,-1\,\rangle &= |\tfrac12,-\tfrac12;\tfrac12,-\tfrac12\,\rangle \end{aligned} \tag{0.89}$$

and

$$\frac{1}{2}\otimes 1\ :\quad \begin{aligned} |(\tfrac12,1)\tfrac32,\tfrac32\,\rangle &= |\tfrac12,\tfrac12;1,1\,\rangle \\ |(\tfrac12,1)\tfrac32,\tfrac12\,\rangle &= \tfrac{1}{\sqrt3}\left(\sqrt2|\tfrac12,\tfrac12;1,0\,\rangle + |\tfrac12,-\tfrac12;1,1\,\rangle\right) \\ |(\tfrac12,1)\tfrac32,-\tfrac12\,\rangle &= \tfrac{1}{\sqrt3}\left(|\tfrac12,\tfrac12;1,-1\,\rangle + \sqrt2|\tfrac12,-\tfrac12;1,0\,\rangle\right) \\ |(\tfrac12,1)\tfrac32,-\tfrac32\,\rangle &= |\tfrac12,-\tfrac12;1,-1\,\rangle \\ |(\tfrac12,1)\tfrac12,\tfrac12\,\rangle &= \tfrac{1}{\sqrt3}\left(-|\tfrac12,\tfrac12;1,0\,\rangle + \sqrt2|\tfrac12,-\tfrac12;1,1\,\rangle\right) \\ |(\tfrac12,1)\tfrac12,-\tfrac12\,\rangle &= \tfrac{1}{\sqrt3}\left(-\sqrt2|\tfrac12,\tfrac12;1,-1\,\rangle + |\tfrac12,-\tfrac12;1,0\,\rangle\right) \end{aligned} \tag{0.90}$$

One notices on the case $\frac12\otimes\frac12$ the property that vectors of total spin $j=1$ are symmetric under the exchange of the two spins, while those of spin 0 are antisymmetric. This is a general property: in the decomposition of the tensor product of two representations of spin $j_1 = j_2$, vectors of spin $j = 2j_1, 2j_1-2, \cdots$ are symmetric, those of spin $2j_1-1, 2j_1-3, \cdots$ are antisymmetric.

This is apparent on the expression (0.88) above, given the announced properties of the 3-j symbols.

In the same circle of ideas, consider the completely antisymmetric product of $2j+1$ copies of a spin j representation. One may show that this representation is of spin 0 (following exercise).

(This has consequences in atomic physics, in the filling of electronic orbitals: a complete shell has a total orbital momentum and a total spin that are both vanishing, hence also a vanishing total angular momentum.)

Exercise. Consider the completely antisymmetric tensor product of $N = 2j + 1$ representations of spin j. Show that this representation is spanned by the vector $\epsilon_{m_1 m_2 \cdots m_N} | j\, m_1, j\, m_2, \cdots, j\, m_N \rangle$, that it is invariant under the action of SU(2) and thus that the corresponding representation has spin $J = 0$.

One also introduces the *6-j symbols* that describe the two possible recombinations of 3 representations of spins j_1, j_2 and j_3

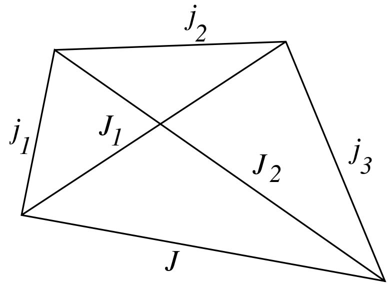


Fig. 4

$$
\begin{aligned}
|j_1\, m_1; j_2\, m_2; j_3\, m_3 \rangle &= \sum \langle\, (j_1\, j_2)\, J_1\, M_1 | j_1\, m_1; j_2\, m_2 \,\rangle \langle\, (J_1\, j_3)\, J\, M | J_1\, M_1; j_3\, m_3 \,\rangle |(j_1 j_2) j_3; J\, M \,\rangle \\
&= \sum \langle\, (j_2\, j_3)\, J_2\, M_2 | j_2\, m_2; j_3\, m_3 \,\rangle \langle\, (j_1\, J_2)\, J'\, M' | j_1\, m_1;\; J_2\, M_2 \,\rangle | j_1 (j_2 j_3); J'\, M' \,\rangle \quad (0.91)
\end{aligned}
$$

depending on whether one composes first j_1 and j_2 into J_1 and then J_1 and j_3 into J, or first j_2 and j_3 into J_2 and then j_1 and J_2 into J'. The matrix of the change of basis is denoted

$$
\langle\, j_1 (j_2 j_3); J\, M | (j_1 j_2) j_3; J'\, M' \,\rangle = \delta_{JJ'} \delta_{MM'} \sqrt{(2J_1+1)(2J_2+1)} (-1)^{j_1+j_2+j_3+J} \begin{Bmatrix} j_1 & j_2 & J_1 \\ j_3 & J & J_2 \end{Bmatrix} . \qquad (0.92)
$$

and the $\{\quad\}$ are the 6-j symbols. One may visualise this operation of "addition" of the three spins by a tetrahedron (see Fig. 4) the edges of which carry j_1, j_2, j_3, J_1, J_2 and J and the symbol is such that two spins carried by a pair of opposed edges lie in the same column. These symbols are tabulated in the literature.

0.5 A physical application: isospin

The group SU(2) appears in physics in several contexts, not only as related to the rotation group of the 3-dimensional Euclidian space. We shall now illustrate another of its avatars by the isospin symmetry.

There exists in nature elementary particles subject to nuclear forces, or more precisely to "strong interactions", and thus called *hadrons*. Some of those particles present similar properties but have different electric charges. This is the case with the two "nucleons", *i.e.* the proton and the neutron, of respective masses M_p =938,28 MeV/c^2 and M_n =939,57 MeV/c^2, and also with the "triplet" of pi mesons, π^0 (mass 134,96 MeV/c^2) and $\pi^\pm$ (139,57 MeV/c^2), with K mesons etc. According to a great idea of Heisenberg these similarities are the manifestation of a symmetry broken by electromagnetic interactions. In the absence of electromagnetic interactions proton and neutron on the one hand, the three π mesons on the other, etc, would have the same mass, differing only by an "internal" quantum number, in the same way as the two spin states of an electron in the absence of a magnetic field. In fact the group behind that symmetry is also SU(2), but a SU(2) group acting in an abstract space differing from the usual space. One gave the name *isotopic spin* or in short, *isospin*, to the corresponding quantum number. To summarize, the idea is that there exists a SU(2) group of symmetry of strong interactions, and that different particles subject to these strong interactions (hadrons) form representations of SU(2) : representation of isospin $I = \frac{1}{2}$ for the nucleon (proton $I_z = +\frac{1}{2}$, neutron $I_z = -\frac{1}{2}$), isospin $I = 1$ pour the pions ($\pi^\pm : I_z = \pm 1$, $\pi^0 : I_z = 0$) etc. The isospin is thus a "good quantum number", conserved in these interactions. Thus the "off-shell" process $N \to N + \pi$, (N for nucleon) important in nuclear physics, is consistent with addition rules of isospins ($\frac{1}{2} \otimes 1$ "contains" $\frac{1}{2}$). The different scattering reactions $N + \pi \to N + \pi$ allowed by conservation of electric charge

$$\begin{array}{ccccc}
p+\pi^+ & \to & p+\pi^+ & & I_z = \frac{3}{2} \\
p+\pi^0 & \to & p+\pi^0 & & I_z = \frac{1}{2} \\
 & \to & n+\pi^+ & & '' \\
p+\pi^- & \to & p+\pi^- & & I_z = -\frac{1}{2} \\
 & \to & n+\pi^0 & & '' \\
n+\pi^- & \to & n+\pi^- & & I_z = -\frac{3}{2}
\end{array}$$

also conserve total isospin I and its I_z component but the hypothesis of SU(2) isospin invariance tells us more. The matrix elements of the transition operator responsible for the two reactions in the channel $I_z = \frac{1}{2}$, for example, must be related by addition rules of isospin. Inverting the relations (0.90), one gets

$$\begin{aligned}
|p, \pi^-\rangle &= \sqrt{\frac{1}{3}}|I = \frac{3}{2}, I_z = -\frac{1}{2}\rangle - \sqrt{\frac{2}{3}}|I = \frac{1}{2}, I_z = -\frac{1}{2}\rangle \\
|n, \pi^0\rangle &= \sqrt{\frac{2}{3}}|I = \frac{3}{2}, I_z = -\frac{1}{2}\rangle + \sqrt{\frac{1}{3}}|I = \frac{1}{2}, I_z = -\frac{1}{2}\rangle
\end{aligned}$$

whereas for $I_z = 3/2$

$$|p, \pi^+\rangle = |I = \frac{3}{2}, I_z = \frac{3}{2}\rangle \ .$$

Isospin invariance implies that $\langle I\, I_z |\mathcal{T}|I'\, I'_z\rangle = \mathcal{T}_I \delta_{II'} \delta_{I_z\, I'_z}$, as we shall see later (Schur lemma or Wigner–Eckart theorem, Chap. 2): not only are I and I_z conserved, but the resulting amplitude depends only on I, not I_z. Calculating then the matrix elements of the transition operator $\mathcal{T}$ between the different states,

$$\begin{aligned}
\langle p\pi^+|\mathcal{T}|p\pi^+\rangle &= \mathcal{T}_{3/2} \\
\langle p\pi^-|\mathcal{T}|p\pi^-\rangle &= \frac{1}{3}\left(\mathcal{T}_{3/2} + 2\mathcal{T}_{1/2}\right) \\
\langle n\pi^0|\mathcal{T}|p\pi^-\rangle &= \frac{\sqrt{2}}{3}\left(\mathcal{T}_{3/2} - \mathcal{T}_{1/2}\right)
\end{aligned} \tag{0.93}$$

one finds that amplitudes satisfy a relation

$$\sqrt{2}\langle n, \pi^0|\mathcal{T}|p, \pi^-\rangle + \langle p, \pi^-|\mathcal{T}|p, \pi^-\rangle = \langle p, \pi^+|\mathcal{T}|p, \pi^+\rangle = \mathcal{T}_{3/2}$$

a non trivial consequence of isospin invariance, which implies triangular inequalities between squared modules of these amplitudes and hence between cross-sections of the reactions

$$\begin{aligned}[\sqrt{\sigma(\pi^- p \to \pi^- p)} &- \sqrt{2\sigma(\pi^- p \to \pi^0 n)}]^2 \le \sigma(\pi^+ p \to \pi^+ p) \le \\ &\le [\sqrt{\sigma(\pi^- p \to \pi^- p)} + \sqrt{2\sigma(\pi^- p \to \pi^0 n)}]^2\end{aligned}$$

which are experimentally well verified. Even better, one finds experimentally that at a certain energy of about 180 MeV, cross sections (proportional to squares of amplitudes) are in the ratios

$$\sigma(\pi^+ p \to \pi^+ p) : \sigma(\pi^- p \to \pi^0 n) : \sigma(\pi^- p \to \pi^- p) = 9 : 2 : 1$$

which is what one would get from (0.93) if $\mathcal{T}_{\frac{1}{2}}$ were vanishing. This indicates that at that energy, scattering in the channel $I = 3/2$ is dominant. In fact, this signals the existence of an intermediate πN state, a very unstable particle called "resonance", denoted Δ, of isospin 3/2 and hence with four states of charge

$$\Delta^{++}, \Delta^+, \Delta^0, \Delta^- ,$$

the contribution of which dominates the scattering amplitude. This particle has a spin 3/2 and a mass $M(\Delta) \approx 1230\ \mathrm{MeV}/c^2$.

In some cases one may obtain more precise predictions. This is for instance the case with the reactions

$$^2\mathrm{H}\, p \to {}^3\mathrm{He}\, \pi^0 \quad \text{and} \quad {}^2\mathrm{H}\, p \to {}^3\mathrm{H}\, \pi^+$$

which involve nuclei of deuterium ^{2}H, of tritium ^{3}H and of helium ^{3}He. To these nuclei too, one may assign an isospin, 0 to the deuteron which is made of a proton and a neutron in an antisymmetric state of their isospins (so that the wave function of these two *fermions*, symmetric in space and in spin, be antisymmetric), $I_z = -\frac{1}{2}$ to ^{3}H and $I_z = \frac{1}{2}$ to ^{3}He which form an isospin $\frac{1}{2}$ representation. Notice that in all cases, the electric charge is related to the I_z component of isospin by the relation $Q = \frac{1}{2}\mathcal{B} + I_z$, with $\mathcal{B}$ the baryonic charge, equal here to the number of nucleons (protons or neutrons).
Exercise: show that the ratio of cross-sections $\sigma(^2\mathrm{H}\, p \to {}^3\mathrm{He}\, \pi^0)/\sigma(^2\mathrm{H}\, p \to {}^3\mathrm{H}\, \pi^+)$ is $\frac{1}{2}$.

Remark : invariance under isospin SU(2) that we just discussed is a symmetry of *strong interactions*. There exists also in the framework of the Standard Model a notion of "weak isospin", a symmetry of *electroweak interactions*, to which we return in Chap. 5.

0.6 Representations of SO(3,1) and SL(2,$\mathbb{C}$)

0.6.1 A short reminder on the Lorentz group

Minkowki space is a $\mathbb{R}^4$ space endowed with a pseudo-Euclidean metric of signature $(+,-,-,-)$. In an orthonormal basis with coordinates $(x^0 = ct, x^1, x^2, x^3)$, the metric is diagonal

$$g_{\mu\nu} = \mathrm{diag}\,(1,-1,-1,-1)$$

and thus the squared norm of a 4-vector reads

$$x.x = x^\mu g_{\mu\nu} x^\nu = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2\,.$$

The isometry group of that quadratic form, called O(1,3) or the Lorentz group $\mathcal{L}$, is such that

$$\Lambda \in \mathrm{O}(1,3) \quad x' = \Lambda x \ :\ x'.x' = \Lambda^\mu{}_\rho x^\rho g_{\mu\nu} \Lambda^\nu{}_\sigma x^\sigma = x^\rho g_{\rho\sigma} x^\sigma$$

i.e.

$$\Lambda^\mu{}_\rho g_{\mu\nu} \Lambda^\nu{}_\sigma = g_{\rho\sigma} \quad \text{or} \quad \Lambda^T g \Lambda = g\,. \tag{0.94}$$

These pseudo-orthogonal matrices satisfy $(\det\Lambda)^2 = 1$ and (by taking the 00 matrix element of (0.94)) $(\Lambda^0{}_0)^2 = 1 + \sum_{i=1}^3 (\Lambda^0{}_i)^2 \geq 1$ and thus $\mathcal{L} \equiv$ O(1,3) has four connected components (or "sheets") depending on whether $\det\Lambda = \pm 1$ and $\Lambda^0{}_0 \geq 1$ or ≤ -1. The subgroup of *proper orthochronous* transformations satisfying $\det\Lambda = 1$ and $\Lambda^0{}_0 \geq 1$ is denoted $\mathcal{L}_+^\uparrow$. Any transformation of $\mathcal{L}_+^\uparrow$ may be written as the product of an "ordinary" rotation of SO(3) and a "special Lorentz transformation" or "boost".

A major difference between the SO(3) and the $\mathcal{L}_+^\uparrow$ groups is that the former is *compact* (the range of parameters is bounded and closed, see (0.2)), whereas the latter is not : in a boost along the 1 direction, say, $x_1' = \gamma(x_1 - vx_0/c), x_0' = \gamma(x_0 - vx_1/c)$, with $\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$, the velocity $|v| < c$ does not belong to a compact domain (or alternatively, the "rapidity" variable β, defined by $\cosh\beta = \gamma$ can run to infinity). This compactness/non-compactness has very important implications on the nature and properties of representations, as we shall see.

The Poincaré group, or inhomogeneous Lorentz group, is generated by Lorentz transformations $\Lambda \in \mathcal{L}$ and space-time translations; generic elements denoted (a, Λ) have an action on a vector x and a composition law given by

$$\begin{aligned} (a,\Lambda) \quad : \quad x \mapsto x' &= \Lambda x + a \\ (a',\Lambda')(a,\Lambda) &= (a' + \Lambda' a, \Lambda'\Lambda)\ ; \end{aligned} \tag{0.95}$$

the inverse of (a, Λ) is $(-\Lambda^{-1}a, \Lambda^{-1})$ (check it !).

0.6.2 Lie algebra of the Lorentz and Poincaré groups

An infinitesimal Poincaré transformation reads $(\alpha^\mu, \Lambda^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu)$. By taking the infinitesimal form of (0.94), one easily sees that the tensor $\omega_{\rho\nu} = \omega^\mu{}_\nu g_{\rho\mu}$ has to be antisymmetric: $\omega_{\nu\rho} + \omega_{\rho\nu} = 0$. This leaves 6 real parameters: the Lorentz group is a 6-dimensional group, and the Poincaré group is 10-dimensional.

To find the Lie algebra of the generators, let us proceed like in § 0.2.3: look at the Lie algebra generated by differential operators acting on functions of space-time coordinates; if $x'^\lambda = x^\lambda + \delta x^\lambda = x^\lambda + \alpha^\lambda + \omega^{\lambda\nu} x_\nu$, $\delta f(x) = f(x^\mu - \alpha^\lambda - \omega^{\lambda\nu} x_\nu) - f(x) = (I - i\alpha^\mu P_\mu - \frac{i}{2}\omega^{\mu\nu} J_{\mu\nu}) f(x)$, (see (0.45)), thus

$$J_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) \qquad\qquad P_\mu = -i\partial_\mu \tag{0.96}$$

the commutators of which are then easily computed

$$\begin{aligned} [J_{\mu\nu}, P_\rho] &= i\,(g_{\nu\rho} P_\mu - g_{\mu\rho} P_\nu) \\ [J_{\mu\nu}, J_{\rho\sigma}] &= i\,(g_{\nu\rho} J_{\mu\sigma} - g_{\mu\rho} J_{\nu\sigma} + g_{\mu\sigma} J_{\nu\rho} - g_{\nu\sigma} J_{\mu\rho}) \\ [P_\mu, P_\nu] &= 0\,. \end{aligned} \tag{0.97}$$

Note the structure of these relations: antisymmetry in $\mu \leftrightarrow \nu$ of the first one, in $\mu \leftrightarrow \nu$, in $\rho \leftrightarrow \sigma$ and in $(\mu,\nu) \leftrightarrow (\rho,\sigma)$ of the second one; the first one shows how a vector (here P_ρ) transforms under the infinitesimal transformation by $J_{\mu\nu}$, and the second then has the same pattern in the indices ρ and σ, expressing that $J_{\rho\sigma}$ is a 2-tensor.

Generators that commute with P_0 (which is the generator of time translations, hence the Hamiltonian) are the P_μ and the J_{ij} but not the J_{0j} : $i[P_0, J_{0j}] = P_j$.

Set

$$J_{ij} = \epsilon_{ijk} J^k \qquad K^i = J_{0i} \ . \tag{0.98}$$

Then

$$\begin{aligned} [J^i, J^j] &= i\epsilon_{ijk} J^k \\ [J^i, K^j] &= i\epsilon_{ijk} K^k \\ [K^i, K^j] &= -i\epsilon_{ijk} J^k \end{aligned} \tag{0.99}$$

and also

$$\begin{aligned} [J^i, P^j] &= i\epsilon_{ijk} P^k \qquad & [K^i, P^j] &= iP^0 \delta_{ij} \\ [J^i, P^0] &= 0 \qquad & [K^i, P^0] &= iP^i \ . \end{aligned} \tag{0.100}$$

Remark. The first two relations (0.99) and the first one of (0.100) express that, as expected, $\mathbf{J} = \{J^j\}$, $\mathbf{K} = \{K^j\}$ and $\mathbf{P} = \{P^j\}$ transform like vectors under rotations of $\mathbb{R}^3$. Now form the combinations

$$M^j = \frac{1}{2}(J^j + iK^j) \qquad N^j = \frac{1}{2}(J^j - iK^j) \tag{0.101}$$

which have the following commutation relations

$$\begin{aligned} [M^i, M^j] &= i\epsilon_{ijk} M^k \\ [N^i, N^j] &= i\epsilon_{ijk} N^k \\ [M^i, N^j] &= 0 \ . \end{aligned} \tag{0.102}$$

By considering the complex combinations M and N of its generators, one thus sees that the Lie algebra of $\mathcal{L} = O(1,3)$ is isomorphic to $su(2) \oplus su(2)$. The introduction of $\pm i$, however, implies that *unitary* representations of $\mathcal{L}$ do not follow in a simple way from those of SU(2)$\times$ SU(2). On the other hand, representations of finite dimension of $\mathcal{L}$, which are *non unitary*, are labelled by a pair (j_1, j_2) of integers or half-integers.

Exercise. Show that this algebra admits *two* independent quadratic Casimir operators, and express them in terms of $\mathbf{M}$ and $\mathbf{N}$ first, and then in terms of $\mathbf{J}$ and $\mathbf{K}$.

0.6.3 Covering groups of $\mathcal{L}_+^\uparrow$ and $\mathcal{P}_+^\uparrow$

We have seen that the study of SO(3) led us to SU(2), its “covering group” (the deep reasons of which will be explained in Chap. 1 and 2). Likewise in the case of the Lorentz group its “covering group” turns out to be SL(2,$\mathbb{C}$).

There is a simple way to see how SL(2,ℂ) and $\mathcal{L}_+^\uparrow$ are related, which is a 4-dimensional extension of the method followed in § 0.1.2. One considers matrices σ_μ made of $\sigma_0 = \mathbf{I}$ and of the three familiar Pauli matrices. Note that

$$\mathrm{tr}\,\sigma_\mu\sigma_\nu = 2\delta_{\mu\nu} \qquad \sigma_\mu^2 = \mathbf{I} \quad \text{with no summation over the index } \mu \ .$$

With any real vector $x \in \mathbb{R}^4$, associate the Hermitian matrix

$$X = x^\mu\sigma_\mu \qquad x^\mu = \frac{1}{2}\mathrm{tr}\,(X\sigma_\mu) \qquad \det X = x^2 = (x^0)^2 - \mathbf{x}^2 \ . \tag{0.103}$$

A matrix $A \in \mathrm{SL}(2,\mathbb{C})$ acts on X according to

$$X \mapsto X' = AXA^\dagger \tag{0.104}$$

which is indeed Hermitian and thus defines a real $x'^\mu = \frac{1}{2}\mathrm{tr}\,(X'\sigma_\mu)$, with $\det X' = \det X$, hence $x^2 = x'^2$. This is a linear transformation of $\mathbb{R}^4$ that preserves the Minkowski norm x^2, and thus a Lorentz transformation, and one checks by an argument of continuity that it belongs to $\mathcal{L}_+^\uparrow$ and that $A \to \Lambda$ is a homomorphism of $\mathrm{SL}(2,\mathbb{C})$ into $\mathcal{L}_+^\uparrow$. In the following we denote $x' = A.x$ if $X' = AXA^\dagger$.

As is familiar from the case of SU(2), the transformations A and $-A \in \mathrm{SL}(2,\mathbb{C})$ give the same transformation of $\mathcal{L}_+^\uparrow$: $\mathrm{SL}(2,\mathbb{C})$ is a covering of order 2 of $\mathcal{L}_+^\uparrow$. For the Poincaré group, likewise, its covering is the ("semi-direct") product of the translation group by $\mathrm{SL}(2,\mathbb{C})$. If one denotes $\underline{a} := a^\mu\sigma_\mu$, then

$$(\underline{a}, A)(\underline{a}', A') = (\underline{a} + A\underline{a}'A^\dagger, AA') \tag{0.105}$$

and one sometimes refers to it as the "inhomogeneous $\mathrm{SL}(2,\mathbb{C})$ group" or $\mathrm{ISL}(2,\mathbb{C})$).

0.6.4 Irreducible finite-dimensional representations of $\mathrm{SL}(2,\mathbb{C})$

The construction of § 0.3.3 yields an explicit representation of GL(2,ℂ) and hence of SL(2,ℂ). For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$, (0.71) gives the following expression for $\mathcal{D}^j_{mm'}(A)$:

$$\mathcal{D}^j_{mm'}(A) = [(j+m)!(j-m)!(j+m')!(j-m')!]^{\frac{1}{2}} \sum_{\substack{n_1,n_2,n_3,n_4 \geq 0 \\ n_1+n_2=j+m\,;\ n_3+n_4=j-m' \\ n_1+n_3=j+m\,;\ n_2+n_4=j-m}} \frac{a^{n_1}b^{n_2}c^{n_3}d^{n_4}}{n_1!n_2!n_3!n_4!} \tag{0.71}$$

Note that $\mathcal{D}^T(A) = \mathcal{D}(A^T)$ (since exchanging $m \leftrightarrow m'$ amounts to $n_2 \leftrightarrow n_3$, hence to $b \leftrightarrow c$) and $(\mathcal{D}(A))^* = \mathcal{D}(A^*)$ (since the numerical coefficients in (0.71) are real) thus $\mathcal{D}^\dagger(A) = \mathcal{D}(A^\dagger)$.

This representation is called $(j,0)$, it is of dimension $2j+1$. There exists another one of dimension $2j+1$, which is non equivalent, denoted $(0,j)$, this is the "contragredient conjugate" representation (in the sense of Chap 2. § 2.1.3.b) $\mathcal{D}^j(A^{\dagger\,-1})$. Replacing A by $A^{\dagger\,-1}$ may be interpreted in the construction of § 0.6.3 if instead of associating $X = x^\mu\sigma_\mu$ with x, one associates $\widetilde{X} = x^0\sigma_0 - \mathbf{x}.\boldsymbol{\sigma}$. Notice that $\sigma_2(\sigma_i)^T\sigma_2 = -\sigma_i$ for $i = 1,2,3$ hence $\widetilde{X} = \sigma_2 X^T\sigma_2$. For the transformation $A\ :\quad X \mapsto X' = AXA^\dagger$, we have

$$\widetilde{X}' = \sigma_2(X')^T\sigma_2 = \sigma_2(AXA^\dagger)^T\sigma_2 = (\sigma_2A^T\sigma_2)^\dagger\widetilde{X}(\sigma_2A^T\sigma_2) \ .$$

Any matrix A of $\mathrm{SL}(2,\mathbb{C})$ may itself be written as $A = a^\mu \sigma_\mu$, with $(a^\mu) \in \mathbb{C}^4$, and as $\det A = (a^0)^2 - \mathbf{a}^2 = 1$ (the "S" of $\mathrm{SL}(2,\mathbb{C})$), one verifies immediately that $A^{-1} = a^0\sigma_0 - \mathbf{a}.\boldsymbol{\sigma}$, donc

$$\sigma_2 A^T \sigma_2 = A^{-1} \ . \tag{0.106}$$

Finally

$$X' = AXA^\dagger \quad \Longleftrightarrow \quad \widetilde{X}' = (A^{-1})^\dagger \widetilde{X} A^{-1} \ . \tag{0.107}$$

Remark. The two representations $(j,0)$ and $(0,j)$ are inequivalent on $\mathrm{SL}(2,\mathbb{C})$, but equivalent on $\mathrm{SU}(2)$. Indeed in $\mathrm{SU}(2)$, $A = U = (U^\dagger)^{-1}$.

Finally, one proves that any finite-dimensional representation of $\mathrm{SL}(2,\mathbb{C})$ is completely reducible and may be written as a direct sum of irreducible representations. The most general finite-dimensional irreducible representation of $\mathrm{SL}(2,\mathbb{C})$ is denoted (j_1, j_2), with j_1 and $j_2 \geq 0$ integers or half-integers; it is defined by

$$(j_1, j_2) = (j_1, 0) \otimes (0, j_2) \ . \tag{0.108}$$

All these representations may be obtained from the representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ by tensoring: $(j_1, 0)$ and $(0, j_2)$ are obtained by symmetrized tensor product of representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$, respectively, as was done for $\mathrm{SU}(2)$. Only representations (j_1, j_2) with j_1 and j_2 simultaneously integers or half-integers are true representations of $\mathcal{L}_\uparrow^+$. The others are representations up to a sign.

Exercise : show that the representation $(0,j)$ is "equivalent" (equal up to a change of basis) to the complex conjugate of representation $(j,0)$. (Hint: show it first for $j = \frac{1}{2}$ by recalling that $(A^{-1})^\dagger = \sigma_2 A^* \sigma_2$, then for representations of arbitrary j obtained by $2j$-th tensor power of $j = \frac{1}{2}$.)

Spinor representations

Return to the "spinor representations" $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$. Those are representations of dimension 2 (two-component spinors). It is traditional to note the indices of components with "pointed" or "unpointed" indices, for representation $(0, \frac{1}{2})$ and $(\frac{1}{2}, 0)$, respectively. With $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$, we thus have

$$\begin{aligned}
&(\frac{1}{2}, 0) \qquad && \xi = (\xi^\alpha) \mapsto \xi' = A\xi = \begin{pmatrix} a\xi^1 + b\xi^2 \\ c\xi^1 + d\xi^2 \end{pmatrix} \\
&(0, \frac{1}{2}) \qquad && \xi = (\xi^{\dot\alpha}) \mapsto \xi' = A^*\xi = \begin{pmatrix} a^*\xi^{\dot 1} + b^*\xi^{\dot 2} \\ c^*\xi^{\dot 1} + d^*\xi^{\dot 2} \end{pmatrix}
\end{aligned} \tag{0.109}$$

Note that the alternating (=antisymmetric) form $(\xi, \eta) = \xi^1\eta^2 - \xi^2\eta^1 = \xi^T(i\sigma_2)\eta$ is invariant in $(\frac{1}{2}, 0)$ (and also in $(0, \frac{1}{2})$), which follows once again from (0.106)

$$(\sigma_2 A^T \sigma_2) A = A^{-1} A = \mathbf{I} \Longleftrightarrow A^T (i\sigma_2) A = i\sigma_2 \ .$$

One may thus use that form to lower indices α (or $\dot\alpha$). Thus

$$\begin{aligned} &\text{in } (\frac{1}{2},0)\ :\ (\xi,\eta)=\xi_\alpha\eta^\alpha \qquad \xi_2=\xi^1 \qquad \xi_1=-\xi^2 \\ &\text{in } (0,\frac{1}{2})\ :\ (\xi,\eta)=\xi_{\dot\alpha}\eta^{\dot\alpha} \qquad \xi_{\dot 2}=\xi^{\dot 1} \qquad \xi_{\dot 1}=-\xi^{\dot 2} \end{aligned} \tag{0.110}$$

(j_1, j_2) **representation**

Tensors $\{\xi^{\alpha_1\alpha_2\cdots\alpha_{2j_1}\dot\beta_1\dot\beta_2\cdots\dot\beta_{2j_2}}\}$ symmetric in $\alpha_1,\alpha_2,\cdots,\alpha_{2j_1}$ and in $\dot\beta_1,\dot\beta_2,\cdots,\dot\beta_{2j_2}$, form the irreducible representation (j_1,j_2). (One cannot lower the rank by taking traces, since the only invariant tensor is the previous *alternating* form). The dimension of that representation is $(2j_1+1)(2j_2+1)$. The most usual representations encountered in field theory are $(0,0)$, $(\frac12,0)$ and $(0,\frac12)$, $(\frac12,\frac12)$. The reducible representation $(\frac12,0)\oplus(0,\frac12)$ describes the (4-component) Dirac fermion; the $(\frac12,\frac12)$ corresponds to 4-vectors, as seen above:

$$x\mapsto X=x^0\sigma_0+\mathbf{x}.\boldsymbol{\sigma} \overset{A\in\mathrm{SL}(2,\mathbb{C})}{\longrightarrow} X'=AXA^\dagger$$

i.e.

$$X=X^{\alpha\dot\beta}\to(X')^{\alpha\dot\beta}=A^{\alpha\alpha'}(A^{\dot\beta\dot\beta'})^*X^{\alpha'\dot\beta'}\ ,$$

which shows that X transforms indeed according to the $(\frac12,\frac12)$ representation .

Exercise. Show that representations $(1,0)$ and $(0,1)$, of dimension 3, describe rank 2 tensors $F^{\mu\nu}$ that are "self-dual" ou "anti-self-dual", *i.e.* satisfy

$$F^{\mu\nu}=\pm\frac{i}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}\ .$$

0.6.5 Irreducible unitary representations of the Poincaré group. One particle states.

According to a theorem of Wigner which will be discussed in Chap, 2, the action of proper orthochronous transformations of the Lorentz or Poincaré groups on state vectors of a quantum theory is described by means of *unitary* representations of these groups, or rather of their "universal covers" SL(2,ℂ) and ISL(2,ℂ). As will be seen below (Chap. 2), unitary representations (of class L^2) of the non compact group $\mathrm{SL}(2,\mathbb{C})$ are necessarily of infinite dimension (with the possible exception of the trivial representation $(0,0)$, which describes a state invariant by rotation and by boosts, *i.e.* the vacuum !,..., and which is in fact not of class L^2!).

Returning to commutation relations of the Lie algebra (0.97), one seeks a maximal set of commuting operators. The four P_μ commute. Let (p_μ) be an eigenvalue for a common eigenvector of P_μ, describing a "one-particle state". We assume that the eigenvector denoted $|p\,\rangle$ is labelled only by p^μ and by discrete indices: (this is indeed the meaning of "one-particle state", in contrast with a two-particle state that would depend on a relative momentum, a continuous variable)

$$P_\mu|p\,\rangle=p_\mu|p\,\rangle\ . \tag{0.111}$$

One also considers the *Pauli-Lubanski tensor*

$$W^\lambda=\frac12\epsilon^{\lambda\mu\nu\rho}J_{\mu\nu}P_\rho \tag{0.112}$$

and one verifies (exercise !) that (0.97) implies

$$[W_\mu,P_\nu]\ =\ 0$$

$$\begin{aligned}[W^\mu, W^\nu] &= -i\epsilon^{\mu\nu\rho\sigma} W_\rho P_\sigma \\ [J_{\mu\nu}, W_\lambda] &= i(g_{\nu\lambda} W_\mu - g_{\mu\lambda} W_\nu) \ .\end{aligned} \tag{0.113}$$

The latter relation means that W is a Lorentz 4-vector (compare with (0.97)). One also notes that $W.P = 0$ because of the antisymmetry of tensor ϵ. One finally shows (check it!) that $P^2 = P_\mu P^\mu$ and $W^2 = W_\mu W^\mu$ commute with all generators P and J : those are the Casimir operators of the algebra. According to a lemma by Schur, (see below Chap. 2, § 2.1.4), these Casimir operators are in any irreducible representation proportional to the identity, in other words, their eigenvalues may be used to label the irreducible representations.

In physics, one encounters only two types of representations for these one-particle states[4]: representations with $P^2 > 0$ and those with $P^2 = 0$, $W^2 = 0$. Their detailed study will be done in Adel Bilal's course.

Bibliography

The historical reference for the physicist is the book by E. Wigner [Wi].
For a detailed discussion of the rotation group, with many formulas and tables, see:
J.-M. Normand, *A Lie group : Rotations in Quantum Mechanics*, North-Holland.
For a deep and detailed study of physical representations of Lorentz and Poincaré groups, see P. Moussa and R. Stora, *Angular analysis of elementary particle reactions*, in *Analysis of scattering and decay*, edited by M. Nikolic, Gordon and Breach 1968.

Problem

One considers two spin $\frac{1}{2}$ representations of the group SU(2) and their direct (or tensor) product. One denotes $\mathbf{J}^{(1)}$ and $\mathbf{J}^{(2)}$ the infinitesimal generators acting in each representation, and $\mathbf{J} = \mathbf{J}^{(1)} + \mathbf{J}^{(2)}$ those acting in their direct product, see (0.75), (0.75').

- What can be said about the operators $\mathbf{J}^{(1)\,2}$, $\mathbf{J}^{(2)\,2}$ and $\mathbf{J}^2$ and their eigenvalues ?
- Show that $\mathbf{J}^{(1)}.\mathbf{J}^{(2)}$ may be expressed in terms of these operators and that operators

$$\frac{1}{4}(3\mathbf{I} + 4\mathbf{J}^{(1)}.\mathbf{J}^{(2)}) \qquad \text{et} \qquad \frac{1}{4}(\mathbf{I} - 4\mathbf{J}^{(1)}.\mathbf{J}^{(2)})$$

  are projectors on spaces to be identified.
- Taking into account the symmetries of the vectors under exchange, what can you say about the operator

$$\frac{1}{2}\mathbf{I} + 2\mathbf{J}^{(1)}.\mathbf{J}^{(2)} \qquad ?$$

Appendix 0. Measure and Laplacian on the S^2 and S^3 spheres

Consider a Riemannian manifold, *i.e.* a manifold endowed with a metric :

$$\mathrm{d}s^2 = g_{\alpha\beta}\,\mathrm{d}\xi^\alpha \mathrm{d}\xi^\beta \tag{0.114}$$

[4] which does not mean that there are no other irreducible representations; for example "unphysical" representations where $P^2 = -M^2 < 0$

with a metric tensor g and (local) coordinates ξ^α, $\alpha = 1, \cdots, n$; n is the dimension of the manifold. This $\mathrm{d}s^2$ must be invariant under changes of coordinates, $\xi \to \xi'$, which dictates the change of the tensor g

$$\xi \mapsto \xi' \, , \quad g \mapsto g' \qquad : \quad g'_{\alpha'\beta'} = \frac{\partial \xi^\alpha}{\partial \xi'^{\alpha'}} \frac{\partial \xi^\beta}{\partial \xi'^{\beta'}} g_{\alpha\beta} \, , \tag{0.115}$$

meaning that g is a covariant rank-2 tensor. The metric tensor is assumed to be non singular, *i.e.* invertible, and its inverse tensor is denoted with upper indices

$$g_{\alpha\beta}\, g^{\beta\gamma} = \delta_\alpha{}^\gamma \, . \tag{0.116}$$

Also, its determinant is traditionnally denoted g

$$g = \det(g_{\alpha\beta}) \, . \tag{0.117}$$

There is then a general method for constructing a volume element on the manifold (*i.e.* an integration measure) and a Laplacian, both invariant under changes of coordinates

$$\begin{aligned} \mathrm{d}\mu(\xi) &= \sqrt{g} \prod_{\alpha=1}^{n} \mathrm{d}\xi^\alpha \\ \Delta &= \frac{1}{\sqrt{g}} \partial_\alpha \sqrt{g} g^{\alpha\beta} \partial_\beta \end{aligned} \tag{0.118}$$

where ∂_α is a shorthand notation for the differential operator $\frac{\partial}{\partial \xi^\alpha}$.
Exercise: check that $\mathrm{d}\mu(\xi)$ and Δ are invariant under a change of coordinates $\xi \mapsto \xi'$.

This may be applied in many contexts, and will be used in Chap. 1 to define an integration measure on compact Lie groups.

Let us apply it here to the n-dimensional Euclidean space $\mathbb{R}^n$. In spherical coordinates, one writes

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2$$

where $\mathrm{d}\Omega$ is a generic notation that collects all the angular variables. The metric tensor is thus of the general form $\begin{pmatrix} 1 & 0 \\ 0 & r^2 A \end{pmatrix}$ with a $(n-1) \times (n-1)$ matrix A which is r-independent and depends only on angular variables. The latter give rise to the Laplacian on the unit sphere S^{n-1}, denoted $\Delta_{S^{n-1}}$; $\sqrt{g} = r^{n-1}\sqrt{\det A}$; and (0.118) tells us that the Laplacian on $\mathbb{R}^n$ takes the general form

$$\Delta_{\mathbb{R}^n} = \frac{1}{r^{n-1}} \frac{\partial}{\partial r} r^{n-1} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{S^{n-1}} = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{S^{n-1}} \, .$$

Let us write more explicit formulae for the S^2 and S^3 unit spheres. Consider first the unit sphere S^2 with angular coordinates $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$ (Fig. 1). We thus have

$$\begin{aligned} ds^2 &= d\theta^2 + \sin^2\theta \, d\phi^2 \\ \sqrt{g} &= \sin\theta \end{aligned}$$

$$\begin{aligned} d\mu(x) &= \sin\theta\, d\theta\, d\phi \\ \Delta_{S^2} &= \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\ . \end{aligned} \tag{0.119}$$

The generators J_i read (see (0.46))

$$\begin{aligned} J_3 &= -i\frac{\partial}{\partial\phi} \\ J_1 &= -i\left[-\cos\phi\,\mathrm{cotg}\,\theta\frac{\partial}{\partial\phi} - \sin\phi\frac{\partial}{\partial\theta}\right] \\ J_2 &= -i\left[-\sin\phi\,\mathrm{cotg}\,\theta\frac{\partial}{\partial\phi} + \cos\phi\frac{\partial}{\partial\theta}\right] \end{aligned} \tag{0.120}$$

and one verifies that $-\Delta_{S^2} = \vec{J}^2 = J_1^2 + J_2^2 + J_3^2$.

For the unit sphere S^3 one finds similar formulas. In the parametrization (0.12), one takes for example

$$\mathrm{d}s^2 = \frac{1}{2}\mathrm{tr}\,\mathrm{d}U\mathrm{d}U^\dagger = \left(\mathrm{d}\frac{\psi}{2}\right)^2 + \sin^2\frac{\psi}{2}\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) \tag{0.121}$$

invariant under $U \to UV$, $U \to VU$ or $U \to U^{-1}$, whence a measure invariant under the same transformations

$$\mathrm{d}\mu(U) = \frac{1}{2}\left(\sin\frac{\psi}{2}\right)^2 \sin\theta\,\mathrm{d}\psi\,\mathrm{d}\theta\,\mathrm{d}\phi\ . \tag{0.122}$$

In the Euler angles parametrization,

$$U = e^{-i\alpha\frac{\sigma_3}{2}} e^{-i\beta\frac{\sigma_2}{2}} e^{-i\gamma\frac{\sigma_3}{2}} \tag{0.123}$$

thus

$$\mathrm{d}s^2 = \frac{1}{2}\mathrm{tr}\,\mathrm{d}U\mathrm{d}U^\dagger = \frac{1}{4}\left(\mathrm{d}\alpha^2 + 2\mathrm{d}\alpha\mathrm{d}\gamma\cos\beta + \mathrm{d}\gamma^2 + \mathrm{d}\beta^2\right) \tag{0.124}$$

and with $\sqrt{g} = \sin\beta$ one computes

$$d\mu(U) = \frac{1}{8}\sin\beta\, d\alpha\, d\beta\, d\gamma \tag{0.125}$$

$$\Delta_{S^3} = \frac{4}{\sin^2\beta}\left[\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\gamma^2} + \frac{\partial^2}{\partial\alpha\partial\gamma}\right] + \frac{4}{\sin\beta}\frac{\partial}{\partial\beta}\sin\beta\frac{\partial}{\partial\sin\beta}\ . \tag{0.126}$$

⋆