Chapter 4

Global symmetries in particle physics

Particle physics offers a wonderful playground to illustrate the various manifestations of symmetries in physics. We will be only concerned in this chapter and the following one with “internal symmetries”, excluding space-time symmetries.

We shall examine in turn various types of symmetries and their realizations, as exact symmetries, or broken explicitly, spontaneously or by quantum anomalies.

4.1 Global exact or broken symmetries. Spontaneous breaking

4.1.1 Overview. Exact or broken symmetries

Transformations that concern us in this chapter are global symmetries and we discuss them in the framework of (classical or quantum) field theory. A group $G$ acts on degrees of freedom of each field $\phi(x)$ in the same way at all points $x$ of space-time. For example, $G$ acts on $\phi$ by a linear representation, and to each element $g$ of the group corresponds a matrix or operator $D(g)$, independent of the point $x$

$$\phi(x) \mapsto D(g)\phi(x). \quad (4.1)$$

In a quantum theory, according to Wigner theorem, one assumes that this transformation is also realized on vector-states of the Hilbert space of the theory by a unitary operator $U(g)$, and, as an operator, $\phi(x) \mapsto U(g)\phi(x)U^\dagger(x)$.

This transformation may be a symmetry of dynamics, in which case $U(g)$ commutes with the Hamiltonian of the system, or in the Lagrangian picture, it leaves the Lagrangian invariant and gives rise to Noether currents $j^\mu_i$ of vanishing divergence (see Chapitre 0\(^1\)) and to conserved

\(^1\)Here and below in this chapter, Chapitre 0 refers to Chap. 0 of the French version, especially in its discussion of Noether’s currents.
Charges \( Q_i = \int dx j^0(x,t) \), \( i = 1, \cdots, \dim G \). These charges act on fields as infinitesimal generators, classically in the sense of the Poisson bracket, \( \{Q_i, \phi(x)\} \delta \alpha^i = \delta \phi(x) \), and if everything goes right in the quantum theory, as operators in the Hilbert space with commutation relations with the fields \( [Q_i, \phi(x)] \delta \alpha^i = -i\hbar \delta \phi(x) \) and between themselves \( [Q_i, Q_j] = iC_{ij}^k Q_k \). An important question will be indeed to know if a symmetry which is manifest at the classical level, say on the Lagrangian, is actually realized in the quantum theory.

- An example of exact symmetry is provided by the U(1) invariance associated with electric charge conservation. A field carrying an electric charge \( q \) (times \( |e| \)) is a complex field, it transforms under the action of the group U(1) according to the irreducible representation labelled by the integer \( q \)

\[
\phi(x) \mapsto e^{iqa} \phi(x) \ ; \quad \phi^\dagger(x) \mapsto e^{-iqa} \phi^\dagger(x),
\]

and there is invariance (of the Lagrangian) if all fields transform that way, with a Noether current \( j^\mu(x) \), sum of contributions of the different charged fields, being divergence-less, \( \partial^\mu j^\mu(x) = 0 \), and the associated charge \( Q = \sum q_i \) is conserved. The quantum theory is quantum electrodynamics, and there one proves that the classical U(1) symmetry, the current conservation (and gauge invariance) are preserved by quantization and in particular by renormalization, for example that all electric charges renormalize in the same way, see the course of Quantum Field Theory.

Other invariances and conservation laws of a similar nature are those associated with baryonic or leptonic charges, which are conserved (until further notice . . .).

- A symmetry may also be broken explicitly. For example the Lagrangian contains terms that are non invariant under the action of \( G \). In that case, the Noether currents are non conserved, but their divergence reads

\[
\partial^\mu j^\mu(x) = \frac{\partial \mathcal{L}(x)}{\partial \dot{\alpha}^i}, \quad (4.2)
\]

(see Chapitre 0, § 4.2). We will see below with flavor SU(3) an example of a broken (or “approximate”) symmetry.

Certain types of breakings, called “soft”, are such that the symmetry is restored at short distance or high energy. This is for example the case of scale invariance (by space dilatations), broken by the presence of any mass scale in the theory, but restored --in a fairly subtle way-- at short distance, see the study of the Renormalization Group in the courses of quantum or statistical field theory.

- A more subtle mechanism of symmetry breaking is that of spontaneous symmetry breaking. This refers to situations where the ground state of the system does not have a symmetry apparent on the Lagrangian or on the equations of motion. The simplest illustration of this phenomenon is provided by a classical system with one degree of freedom, described by the “double well potential” of Fig. 4.1(a). Although the potential exhibits a manifest \( \mathbb{Z}_2 \) symmetry under \( x \rightarrow -x \), the system chooses a ground state in one of the two minima of the potential, which breaks symmetry. This mechanism plays a fundamental role in physics, with diverse manifestations ranging from condensed matter –ferromagnetism, superfluidity, supraconductivity. . .– to particle physics –chiral symmetry, Higgs phenomenon– and cosmology.
4.1. Global exact or broken symmetries. Spontaneous breaking

Example. Spontaneous breaking in the $O(n)$ model

The Lagrangian of the bosonic (and Minkovskian, here) “$O(n)$ model" for a real $n$-component field $\phi = \{\phi^i\}$

$$\mathcal{L} = \frac{1}{2}(\partial \phi)^2 - \frac{1}{2}m^2 \phi^2 - \frac{\lambda}{4}(\phi^2)^2$$  \hspace{1cm} (4.3)

is invariant under the $O(n)$ rotation group. The Noether current $j^a_\mu = \partial_\mu \phi^i(T^a)_{ij}\phi^j$ (with $T^a$ real antisymmetric) has a vanishing divergence, which implies the conservation of a “charge” etc.

The minimum of the potential corresponds to the ground state, alias the vacuum, of the theory. If the parameter $m^2$ is taken negative, the minimum of the potential $V = \frac{1}{2}m^2 \phi^2 + \frac{1}{4}(\phi^2)^2$ is no longer at $\phi^2 = 0$ but at some value $v^2$ of $\phi^2$ such that $-m^2 = \lambda v^2$, see Fig. 4.1(b). The field $\phi$ “chooses” spontaneously a direction $\hat{n}$ ($\hat{n}^2 = 1$) in the internal space, in which its vacuum expectation value (“vev” in the jargon) is non vanishing

$$\langle 0 | \phi | 0 \rangle = v \hat{n}.$$  \hspace{1cm} (4.4)

This “vev” breaks the initial invariance group $G = O(n)$ down to its subgroup $H$ that leaves invariant the vector $\langle 0 | \phi | 0 \rangle = v \hat{n}$, hence a group isomorphic to $O(n - 1)$. The fact that a vacuum expectation value of a non invariant field be non zero, $\langle 0 | \phi | 0 \rangle \neq 0$, signals that the vacuum is not invariant: this is a case of spontaneous symmetry breaking. This is the mechanism at work in a low temperature ferromagnet, for example, in which the non zero magnetization signals the spontaneous breaking of the space rotation symmetry.

Exercise (see F. David’s course): Set $\phi = (v + \sigma)\hat{n} + \pi$, where $\pi$ denote the $n - 1$ components of the field $\phi$ orthogonal to $\langle \phi \rangle = v \hat{n}$ and determine the terms of $V(\sigma, \pi)$ that are linear and quadratic in the fields $\sigma$ and $\pi$; check that the linear term in $\sigma$ vanishes (minimum of the potential), that $\sigma$ has a non-zero mass term, but that the $\pi$ are massless, they are the Nambu–Goldstone bosons of the spontaneously broken symmetry. This is a general phenomenon: any continuous spontaneously broken symmetry is accompanied by the appearance of massless excitations whose number equals that of the generators of the broken symmetry (Goldstone theorem). More precisely when a group $G$ is spontaneously broken into a subgroup $H$ (group of residual symmetry, invariance group of the ground state), a number $d(G) - d(H)$ of massless Goldstone bosons appears. In the previous example, $G = O(n)$, $H = O(n - 1)$, $d(G) - d(H) =$
Let us give a simple proof of that theorem in the case of a Lagrangian field theory. We write $L = \frac{1}{2} (\partial \phi)^2 - V(\phi)$ with quite generic notations, $\phi$ denotes a set of fields $\{\phi_i\}$ on which acts a continuous transformation group $G$. The potential $V$ is assumed to be invariant under the action infinitesimal transformations $\delta^a \phi_i$, $a = 1, \cdots, \dim G$. For example for linear transformations: $\delta^a \phi_i = T^a_{ij} \phi_j$. We thus have

$$\frac{\partial V(\phi(x))}{\partial \phi_i(x)} \delta^a \phi_i(x) = 0.$$ 

Differentiate this equation with respect to $\phi_j(x)$ (omitting everywhere the argument $x$)

$$\frac{\partial V}{\partial \phi_i} \frac{\partial \delta^a \phi_i}{\partial \phi_j} + \frac{\partial^2 V}{\partial \phi_i \partial \phi_j} \delta^a \phi_i = 0$$

and evaluate it at $\phi(x) = v$, a (constant, $x$-independent) minimum of the potential: the first term vanishes, the second tells us

$$\frac{\partial^2 V}{\partial \phi_i \partial \phi_j} \bigg|_{\phi=v} \delta^a v_i = 0,$$

where we write (with a little abuse of notation) $\delta^a v_i = \delta^a \phi_i|_{\phi=v}$. On the other hand, the theory is quantized near that minimum $v$ (“vacuum” of the theory) by writing $\phi(x) = v + \varphi(x)$ and by expanding

$$V(\phi) = V(v) + \frac{1}{2} \frac{\partial^2 V}{\partial \phi_i \partial \phi_j} \bigg|_{\phi=v} \varphi_i \varphi_j + \cdots$$

and the masses of the fields $\varphi$ are then read off the quadratic form. But (4.5) tells us that the “mass matrix” $\frac{\partial^2 V}{\partial \phi_i \partial \phi_j} \bigg|_{\phi=v}$ has as many “zero modes” (eigenvectors of vanishing eigenvalue) as there are independent variations $\delta^a v_i \neq 0$. If $H$ is the invariance group of $v$, $\delta^a v_i \neq 0$ for the generators of $G$ that are not generators of $H$, and there are indeed $\dim G - \dim H$ massless modes, qed.

### 4.1.2 Chiral symmetry breaking

Consider a Lagrangian that involves massless fermions

$$L = \bar{\psi} i \partial_\mu \psi + g (\bar{\psi} \gamma^\mu \psi) (\bar{\psi} \gamma_\mu \psi),$$

where $\psi = \{\psi_a\}_{a=1,\cdots,N}$ is a $N$-component vector of 4-spinor fields. Note the absence of a mass term $\bar{\psi} \psi$ in (4.6). That Lagrangian is invariant under the action of two types of infinitesimal transformations

$$\delta_A \psi(x) = \delta A \psi(x)$$

$$\delta_B \psi(x) = \delta B \gamma_5 \psi(x),$$

where the matrices $A$ and $B$ are infinitesimal $N \times N$ antihermitian, that act on the “flavor” indices $\alpha$ but on spinor indices and hence commute with $\gamma$ matrices. Recall that $\gamma_5$ is Hermitian and anticommutes with the $\gamma_\mu$ and check that $\delta_A \bar{\psi} = -\bar{\psi} \delta A$, $\delta_B \bar{\psi} = \bar{\psi} \delta B \gamma_5$. The conserved Noether currents are respectively

$$J^a_\mu = \bar{\psi} T^a \gamma_\mu \psi$$

$$J^{a(5)}_\mu = \bar{\psi} T^a \gamma_5 \gamma_\mu \psi,$$

with $T^a$ infinitesimal generators of the unitary group $U(N)$. The transformations of the first line are called “vector”, those of the second, which involve $\gamma_5$, are “axial”. One may also rephrase it
in terms of independent transformations of $\psi_L := \frac{1}{2}(I-\gamma_5)\psi$ and of $\psi_R := \frac{1}{2}(I+\gamma_5)\psi$; one recalls that $(\gamma_5)^2 = I$ and that $\frac{1}{2}(I \pm \gamma_5)$ are thus projectors; one has thus $\bar{\psi}_L = \psi_L^\dagger \gamma_0 = \frac{1}{2} \bar{\psi}(I + \gamma_5)$, etc, and

$$\mathcal{L} = \bar{\psi}_L i\partial \psi_L + \bar{\psi}_R i\partial \psi_R + g(\bar{\psi}_L \gamma_\mu \psi_L + \bar{\psi}_R \gamma_\mu \psi_R)(\bar{\psi}_L \gamma^\mu \psi_L + \bar{\psi}_R \gamma^\mu \psi_R)$$

which is clearly invariant under the finite unitary transformations $\psi_L \rightarrow U_1 \psi_L$, $\psi_R \rightarrow U_2 \psi_R$, with $U_1, U_2 \in U(N)$. The group of chiral symmetry is thus $U(N) \times U(N)$.

If we now introduce a mass term $\delta \mathcal{L} = -m\bar{\psi}\psi$ (which “couples” the left and right components $\psi_L$ and $\psi_R$: $\delta \mathcal{L} = -m(\bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R)$), the “vector” symmetry is preserved, but the axial one is not and gives rise to a divergence

$$\partial^\mu J_\mu^{(5)}(x) \propto m\bar{\psi} T^a \gamma_5 \psi .$$

The residual symmetry group is $U(N)$, “diagonal” subgroup of $U(N) \times U(N)$ (diagonal in the sense that one takes $U_1 = U_2$ in the transformations of $\psi_{L,R}$).

The axial symmetry may also be spontaneously broken. Let us start from a Lagrangian, sum of terms of the type (4.6) with $N = 2$ and (4.3) for $n = 4$, with a coupling term between the fermions and the four bosons, traditionally denoted $\sigma$ and $\pi$

$$\mathcal{L} = \bar{\psi}(i\partial + g(\sigma + i\pi.\tau \gamma_5))\psi + \frac{1}{2}((\partial \pi)^2 + (\partial \sigma)^2) - \frac{1}{2} m^2(\sigma^2 + \pi^2) - \frac{\lambda}{4}(\sigma^2 + \pi^2)^2 ,$$

in which the Pauli matrices have been exceptionally denoted by $\tau$ not to confuse them with the field $\sigma$. The symmetry group is $U(2) \times U(2)$, with fields $\psi_L$, $\psi_R$ and $\sigma + i\pi.\tau$ transforming respectively by the representations $(\frac{1}{2},0)$, $(0,\frac{1}{2})$ and $(\frac{1}{2},\frac{1}{2})$ of $SU(2) \times SU(2)$ (see exercise A).

If $m^2 < 0$, the field $\phi = (\sigma, \pi)$ develops a non-zero vev, that may be oriented in the direction $\sigma$ if one has initially introduced a small explicit breaking term $\delta \mathcal{L} = \sigma \phi$, the analogue of a small magnetic field, which is then turned off. The vev is given as above by $\nu = -m^2/\lambda$, and, rewriting $\sigma(x) = \sigma'(x) + \nu$, where the field $\sigma'$ has now a vanishing vev, one sees that the fermions have acquired a mass $m_\psi = -g\nu$, whereas the $\pi$ are massless. This Lagrangian, the $\sigma$-model of Gell-Mann–Lévy, has been proposed as a model explaining the chiral symmetry breaking and the low mass of the $\pi$ mesons, regarded as “quasi Nambu–Goldstone quasi-bosons” (“quasi” in the sense that the chiral symmetry is only approximate before being spontaneously broken). Some elements of that model will reappear in the standard model.

### 4.1.3 Quantum symmetry breaking. Anomalies

Another mode of symmetry breaking, of purely quantum nature, manifests itself in anomalies of quantum field theories. A symmetry, which is apparent at the classical level of the Lagrangian, is broken by the effect of “quantum corrections". This is for instance what takes place with some chiral symmetries of the type just studied: an axial current which is classically divergenceless may acquire by a “one-loop effect" a divergence $\partial_\mu J_\mu^{(a)} \neq 0$. If the “anomalous" current is the Noether current of an internal classical symmetry, that symmetry is broken by the quantum anomaly, which may cause interesting physical effects (see discussion of the decay $\pi^0 \rightarrow \gamma\gamma$, for example in [IZ] chap 11). But in a theory like a gauge theory where the conservation of the axial current is crucial to ensure consistency –renormalizability, unitarity–, the anomaly constitutes a potential threat that must be controlled. This is what happens in the standard model, and we return to it in Chap. 5. Another example is provided by dilatation (scale) invariance of a massless theory, see the study of the renormalization group in F. David’s course.
4.2 The SU(3) flavor symmetry and the quark model.

An important approximate symmetry is the “flavor” SU(3) symmetry, to which we devote the rest of this chapter.

4.2.1 Why SU(3)?

We saw (Chap. 0) that if the weak and electromagnetic interactions are neglected, hadrons, i.e. particles subject to strong interactions such as proton and neutron, π mesons etc, fall into “multiplets” of a SU(2) group of isospin. Or said differently, the Hamiltonian (or Lagrangian) of strong interactions is invariant under the action of that SU(2) group and consequently, the SU(2) group is represented in the space of hadronic states by unitary representations. Proton and neutron belong to a representation of dimension 2 and of isospin $\frac{1}{2}$, the three pions $\pi^\pm, \pi^0$ form a representation of dimension 3 and isospin 1, etc. The electric charge $Q$ of each of these particles is related to the eigenvalue of the third component $I_z$ of isospin by

$$Q = \frac{1}{2} B + I_z$$  \hspace{1cm} \text{[for SU(2)]} \tag{4.11}$$

where a new quantum number $B$ appears, the baryonic charge, supposed to be (additively) conserved in all interactions (until further notice). $B$ is 0 for $\pi$ mesons, 1 for “baryons” as proton or neutron, $-1$ for their antiparticles, 4 for an $\alpha$ particle (Helium nucleus), etc.

This relation between $Q$ and $I_z$ must be amended for a new family of mesons ($K^\pm, K^0, \bar{K}^0, \cdots$) and baryons $\Lambda^0, \Sigma, \Xi, \ldots$ discovered at the end of the fifties. One assigns them a new quantum number $S$, strangeness. This strangeness is assumed to be additively conserved in strong interactions. Thus, if $S$ is $-1$ for the $\Lambda^0$ and +1 for the $K^+$ and the $K^0$, the reaction $p + \pi^- \rightarrow \Lambda^0 + K^0$ conserves strangeness, whereas the observed decay $\Lambda^0 \rightarrow p + \pi^-$ violates that conservation law, as it proceeds through weak interactions. Relation (4.11) must be modified into the Gell-Mann–Nishima relation

$$Q = \frac{1}{2} B + \frac{1}{2} S + I_z = \frac{1}{2} Y + I_z$$, \hspace{1cm} \text{where we introduced the hypercharge $Y$}, \hspace{1cm} \text{at this stage, equals $Y = B + S$.} \tag{4.12}$$

These conservation laws and different properties of mesons and baryons discovered then, in particular their organisation into “octets”, led at the beginning of the sixties M. Gell-Mann and Y. Ne’eman to postulate the existence of a group SU(3) of approximate symmetry of strong interactions. The quantum numbers $I_z$ and $Y$ that are conserved and simultaneously measurable are interpreted as eigenvalues of two commuting charges, hence of two elements of a Cartan algebra of rank 2, and the algebra of SU(3) is the natural candidate, as it possesses an irreducible 8-dimensional representation (see also exercise C of Chap. 3).

In the defining representation 3 of SU(3), one constructs a basis of the Lie algebra su(3), made of 8 Hermitian matrices $\lambda_a$ that play the role of Pauli matrices $\sigma_i$ for su(2). These matrices are normalised by

$$\text{tr} \lambda_a \lambda_b = 2 \delta_{ab}$$ \hspace{1cm} \text{[4.13]}$$
4.2. The SU(3) flavor symmetry and the quark model.

Figure 4.2: Octets of pseudoscalar \((J^P = 0^-)\) and of vector mesons \((J^P = 1^-)\)

\[
\begin{array}{c}
\pi^- & \pi^0 & \pi^+ \\
-1 & 0 & 1 \\
K^- & K^0 & K^+ \\
-1 & 0 & 1 \\
\end{array}
\]

Figure 4.3: Baryon octet \((J^P = \frac{1}{2}^+)\) and decuplet \((J^P = \frac{3}{2}^+)\)

\[
\begin{array}{c}
\eta & \xi & \rho \\
1 & 0 & 1 \\
\Sigma^- & \Sigma^0 & \Sigma^+ \\
-1 & 0 & 1 \\
\Xi^- & \Xi^0 & \Omega^- \\
-1 & 0 & 1 \\
\end{array}
\]

\[
\begin{array}{c}
\Delta^- & \Delta^0 & \Delta^+ & \Delta^{++} \\
-1 & 0 & 1 & 1 \\
\Sigma^{--} & \Sigma^{00} & \Sigma^{++} \\
-1 & 0 & 1 & 1 \\
\end{array}
\]

\[
\lambda_1, \lambda_2, \lambda_4, \lambda_5, \lambda_6, \text{and} \lambda_7 \text{ have the same matrix elements as } \sigma_1 \text{ and } \sigma_2 \text{ at the } \ast \text{ locations (} \ast \ldots \ast) \text{ and (} \ast \ldots \ast) \text{ respectively, where dots stand for zeros. The two generators of the Cartan algebra are}
\]

\[
\lambda_3 = \begin{pmatrix}
1 & . & . \\
. & -1 & . \\
. & . & .
\end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix}
1 & . & . \\
. & 1 & . \\
. & . & -2
\end{pmatrix}. \quad (4.14)
\]

The charges \(I_z\) and \(Y\) are then representatives in the representation under study of \(\frac{1}{2} \lambda_3\) and \(-\frac{1}{\sqrt{3}} \lambda_8\). See exercise B for the change of coordinates from \((\lambda_1, \lambda_3)\) (Dynkin labels of some representation, not to be confused with the above matrices !) to \((I_z, Y)\).

The matrices \(\lambda_a\) satisfy commutation relations

\[
[\lambda_a, \lambda_b] = 2i f_{abc} \lambda_c \quad (4.15)
\]

with structure constants (real and completely antisymmetric) \(f_{abc}\) of the \(su(3)\) Lie algebra. It is useful to also consider the anticommutators

\[
\{\lambda_a, \lambda_b\} = \frac{4}{3} \delta_{ab} + 2d_{abc} \lambda_c. \quad (4.16)
\]
Thanks to (4.13), (4.15) and (4.16) may be rewritten as tr(λₐ, λₖ, λₙ) = 4fₐₖₙ, tr({λₐ, λₖ}) = 4dₐₖ. These numbers f and d are tabulated in the literature . . . but they are easily computable! Beware that in contrast with (4.15), relation (4.16) and the (real, completely symmetric) constants dₐₖₙ are proper to the 3-dimensional representation.

Hadrons are then organized in SU(3) representations. Each multiplet gathers particles with the same spin \(J\) and parity \(P\). For instance two octets of mesons with \(JP\) equal to \(0^-\) or \(1^-\) and one octet and one “decuplet” of baryons of baryonic charge \(B = 1\) are easily identified. Contrary to isospin symmetry, the SU(3) symmetry \(^2\) is not an exact symmetry of strong interactions. The conservation laws and selection rules that follow are only approximate.

At this stage one may wonder about the absence of other representations of zero triality, such as the 27, or of those of non zero triality, like the 3 and the \(\bar{3}\). We return to that point in § 4.2.5.

4.2.2 Consequences of the SU(3) symmetry

The octets of fields

Let us look more closely at the two octets of baryons \(\mathcal{N} = (N, \Sigma, \Xi, \Lambda)\) and of pseudoscalar mesons \(\mathcal{P} = (\pi, K, \eta)\). Recalling what was said in Chap. 3, § 4.2, namely that the adjoint representation is made of traceless tensors of rank \((1,1)\), it is natural to group the 8 fields associated to these particles in the form of traceless matrices

\[
\Phi = \begin{pmatrix}
\frac{1}{\sqrt{2}}\pi^0 - \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\
\pi^- & -\frac{1}{\sqrt{2}}\pi^0 - \frac{1}{\sqrt{6}}\eta & K^0 \\
\bar{K}^- & \bar{K}^0 & \sqrt{\frac{2}{3}}\eta
\end{pmatrix},
\]

(4.17)

and

\[
\Psi = \begin{pmatrix}
\frac{1}{\sqrt{2}}\Sigma^0 - \frac{1}{\sqrt{6}}\Lambda & \Sigma^+ & p \\
\Sigma^- & -\frac{1}{\sqrt{2}}\Sigma^0 - \frac{1}{\sqrt{6}}\Lambda & n \\
\Xi^- & \Xi^0 & \sqrt{\frac{2}{3}}\Lambda
\end{pmatrix}.
\]

(4.18)

To make sure that the assignments of fields/particles to the different matrix elements are correct, it suffices to check their charge and hypercharge. The generators of charge \(Q\) and hypercharge \(Y\)

\[
Q = I_z + \frac{1}{2} Y = \frac{1}{3} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad Y = \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
\]

(4.19)

act in the adjoint representation by commutation and one has indeed

\[
[Q, \Phi] = \begin{pmatrix} 0 & \pi_+ & \bar{K}_+ \\ -\pi_- & 0 & 0 \\ -K_- & 0 & 0 \end{pmatrix} \quad [Y, \Phi] = \begin{pmatrix} 0 & 0 & K_+ \\ 0 & 0 & K_0 \\ -K_- & -\bar{K}_0 & 0 \end{pmatrix}.
\]

\(^2\)said to be “of flavour”, according to the modern terminology, but called “unitary symmetry” or “eightfold way” at the time of Gell-Mann and Ne’eman...
Exercises: (i) with no further calculation, what is $[I_z, \Phi]$? Check.  
(ii) Compute $\text{tr} \Phi^2$, and explain why the result justifies the choice of normalization of the matrix elements in (4.17). See also Problem 2.c.

Tensor products in SU(3) and invariant couplings

Recall that in SU(3), with notations of Chap. 3,

$$8 \otimes 8 = 1 \oplus 8 \oplus 8 \oplus 10 \oplus \overline{10} \oplus 27 .$$  

(4.20)

Let us show that this has immediate implications on the number of invariant couplings between fields.

- We want to write an SU(3) invariant Lagrangian involving the previous octets of fields $\Phi$ and $\Psi$. What is the number of independent “Yukawa couplings”, i.e. of the form $\bar{\Psi} \Phi \Psi$, that are invariant under SU(3)? In other words, what is the number of (linearly independent) invariants in $8 \otimes 8 \otimes 8$? According to a reasoning already done in Chap. 2. § 3.2, this number equals the number of times the representation 8 appears in $8 \otimes 8$, hence, according to (4.20), 2. There are thus two independent invariant Yukawa couplings. If the two octets $\Psi$ and $\Phi$ are written as traceless $3 \times 3$ matrices as in (4.17) and (4.18), $\Psi = \{\psi_j^i\}$ and $\Phi = \{\phi_k^i\}$, these two couplings read

$$\text{tr} \bar{\Psi} \Phi \Psi = \bar{\psi}_j^i \psi_k^j \phi_k^i \phi_j^k$$  

(4.21)

(this compact writing omits indices of Dirac spinors, a possible $\gamma_5$ matrix, etc). An often preferred expression uses the sum and difference of these two terms, hence $\text{tr} \bar{\Psi} \Phi \Psi$ and $\text{tr} \bar{\Psi} \bar{\Phi} \Phi \Psi$, traditionally called $f$ term and $d$ term, by reference to (4.15) and (4.16).

- Another question of the same nature is: what is $a \text{ priori}$ the number of SU(3) invariant amplitudes in the scattering of two particles belonging to the octets $\mathcal{N}$ and $\mathcal{P}: \mathcal{N}_i + \mathcal{P}_1 \rightarrow \mathcal{N}_f + \mathcal{P}_f$? (One takes only SU(3) invariance into account and does not consider possible discrete symmetries.) The issue is thus the number of invariants in the fourth tensor power of representation 8. Or equivalently, the number of times the same representation appears in the two products $8 \otimes 8$ and $8 \otimes 8$. If $m_i$ are the multiplicities appearing in $8 \otimes 8$, namely $m_1 = 1, m_8 = 2$, etc, see (4.20), this number is $\sum_i m_i^2 = 8$. There are thus eight invariant amplitudes. In other words, one may write $a \text{ priori}$ the scattering amplitude in the form

$$\langle \mathcal{N}_f \mathcal{P}_f | T | \mathcal{N}_i \mathcal{P}_i \rangle =$$  

(4.22)

$$\sum_{r=1}^8 A_r(s,t) \langle (I, I_z, Y)_{(N_r)}, (I, I_z, Y)_{(P_r)} | r, (I, I_z, Y)_{(r)} \rangle \langle r, (I, I_z, Y)_{(r)} | (I, I_z, Y)_{(N_r)}, (I, I_z, Y)_{(P_r)} \rangle$$

(with $s$ and $t$ the usual relativistic invariants $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$), and all the dependence in the nature of the scattered particles, identified by the values of their isospin and hypercharge, is contained in SU(3) Clebsch-Gordan coefficients.

- Let $\Phi_i$, $i = 1, 2, 3, 4$ be four distinct octet fields. How many quartic (degree 4) SU(3) invariant couplings may be formed with these four fields? On the one hand, the previous argument gives 8 couplings; on the other hand, it is clear that terms $\text{tr} (\Phi_{P_1} \Phi_{P_2} \Phi_{P_3} \Phi_{P_4})$ and $\text{tr} (\Phi_{P_1} \Phi_{P_2} \text{tr} (\Phi_{P_3} \Phi_{P_4}))$ are invariant under all permutations $P$. A quick counting leads to 9 different terms, in contradiction with the previous argument. Where is the catch? For more, go to the Problem 1 at the end of this chapter...
4.2.3 Electromagnetic breaking of the SU(3) symmetry

The SU(3) symmetry is broken, as we said, by strong interactions. Of course, just like the isospin SU(2) symmetry, it is also broken by electromagnetic and by weak interactions. We won’t examine the latter but describe now two consequences of the strong and electromagnetic breakings.

The interaction Lagrangian of a particle of charge \( q \) with the electromagnetic field \( A \) reads

\[
\mathcal{L}_{\text{em}} = -q j^\mu A_\mu
\]

(4.23)

where \( j \) is the electric current. The field \( A \) is invariant under SU(3) transformations, but how does \( j \) transform? One knows the transformation of its charge \( Q = R_3 x j_0(x, t) \), since following (4.12), \( Q \) is a linear combination of two generators \( Y \) and \( I_z \). \( Q \) thus transforms according to the adjoint representation (8, alias (1, 1) in terms of Dynkin labels). And it is natural to assume that the current \( j \) also transforms in the same way. This is indeed what is found when the current \( j^\mu \) is regarded as the Noether current of the U(1) symmetry (exercise, check it).

Magnetic moments

The electromagnetic form factors of the baryon octet are defined as

\[
\langle B | j_\mu(x) | B' \rangle = e^{ikx} \bar{u}(F_e^{BB'}(k^2)\gamma_\mu + F_m^{BB'}(k^2)\sigma_{\mu\nu}k^\nu)u'
\]

(4.24)

where \( \bar{u} \) and \( u' \) are Dirac spinors which describe respectively the baryons \( B \) and \( B' \); \( k \) is the four-momentum transfer from \( B' \) to \( B \). \( F_e \) is the electric form factor, if \( B = B' \), \( F_e(0) = q_B \), the electric charge of \( B \), whereas \( F_m \) is the magnetic form factor and \( F_m^{BB'}(0) \) gives the magnetic moment of baryon \( B \). One wants to compute these form factors to first order in the electromagnetic coupling and to zeroth order in the other terms that might break the SU(3) symmetry.

From a group theoretical point of view, the matrix element \( \langle B | j_\mu(x) | B' \rangle \) comes under the Wigner-Eckart theorem: there are two ways to project \( 8 \times 8 \) on \( 8 \) (see (4.20)), (or also, there are two ways to construct an invariant with \( 8 \otimes 8 \otimes 8 \)). There are thus two “reduced matrix elements”, hence two independent amplitudes for each of the two form factors, dressed with SU(3) Clebsch-Gordan coefficients. By an argument similar to (4.21), one finds that one may write

\[
F_{e,m}^{BB'}(k^2) = F_{e,m}^{(1)}(k^2) \text{tr} \bar{B}QB' + F_{e,m}^{(2)}(k^2) \text{tr} \bar{B}B'Q
\]

where \( Q \) is the matrix of (4.19)

\[
Q = \begin{pmatrix}
\frac{2}{3} & 0 & 0 \\
0 & -\frac{1}{3} & 0 \\
0 & 0 & -\frac{1}{3}
\end{pmatrix}
\]

and \( \text{tr} \bar{B}QB' \) means the coefficient of \( \bar{B}B' \) in the matrix trace \( \text{tr} \bar{\Psi}Q\Psi \), and likewise for \( \text{tr} \bar{B}B'Q \).

For example, the magnetic moment of the neutron \( \mu(n) \) is proportional to the magnetic term in
\(\bar{n}n\), namely \(-\frac{1}{3}(F_{m}^{(1)} + F_{m}^{(2)})\). The four functions \(F_{e,m}^{(1,2)}\) are unknown (their computation would involve the theory of strong interactions) but one may eliminate them and find relations

\[
\begin{align*}
\mu(n) &= \mu(\Sigma^{0}) = 2\mu(\Lambda) = -2\mu(\Sigma^{0}) & \mu(\Sigma^{+}) &= \mu(p) \\
\mu(\Xi^{-}) &= \mu(\Sigma^{-}) = -(\mu(p) + \mu(n)) & \mu(\Sigma^{0} \to \Lambda) &= \frac{\sqrt{3}}{2}\mu(n),
\end{align*}
\]

where the last quantity is the transition magnetic moment \(\Sigma^{0} \to \Lambda\). These relations are in qualitative agreement with experimental data.

The magnetic moments of “hyperons” (baryons of higher mass than the nucleons) are measured by their spin precession in a magnetic field or in transitions within “exotic atoms” (i.e. in the nucleus of which a nucleon has been substituted for a hyperon). The transition magnetic moment \(\Sigma^{0} \to \Lambda\) is determined from the cross-section \(\Lambda \to \Sigma^{0}\) in the Coulomb field of a heavy nucleus. One reads in tables

\[
\begin{align*}
\mu(p) &= 2.792847351 \pm 0.000000028 \mu_N & \mu(n) &= -1.9130427 \pm 0.0000005 \mu_N \\
\mu(\Lambda) &= -0.613 \pm 0.004 \mu_N & |\mu(\Sigma^{0} \to \Lambda)| &= 1.61 \pm 0.08 \mu_N \\
\mu(\Sigma^{+}) &= 2.458 \pm 0.010 \mu_N & \mu(\Sigma^{-}) &= -1.160 \pm 0.025 \mu_N \\
\mu(\Xi^{0}) &= -1.250 \pm 0.014 \mu_N & \mu(\Xi^{-}) &= -0.6507 \pm 0.0025 \mu_N
\end{align*}
\]

where \(\mu_N\) is the nuclear magneton, \(\mu_N = \frac{e\hbar}{2m_p} = 3.152 \times 10^{-14}\) MeV T\(^{-1}\).

**Electromagnetic mass splittings**

With similar assumptions and methods, one may also find relations between mass splittings of particles with same hypercharge and isospin \(I\) but different charge, due to electromagnetic interactions, see Problem 3.

### 4.2.4 “Strong” mass splittings. Gell-Mann–Okubo mass formula

In view of the discrepancies between masses within a SU(3) multiplet, the mass term in the Lagrangian (or Hamiltonian) cannot be an invariant of SU(3). Gell-Mann and Okubo made the assumption that the non invariant term \(\Delta M\) transforms under the representation 8, more precisely, since it must have vanishing isospin and hypercharge, that it transforms like the \(\eta\) or \(\Lambda\) component of octets. One is thus led to consider matrix elements \(\langle H|\Delta M|H \rangle\) for the hadrons \(H\) of a multiplet, and to appeal once more to Wigner–Eckart theorem. According to the decomposition rules of tensor products given in Chap. 3, the representation 8 appears at most twice in the product of an irreducible representation of SU(3) by its conjugate, (check it, recalling that \(8 = 3 \otimes 3 \oplus 1\)); there are at most two independent amplitudes describing mass splittings within the multiplet, which leads to relations between these mass splittings.

An elegant argument enables one to avoid the computation of Clebsch–Gordan coefficients and to find these two amplitudes in any representation. As the eight infinitesimal generators transform themselves according to the representation 8 (adjoint representation), they may be set as before into a \(3 \times 3\) matrix

\[
G = \begin{pmatrix}
\frac{1}{2}Y + I_z & \sqrt{2}I_+ & * \\
\sqrt{2}I_- & \frac{1}{2}Y - I_z & * \\
* & * & -Y
\end{pmatrix}
\]

where the * stand for strangeness-changing generators that are of no concern to us here. (Note that \(G_{11} = I_z + \frac{1}{2}Y = Q\), the electric charge, is invariant under the action (by commutation with \(G\)) of generators \(X =\)
which preserve the electric charge.) One seeks two combinations of the generators $I_z$ and $Y$ transforming like the element $(3, 3)$ of that matrix. One is of course $Y$ itself, the other is given by the element $(3, 3)$ of the cofactor of $G$, $\text{cof}G_{33} = \frac{1}{4}Y^2 - I_z^2 - 2I_yL_y = \frac{1}{4}Y^2 - \vec{P}^2$.

One gets in that way a mass formula for any representation (any multiplet)

$$M = m_1 + m_2Y + m_3(I(I + 1) - \frac{1}{4}Y^2)$$

which leaves three undetermined constants (that depend on the multiplet). For example for the baryon octet, one has four experimental masses, which leads to a sum rule

$$\frac{M_\Xi + M_N}{2} = \frac{3M_\Lambda + M_\Sigma}{4}$$

which is experimentally well verified: one finds 1128.5 MeV/$c^2$ in the left hand side, 1136 MeV/$c^2$ in the rhs\(^3\). For the decuplet, show that the same formula gives equal mass differences between the four particles $\Delta$, $\Sigma^*$, $\Xi^*$ and $\Omega^-$. The latter result led to an accurate prediction of the existence and mass of the $\Omega^-$ particle, which was regarded as one of the major achievements of SU(3). For the octet of pseudoscalar mesons, the mass formula is empirically better verified in terms of the square masses

$$m_K^2 = \frac{3m_\eta^2 + m_\pi^2}{4}.$$  

### 4.2.5 Quarks

The representations $3$ and $\bar{3}$ are so far absent from the scene: among the observed particles, no “triplet” seems to show up. The Gell-Mann–Zweig model makes the assumption that a triplet (representation $3$) of quarks $(u, d, s)$ (“up”, “down” and “strange”) and its conjugate representation $\bar{3}$ of antiquarks $(\bar{u}, \bar{d}, \bar{s})$ encompass the elementary constituents of all hadrons (known at the time). Their charges and hypercharges are respectively

<table>
<thead>
<tr>
<th>Quarks</th>
<th>$u$</th>
<th>$d$</th>
<th>$s$</th>
<th>$\bar{u}$</th>
<th>$\bar{d}$</th>
<th>$\bar{s}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Isospin $I_z$</td>
<td>$\frac{1}{2}$</td>
<td>$-\frac{1}{2}$</td>
<td>$0$</td>
<td>$-\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$0$</td>
</tr>
<tr>
<td>Baryonic charge $B$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
<td>$-\frac{1}{3}$</td>
<td>$-\frac{1}{3}$</td>
<td>$-\frac{1}{3}$</td>
</tr>
<tr>
<td>Strangeness $S$</td>
<td>0</td>
<td>0</td>
<td>$-1$</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Hypercharge $Y$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
<td>$-\frac{2}{3}$</td>
<td>$-\frac{1}{3}$</td>
<td>$-\frac{1}{3}$</td>
<td>$\frac{2}{3}$</td>
</tr>
<tr>
<td>Electric charge $Q$</td>
<td>$\frac{2}{3}$</td>
<td>$-\frac{1}{3}$</td>
<td>$-\frac{1}{3}$</td>
<td>$-\frac{2}{3}$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
</tr>
</tbody>
</table>

Table 1. Quantum numbers of quarks $u$, $d$, $s$

One recalls (Chap. 3 §4) that any irreducible representation of SU(3) appears in the decomposition of iterated tensor products of representations $3$ and $\bar{3}$; in particular, $3 \otimes \bar{3} = 1 \oplus 8$\(^3\)

\(^3\)The observed masses of these hadrons are $M_N \approx 939$ MeV/$c^2$, $M_\Lambda = 1116$ MeV/$c^2$, $M_\Sigma \approx 1195$ MeV/$c^2$, $M_\Xi \approx 1318$ MeV/$c^2$; those of pseudoscalar mesons $m_\pi \approx 137$ MeV/$c^2$, $m_K \approx 496$ MeV/$c^2$ and $m_\eta = 548$ MeV/$c^2$. For the decuplet, $M_\Delta \approx 1232$ MeV/$c^2$, $M_{\Sigma^*} \approx 1385$ MeV/$c^2$, $M_{\Xi^*} \approx 1530$ MeV/$c^2$, $M_\Omega \approx 1672$ MeV/$c^2$. 

\(J.-B. Z\)  M2 ICFP/Physique Théorique 2012  November 20, 2012
and $3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$. Mesons and baryons observed in Nature and classified as above in representations 8 and 10 of SU(3) are bound states of pairs $q\bar{q}$ or $qqq$, respectively. More generally, one assumes that only representations of zero triality may give rise to observable particles. Thus

$$p = uud, \ n = udd, \ \Omega^- = sss, \ \Delta^{++} = uuu, \ \cdots, \ \Delta^- = ddd,$$

where $\pi^+ = u\bar{d}$, $\pi^0 = \frac{(u\bar{u} - d\bar{d})}{\sqrt{2}}$, $\pi^- = d\bar{u}$, $\eta_8 = \frac{(u\bar{u} + d\bar{d} - 2s\bar{s})}{\sqrt{6}}$, $K^+ = u\bar{s}$, $K^0 = d\bar{s}$ etc.

The quark model interprets the singlet that appears in the product $3 \times 3$ as a bound state $\eta_1 = \frac{(u\bar{u} + d\bar{d} + s\bar{s})}{\sqrt{3}}$. The physically observed particles $\eta$ (mass 548 MeV) and $\eta'$ (958 MeV) result from a “mixing” (i.e. a linear combination) due to SU(3) breaking interactions of these $\eta_1$ and $\eta_8$. Exercise: complete on Fig. 4.3 the interpretations of baryons as bound states of quarks, making use of the knowledge of their charges and other quantum numbers.

### 4.2.6 Hadronic currents and weak interactions

The weak interactions are phenomenologically well described by an effective “current–current” Lagrangian (Fermi)

$$L_{\text{Fermi}} = -\frac{G}{\sqrt{2}} J^\mu(x) J^\dagger_\mu(x)$$

where $G$ is the Fermi constant, whose value (in units where $\hbar = c = 1$) is

$$G = (1,026 \pm 0,001) \times 10^{-5} m_p^{-2}.$$  

(But this interaction Lagrangian has the major flaw of being non renormalisable, a flaw which will be corrected by the gauge theory of the Standard Model. At low energy, however, $L_{\text{Fermi}}$ offers a good description of physics, whence the name “effective”.) The current $J_\rho$ is the sum of a leptonic and a hadronic contributions

$$J_\rho(x) = l_\rho(x) + h_\rho(x)$$

The leptonic current

$$l_\rho(x) = \bar{\psi}_e(x) \gamma_\rho (1 - \gamma_5) \psi_\nu + \bar{\psi}_\mu(x) \gamma_\rho (1 - \gamma_5) \psi_\nu + \cdots [ + \bar{\psi}_\tau(x) \gamma_\rho (1 - \gamma_5) \psi_\nu]$$
is the sum of contributions of the lepton families (or *generations*), e, μ (and τ that we omit in this first approach). The hadronic current, if one restricts to the first two generations, reads

\[ h_\rho = \cos \theta_C \ h_\rho^{(\Delta S=0)} + \sin \theta_C \ h_\rho^{(\Delta S=1)} \]  

(4.33)

i.e. a combination of strangeness-conserving and non-conserving currents, weighted by the *Cabibbo angle* \( \theta_C \approx 0.25 \). (This “mixing” extends to the introduction of the third generation, see next Chapter.) Finally each of these currents \( h_\rho^{(\Delta S=0)}, h_\rho^{(\Delta S=1)} \) has the “\( V - A \)” form, following an idea of Feynman and Gell-Mann, *i.e.* is a combination of vector and axial currents,

\[ h_\rho^{(\Delta S=0)} = (V_\rho^1 - i V_\rho^2) - (A_\rho^1 - i A_\rho^2) \]  

(4.34)

\[ h_\rho^{(\Delta S=1)} = (V_\rho^4 - i V_\rho^5) - (A_\rho^4 - i A_\rho^5) \]  

(4.35)

The vector currents \( V_\rho^{1,2,3} \) are the Noether currents of isospin, the other components of \( V_\rho \) are those of the SU(3) symmetry. One shows that their conservation (exact for isospin, approximate for the others) implies that in the matrix element \( G(p|h_\rho^{(\Delta S=0)}|n) = \bar{u}_p \gamma_\mu (G_V(q^2) - G_A(q^2)\gamma_5)u_n \) measured in beta decay at quasi-vanishing momentum transfer, the vector form factor \( G_V(0) = G \). On the contrary, the axial currents are non conserved and \( G_A(0) \) is “renormalized” (that is, dressed) by strong interactions, \( G_A/G_V \approx 1.22 \). The electromagnetic current is nothing other that the combination \( j_\rho = V_\rho^3 + \frac{1}{\sqrt{3}} V_\rho^8 \). In the quark model, these hadronic currents have the form

\[ V_\rho^a(x) = \bar{q}(x) \frac{\lambda^a}{2} \gamma_\mu g(x) \quad A_\rho^a(x) = \bar{q}(x) \frac{\lambda^a}{2} \gamma_\mu \gamma_5 g(x) \]  

(4.36)

We will meet them again in the Standard Model.

### 4.3 From SU(3) to SU(4) to six flavors

#### 4.3.1 New flavors

The discovery in the mid 70’s of particles of a new type revived the game: these particles carry another quantum number, “charm” (whose existence had been postulated beforehand by Glashow, Iliopoulos and Maiani and by Kobayashi and Maskawa for two different reasons). This introduces a third direction in the space of internal symmetries, on top of isospin and strangeness (or hypercharge). The relevant group is SU(4), which is more severely broken than SU(3). Particles fall into representations of that SU(4), etc. A fourth flavor, charm, is thus added, and a fourth charmed quark \( c \) constitutes with \( u, d, s \) the representation 4 of SU(4), as inobservable as the 3 of SU(3), according to the same principle.

As for today, one believes there are in total six flavors, the last two being *beauty* or *bottomness* and *truth* (or *topness* ??), hence two additional quarks \( b \) and \( t \). B mesons, which are bound states \( u \bar{b}, d \bar{b} \) etc, are observed in everyday experiments, in particular at LHCb, whereas the experimental evidence for the existence of the \( t \) quark is more indirect. The hypothetical flavor
§ 4.3. From SU(3) to SU(4) to six flavors

4.3. From SU(3) to SU(4) to six flavors

Figure 4.5: Mesons of spin $J^P = 0^-$ of the representation 15 of SU(4)

The SU(6) group SU(6) is very strongly broken, as attested by masses of the 6 quarks:

\[ m_u \approx 1.5 - 4 \text{MeV}, \quad m_d \approx 4 - 8 \text{MeV}, \quad m_s \approx 80 - 130 \text{MeV} \tag{4.37} \]

\[ m_c \approx 1.15 - 1.35 \text{GeV}, \quad m_b \approx 4 - 5 \text{GeV}, \quad m_t \approx 175 \text{GeV} \]

and this limits its usefulness. One may however rewrite (4.12) in the form

\[ Q = \frac{1}{2} Y + I_z \quad Y = B + S + C + B + T \]

with different quantum numbers contributing additively to hypercharge. The convention is that the flavor $S, C, B, T$ of a quark vanishes or is of the same sign as its electric charge $Q$. Thus $C(c) = 1, B(b) = -1$ etc. Table 1 must now be extended as follows

<table>
<thead>
<tr>
<th>Quarks</th>
<th>$u$</th>
<th>$d$</th>
<th>$s$</th>
<th>$c$</th>
<th>$b$</th>
<th>$t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Isospin $I_z$</td>
<td>$\frac{1}{2}$</td>
<td>$-\frac{1}{2}$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>Baryonic charge $B$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>Strangeness $S$</td>
<td>$0$</td>
<td>$0$</td>
<td>$-1$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>Charm $C$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$1$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>Beauty $B$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$-1$</td>
<td>$0$</td>
</tr>
<tr>
<td>Truth $T$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$1$</td>
</tr>
<tr>
<td>Hypercharge $Y$</td>
<td>$\frac{2}{3}$</td>
<td>$-\frac{1}{3}$</td>
<td>$-\frac{1}{3}$</td>
<td>$\frac{2}{3}$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>Electric charge $Q$</td>
<td>$\frac{1}{3}$</td>
<td>$-\frac{2}{3}$</td>
<td>$-\frac{2}{3}$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
</tr>
</tbody>
</table>

Table 2. Quantum numbers of quarks $u, d, s, c, b, t$

4.3.2 Introduction of color

Various problems with the original quark model have led to the hypothesis (Han–Nambu) that each flavor comes with a multiplicity 3, which reflects the existence of a group SU(3), different from the previous one, the color group SU(3)$_c$.

One should of course make precise what is meant by mass of an invisible particle, and this may be done in an indirect way and with several definitions, whence the range of given values.
Considerations leading to that triplicating hypothesis are first the study of the $\Delta^{++}$ particle, with spin $3/2$, made of 3 quarks $u$. This system of 3 quarks has a spin $3/2$ and an orbital angular momentum $L = 0$, which give it a symmetric wave function, in contradiction with the fermionic character of quarks. The additional color degree of freedom allows an extra antisymmetrization, (which leads to a singlet state of color), and thus removes the problem. On the other hand, the decay amplitude of $\pi^0 \rightarrow 2\gamma$ is proportional to the sum $\sum Q^2 I_z$ over the set of fermionic constituents of the $\pi^0$. The proton, with its charge $Q = 1$ and $I_z = 1/2$, gives a value in agreement with experiment. Quarks ($u,d,s$) with $Q = (2/3, 1/3, 0)$ and $I_z = (1/2, -1/2, 0)$ lead to a result three times too small, and color multiplicity corrects it to the right value.

According to the confinement hypothesis, only states of the representation 1 of SU(3)$_c$ are observable. The other states, which are said to be “colored”, are bound in a permanent way inside hadrons. This applies to quarks, but also to gluons, which are vector particles (spin 1) transforming by the representation 8 of SU(3)$_c$, whose existence is required by the construction of the gauge theory of strong interactions, Quantum Chromodynamics (QCD), see Chap. 5.

To be more precise, the confinement hypothesis applies to zero or low temperature, and quark or gluon deconfinement may occur in hadronic matter at high temperature or high density (within the “quark gluon plasma”).

The quark model with its color group SU(3)$_c$ is now regarded as part of quantum chromodynamics. The six flavors of quarks are grouped into three “generations”, $(u,d), (c,s), (t,b)$, which are in correspondence with three generations of leptons, $(e^-, \nu_e), (\mu^-, \nu_\mu), (\tau^-, \nu_\tau)$. That correspondence is important for the consistency of the Standard Model (anomaly cancellation), see next chapter.

Further references for Chapter 4

On flavor SU(3), the standard reference containing all historical papers is M. Gell-Mann and Y. Ne’eman, *The Eightfold Way*, Benjamin 1964.

In particular one finds there tables of SU(3) Clebsch-Gordan coefficients by J.J. de Swart.


All the properties of particles mentioned in this chapter may be found in the tables of the Particle Data Group, on line on the site [http://pdg.lbl.gov/2012/reviews/contents_sports.html](http://pdg.lbl.gov/2012/reviews/contents_sports.html)
Exercises for chapter 4

A. Sigma model and chiral symmetry breaking
Consider the Lagrangian (4.10) and define $W = \sigma + i\pi$.

1. Compute $\det W$. Show that one may write $L$ in terms of $L, R$ and $W$ as

$$L = \bar{R} i / @ R + \bar{L} i / @ L + g (\bar{L} W R + \bar{R} W^\dagger L) + L_K - \frac 1 2 m^2 \det W - \frac \lambda 4 (\det W)^2$$

where $L_K$ is the kinetic term of the fields $(\sigma, \pi)$. One may also give that term the form $L_K = \frac 1 2 (\det \partial_0 W - \sum_{i=1}^3 \det \partial_i W)$ (which looks a bit odd, but which is indeed Lorentz invariant!).

2. Show that $L$ is invariant under transformations of SU(2) $\times$ SU(2) with $L \rightarrow U L, R \rightarrow V R$, provided $W$ transforms in a way to be specified. Justify the assertion made in §4.1.2: $L, R$ and $W$ transform respectively under the representations $(1/2, 0), (0, 1/2)$ and $(1/2, 1/2)$.

3. If the field $W$ acquires a vev $v$, for example along the direction of $\sigma, \langle \sigma \rangle = v$, show that the field $\psi$ acquires a mass $M = -g v$.

B. Changes of basis in SU(3)
In SU(3), write the change of basis which transforms the weights $\lambda_1, \lambda_2$ of Chap. 3 into the axes used in figures 2, 3 and 4. Derive the transformation of the coordinates $(\lambda_1, \lambda_2)$ (Dynkin labels) into the physical coordinates $(I_z, Y)$. What is the dimension of the representation of SU(3) expressed in terms of the isospin and hypercharge of its highest weight?

C. Gell-Mann–Okubo formula
Complete and justify all the arguments sketched in §4.2.2, 4.2.3 and 4.2.4. In particular check that the formula (4.27) does lead for the baryon octet to the rule (4.28), and for the decuplet, to constant mass splittings.

D. Counting amplitudes
How many independent amplitudes are necessary to describe the scattering $BD \rightarrow BD$, where $B$ and $D$ refer to the baryonic octet and decuplet?

Problems

1. SU(3) invariant four-field couplings
Consider a Hermitian, $3 \times 3$ and traceless matrix $A$.

   a. Show that its characteristic equation

   $$A^3 - (\text{tr } A) A^2 + \frac 1 2 ( (\text{tr } A)^2 - \text{tr } A^2 ) A - \det A = 0$$

   implies a relation between $\text{tr } A^4$ and $(\text{tr } A^2)^2$.

   b. If the group SU(3) acts on $A$ by $A \rightarrow U A U^\dagger$, show that any sum of products of traces of powers of $A$ is invariant. We call such a sum an “invariant polynomial in $A$”. How many linearly independent such invariant polynomials in $A$ of degree 4 are there?
c. One then “polarises” the identity found in a., which means one writes \( A = \sum_{i=1}^{4} x_i A_i \)
with 4 matrices \( A_i \) of the previous type and 4 arbitrary coefficients \( x_i \), and one identifies the
coefficient of \( x_1 x_2 x_3 x_4 \). Show that this gives an identity of the form (Burgoyne’s identity)
\[
\sum_P \text{tr} (A_P A_P A_P A_P) = a \sum_P \text{tr} (A_P A_P) \text{tr} (A_P A_P)
\] (4.38)
with sums over permutations \( P \) of 4 elements and a coefficient \( a \) to be determined. How many
distinct terms appear in each side of that identity?

d. How many polynomials of degree 4, quadrilinear in \( A_1, \cdots, A_4 \), invariant under the action
of SU(3) \( A_i \rightarrow UA_i U^\dagger \) and linearly independent, can one write? Why is the identity (4.38)
useful?

2. Hidden invariance of a bosonic Lagrangian

One wants to write a Lagrangian for the field \( \Phi \) of the pseudoscalar meson octet, see (4.17).

a. Why is it natural to impose that this Lagrangian be even in the field \( \Phi \) ?

b. Using the results of Problem 1., write the most general form of an SU(3) invariant
Lagrangian, of degree less or equal to 4 (for renormalizability) and even in \( \Phi \).

c. One then writes each complex field by making explicit its real and imaginary parts, for
example \( K^+ = \frac{1}{\sqrt{2}} (K_1 - iK_2) \), \( K^- = \frac{1}{\sqrt{2}} (K_1 + iK_2) \), and likewise with \( K^0, \bar{K}^0 \) and with \( \pi^\pm \).
Compute \( \Phi^2 \) with that parametrization and show that one gets a simple quadratic form in
the 8 real components. What is the invariance group \( G \) of that quadratic form? Is \( G \) a subgroup
of SU(3)?

d. Conclude that any Lagrangian of degree 4 in \( \Phi \) which is invariant under SU(3) is in fact
invariant by this group \( G \).

3. Electromagnetic mass splittings in an SU(3) octet

Preliminary question.

Given a vector space \( E \) of dimension \( d \), we denote \( E \otimes E \) the space of rank 2 tensors and
\((E \otimes E)_S, \) resp. \((E \otimes E)_A \), the spaces of symmetric, resp. antisymmetric, rank 2 tensors, also
called (anti-)symmetrized tensor product. What is the dimension of spaces \( E \otimes E, \) \((E \otimes E)_S, \)
\((E \otimes E)_A \) ?

One assumes that SU(3) is an exact symmetry of strong interactions, and one wants to study
mass splittings due to electromagnetic effects.

a. How many independent mass differences between baryons with the same quantum num-
bers \( I \) and \( Y \) but different charges \( Q \) (or \( I_z \) component), are there in the baryon octet \( J^P = \frac{1}{2}^+ \)?

We admit that these electromagnetic effects result from second order perturbations in the
Lagrangian \( \mathcal{L}_{em}(x) = -g j^\mu(x) A_\mu(x) \). If \( |B\rangle \) is a baryon state, one should thus compute
\[
\delta M_B = \langle B | (\int d^4x \mathcal{L}_{em})^2 | B \rangle .
\] (4.39)

For lack of a good way of computing that matrix element, one wants to determine the number
of independent amplitudes that contribute.