

Chapter 2

Linear representations of groups

The action of a group in a set has been mentioned in the previous chapter (see exercise A and TD). We now focus our attention on the *linear* action of a group in a vector space. This situation is frequently encountered in geometry and in physics (quantum mechanics, statistical physics, field theory, ...). One should keep in mind, however, that other group actions may have some physical interest: for instance the rotation group $SO(n)$ acts on the sphere S^{n-1} in a non-linear way, and this is relevant for example in models of ferromagnetism and field theories called non linear σ models, see the course of F. David.

2.1 Basic definitions and properties

2.1.1 Basic definitions

A group G is said to be represented in a vector space E (on a field which for us is always \mathbb{R} or \mathbb{C}), or stated differently, E carries a representation of G , if one has a homomorphism D of the group G into the group $GL(E)$ of linear transformations of E :

$$\begin{aligned}\forall g \in G \quad g &\mapsto D(g) \in GL(E) \\ \forall g, g' \in G \quad D(g.g') &= D(g).D(g') \\ D(e) &= I \\ \forall g \in G \quad D(g^{-1}) &= (D(g))^{-1}\end{aligned}\tag{2-1}$$

where I denotes the identity operator in $GL(E)$. If the representation space E is of finite dimension p , the representation itself is said to be of dimension p . The representation which to any $g \in G$ associates 1 (considered as $\in GL(\mathbb{R})$) is called *trivial* or *identity representation*; it is of dimension 1.

The representation is said to be *faithful* if $\ker D = \{e\}$, or equivalently if $D(g) = D(g') \Leftrightarrow g = g'$. Else, the kernel of the homomorphism is an invariant subgroup H , and the representation of the quotient group G/H in E is faithful (check!). Consequently, any non trivial representation

of a simple group is faithful. Conversely, if G has an invariant subgroup H , any representation of G/H gives a degenerate (*i.e.* non faithful) representation of G .

If E is of finite dimension p , one may choose a basis e_i , $i = 1, \dots, p$, and associate with any $g \in G$ the representative matrix of $D(g)$, denoted with a curly letter :

$$D(g)e_j = e_i \mathcal{D}_{ij}(g) \quad (2-2)$$

with, as (almost) always in these notes, the convention of summation over repeated indices. The setting of indices (i : row index, j column index) is dictated by (2-1). Indeed we have

$$\begin{aligned} D(g.g')e_k &= e_i \mathcal{D}_{ik}(g.g') \\ &= D(g)(D(g')e_k) = D(g)e_j \mathcal{D}_{jk}(g') \\ &= e_i \mathcal{D}_{ij}(g) \mathcal{D}_{jk}(g') \\ \text{hence } \mathcal{D}_{ik}(g.g') &= \mathcal{D}_{ij}(g) \mathcal{D}_{jk}(g') . \end{aligned} \quad (2-3)$$

Examples : The group $SO(2)$ of rotations in the plane admits a dimension 2 representation, with matrices

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad (2-4)$$

which describe indeed rotations of angle θ around the origin.

The group $SU(3)$ is defined as the set of unitary, unimodular 3×3 matrices U . These matrices form by themselves a representation of $SU(3)$, it is the “defining representation”. Show that the complex conjugate matrices form another representation of $SU(3)$.

Of which group do the matrices $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ form a representation?

2.1.2 Equivalent representations. Characters

Take two representations D et D' of G in spaces E and E' , and suppose that there exists a linear operator V from E into E' such that

$$\forall g \in G \quad VD(g) = D'(g)V . \quad (2-5)$$

Such a V is called an *intertwining operator*, or “intertwiner” in short. If V is invertible (and hence if E and E' have equal dimension, if finite), we say that the representations D et D' are *equivalent*. (It is an equivalence relation between representations!).

In the case of finite dimension, where one identifies E and E' , the representative matrices of D et D' are related by a similarity transformation and may be considered as differing by a change of basis. There is thus no fundamental distinction between two equivalent representations, and in representation theory, one strives to study inequivalent representations.

One calls *character* of a finite dimension representation the trace of the operateur $D(g)$:

$$\chi(g) = \text{tr } D(g) . \quad (2-6)$$

It is a function of G in \mathbb{R} or \mathbb{C} which satisfies the following properties (check!) :

- The character is independent of the choice of basis in E .
- Two equivalent representations have the same character.
- The character takes the same value for all elements of a same conjugacy class of G : one says that the character is a *class function*: $\chi(g) = \chi(hgh^{-1})$.

The converse property, namely whether any class function may be expressed in terms of characters, is true for any finite group, and for any compact Lie group and continuous (or L^2) function on G : this is the Peter-Weyl theorem, see below §2.3.1.

Note also that the character, evaluated for the identity element in the group, gives the dimension of the representation

$$\chi(e) = \dim D . \quad (2-7)$$

2.1.3 Reducible and irreducible, conjugate, unitary representations...

a. Reducible and irreducible representations

Another redundancy is due to direct sums of representations. Assume that we have two representations D_1 and D_2 of G in two spaces E_1 and E_2 . One may then construct a representation in the space $E = E_1 \oplus E_2$ and the representation is called direct sum of representations D_1 and D_2 and denoted $D_1 \oplus D_2$. (Recall that any vector of $E_1 \oplus E_2$ may be written in a unique way as a linear combination of a vector of E_1 and of a vector of E_2). The two subspaces E_1 and E_2 of E are clearly left separately invariant by the action of $D_1 \oplus D_2$.

Inversely, if a representation of G in a space E leaves invariant a subspace of E , it is said to be *reducible*. Else, it is *irreducible*. If D is reducible and leaves both the subspace E_1 and its complementary subspace E_2 invariant, one says that the representation est *completely reducible* (or *decomposable*); one may then consider E as the direct sum of E_1 and E_2 and the representation as a direct sum of representations in E_1 and E_2 .

If E is finite dimensional, this means that the matrices of the representation take the following form (in a basis adapted to the decomposition!) with blocks of dimensions $\dim E_1$ and $\dim E_2$

$$\forall g \in G \quad \mathcal{D}(g) = \begin{pmatrix} \mathcal{D}_1(g) & 0 \\ 0 & \mathcal{D}_2(g) \end{pmatrix} . \quad (2-8)$$

If the representation is reducible but not completely reducible, its matrix takes the following form, in a basis made of a basis of E_1 and a basis of some complementary subspace

$$\mathcal{D}(g) = \begin{pmatrix} \mathcal{D}_1(g) & \mathcal{D}'(g) \\ 0 & \mathcal{D}_2(g) \end{pmatrix} . \quad (2-9)$$

This is the case of representations of the translation group in one dimension. The representation

$$\mathcal{D}(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \quad (2-10)$$

is reducible, since it leaves invariant the vectors $(X, 0)$ but it has no invariant supplementary subspace.

On the other hand, if the reducible representation of G in E leaves invariant the subspace E_1 , there exists a representation in the subspace $E_2 = E/E_1$. In the notations of equ. (2-9), its matrix is $\mathcal{D}_2(g)$.

One should stress the importance of the number field in that discussion of irreducibility. For instance the representation (2-4) which is irreducible on a space over \mathbb{R} is not over \mathbb{C} : it may be rewritten by a (complex) change of basis in the form

$$\begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}. \quad (2-11)$$

b. Conjugate and contragredient representations

Given a representation D , \mathcal{D} its matrix in some basis, the complex conjugate matrices \mathcal{D}^* form another representation D^* , called conjugate representation, since they also satisfy (2-3)

$$\mathcal{D}_{ik}^*(g \cdot g') = \mathcal{D}_{ij}^*(g) \mathcal{D}_{jk}^*(g'). \quad (2-12)$$

The representation D is said to be *real* if there exists a basis where $\mathcal{D} = \mathcal{D}^*$. This implies that its character χ is real. Conversely if χ is real, the representation D is equivalent to its conjugate D^* ¹. If the representations D and D^* are equivalent but if there no basis where $\mathcal{D} = \mathcal{D}^*$, the representations are called *pseudoreal*. (This is for example the case of the spin $\frac{1}{2}$ representation of $SU(2)$.) For alternative and more canonical definitions of these notions of real representations, see the Problem III.

This concept plays a key role in the study of the “chiral non-singlet anomaly” in gauge theories: if fermions belong to a real or pseudoreal representation of the gauge group, their potential anomaly cancels, which is determinant for the consistency of the theory. In the standard model, this comes from a balance between contributions of quarks and leptons, see chap 5.

The *contragredient* representation of D is defined by

$$\bar{D}(g) = D^{-1T}(g) \quad (2-13)$$

or alternatively, $\bar{D}_{ij}(g) = \mathcal{D}_{ji}(g^{-1})$, which does satisfy (2-3). For a unitary representation, see next paragraph, $\bar{D}_{ij}(g) = \mathcal{D}_{ij}^*(g)$, and the contragredient representation equals the conjugate. The representations D , D^* and \bar{D} are simultaneously reducible or irreducible.

c. Unitary representations

Suppose that the vector space E is “prehilbertian”, *i.e.* is endowed with a scalar product, (*i.e.* a form $J(x, y) = \langle x|y \rangle = \langle y|x \rangle^*$, bilinear symmetric if we work on \mathbb{R} , or sesquilinear on \mathbb{C}),

¹This is true at least for the irreducible representations of finite and compact groups, for which we see below (§2.3) that two non irreducible representations are equivalent iff they have the same character.

such that the norm be positive definite: $x \neq 0 \Rightarrow \langle x|x \rangle > 0$. If the dimension of E is finite, one may find an orthonormal basis where the matrix of J reduces to I and then define unitary operators U such that $U^\dagger U = I$. If the space is infinite dimensional, (and is assumed to be a separable prehilbertian space²), one proves that one may find a countable orthonormal basis, thus labelled by a discrete index. A representation of G in E is called *unitary* if for any $g \in G$, the operator $D(g)$ is unitary. Then for any $g \in G$ and $x, y \in E$

$$\langle x|y \rangle = \langle D(g)x|D(g)y \rangle \tag{2-14}$$

$$\text{hence } D(g)^\dagger D(g) = I \tag{2-15}$$

and

$$D(g^{-1}) = D^{-1}(g) = D^\dagger(g) \ . \tag{2-16}$$

The following important properties hold:

(i) *Any unitary reducible representation is completely reducible.*

Proof: let E_1 be an invariant subspace, its complementary subspace $E_2 = (E_1)_\perp$ is invariant since for all $g \in G$, $x \in E_1$ et $y \in E_2$

$$\langle x|D(g)y \rangle = \langle D(g^{-1})x|y \rangle = 0 \tag{2-17}$$

which proves that $D(g)y \in E_2$.

(ii) *Any representation of a finite or compact group on a prehilbertian space is “unitarisable”, i.e. equivalent to a unitary representation.*

Proof: consider first a finite group and define

$$Q = \sum_{g' \in G} D^\dagger(g')D(g') \tag{2-18}$$

which satisfies

$$D^\dagger(g)QD(g) = \sum_{g' \in G} D^\dagger(g'.g)D(g'.g) = Q \tag{2-19}$$

where the “rearrangement” of $\sum_{g'}$ by $\sum_{g'.g}$ has been used (see § 1.2.4). The self-adjoint operator Q is positive definite (why?) and may thus be written

$$Q = V^\dagger V \tag{2-20}$$

with V invertible. (For example, by diagonalisation of the operator Q by a unitary operator, $Q = U\Lambda^2U^\dagger$, with Λ diagonal real, one may construct the “square root” $V = U\Lambda U^\dagger$.) The intertwiner V defines a representation D' equivalent to D and unitary:

$$\begin{aligned} D'(g) &= VD(g)V^{-1} \\ D'^\dagger(g)D'(g) &= V^{\dagger-1}D^\dagger(g)V^\dagger VD(g)V^{-1} \\ &= V^{\dagger-1}D^\dagger(g)QD(g)V^{-1} \stackrel{(2-19)}{=} V^{\dagger-1}QV^{-1} = I. \end{aligned} \tag{2-21}$$

²A space is *separable* if it contains a dense countable subset.

In the case of a continuous compact group, the existence of the invariant Haar measure (see § 1.2.4) allows us to repeat the same argument with $Q = \int d\mu(g')D^\dagger(g')D(g')$. \square

As a corollary of the two previous properties, any reducible representation of a finite or compact group on a prehilbertian space is equivalent to a unitary completely reducible representation. It thus suffices to construct and classify unitary irreducible representations. We show below that, for a finite or compact group, these irreducible representations are finite dimensional.

Counter-example for a non compact group: the matrices $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ form an indecomposable (=non completely reducible) representation of the group \mathbb{R} .

2.1.4 Schur lemma

Consider two irreducible representations D in E and D' in E' and an intertwiner between them, as defined in (2-5). We then have the important

Schur lemma: *either $V = 0$, or V is a bijection and the representations are equivalent.*

Proof: Suppose $V \neq 0$. Then $VD(g) = D'(g)V$ implies that $\ker V$ is a subspace of E invariant under D ; by the assumption of irreducibility, it reduces to 0 (it cannot be equal to the whole E otherwise V would vanish). Likewise, the image of V is a subspace of E' invariant under D' , it cannot be $\{0\}$ and thus equals E' . A classical theorem on linear operators between vector spaces then asserts that V is a bijection from E to E' and the representations are thus equivalent. *q.e.d.*

Note that if the two representations are not irreducible, this result is generally false. A counter-example is given by the representation (2-10) which commutes with matrices $V = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$.

Corollary 1. *Any intertwining operator of an irreducible representation on \mathbb{C} with itself, i.e. any operator that commutes with all the representatives of the group, is a multiple of the identity.*

Proof: on \mathbb{C} , V has at least one eigenvalue λ ; $\lambda \neq 0$ since V is invertible by Schur lemma. The operator $V - \lambda I$ is itself an intertwining operator, but it is singular and thus vanishes.

Corollaire 2. *An irreducible representation on \mathbb{C} of an abelian group is necessarily of dimension 1.*

Proof: take $g' \in G$, $D(g')$ commutes with all $D(g)$ since G is abelian. Thus (corollary 1) $D(g') = \lambda(g')I$. The representation decomposes into $\dim D$ copies of the representation of dimension 1 : $g \mapsto \lambda(g)$, and irreducibility imposes that $\dim D = 1$.

Let us insist on the importance of the property of the complex field \mathbb{C} to be algebraically closed, in contrast with \mathbb{R} , in these two corollaries. The representation on \mathbb{R} of the group $SO(2)$ by matrices $\mathcal{D}(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ provides counterexamples to both propositions: any matrix $\mathcal{D}(\alpha)$ commutes with $\mathcal{D}(\theta)$ but has no real eigenvalue (for $\theta \neq 0, \pi$) and the representation is irreducible on \mathbb{R} , although of dimension 2.

Application of Corollary 1: in the Lie algebra of a Lie group, the Casimir operators defined in Chap. 1 commute with all infinitesimal generators and thus with all the group elements. Anticipating a little bit on a forthcoming discussion of representations of a Lie algebra, in a unitary representation these Casimir operators may be chosen hermitian hence diagonalisable, which allows one to apply the argument of Corollary 1: in an irreducible representation, they are multiples of the identity. Thus for $SU(2)$, $\mathbf{J}^2 = j(j+1)I$ in the spin j representation.

2.1.5 Tensor product of representations. Clebsch-Gordan decomposition

Tensor product of representations

A very commonly used method to construct irreducible representations of a given group consists in building the tensor product of known representations and decomposing it into irreducible representations. This is the situation encountered in Quantum Mechanics, when the transformation properties of the components of a system are known and one wants to know how the system transforms as a whole (a system of two particles of spins j_1 and j_2 for example).

Let E_1 et E_2 be two vector spaces carrying representations D_1 and D_2 of a group G . The tensor product³ $E = E_1 \otimes E_2$ is the space generated by linear combinations of (tensor) “products” of a vector of E_1 and a vector of E_2 : $z = \sum_i x^{(i)} \otimes y^{(i)}$. The space E carries also a representation, denoted $D = D_1 \otimes D_2$, the *tensor product* (one says also *direct product*) of representations D_1 and D_2 . (See Chap. 0 for the example of the group $SU(2)$). On the vector z above

$$D(g)z = \sum_i D_1(g)x^{(i)} \otimes D_2(g)y^{(i)}. \quad (2-22)$$

One readily checks that the character of representation D is the product of characters χ_1 et χ_2 de D_1 et D_2

$$\chi(g) = \chi_1(g)\chi_2(g) \quad (2-23)$$

In particular, evaluating this relation for $g = e$, one has for finite dimensional representations

$$\dim D = \dim(E_1 \otimes E_2) = \dim E_1 \cdot \dim E_2 = \dim D_1 \cdot \dim D_2 \quad (2-24)$$

as is well known for a tensor product.

Clebsch-Gordan decomposition

The tensor product representation of two irreducible representations D et D' is in general not irreducible. If it is fully reducible (as is the case for the unitary representations that are our chief concern), one performs the *Clebsch-Gordan decomposition* into irreducible representations

$$D \otimes D' = \oplus_j D_j \quad (2-25)$$

where in the right hand side, certain irreducible representations D_1, \dots appear. The notation \oplus_j encompasses very different situations: summation over a finite set (for finite groups), on a

³The reader will find in Appendix D a short summary on tensor products and tensors.

finite subset of an a priori infinite but discrete set (compact groups) or on possibly continuous variables (non compact groups).

If G is finite or compact and if its inequivalent irreducible representations are classified and labelled: $D^{(\rho)}$, one may rather rewrite (2-25) in a way showing which of these inequivalent representations appear, *and with which multiplicity*

$$D \otimes D' = \bigoplus_{\rho} m_{\rho} D^{(\rho)} . \quad (2-26)$$

A more correct expression would be $E \otimes E' = \bigoplus_{\rho} F_{\rho} \otimes E^{(\rho)}$ where F_{ρ} is a vector space of dimension m_{ρ} , the “multiplicity space”.

The integers m_{ρ} are all non negative. The equations (2-25) and (2-26) imply simple rules on characters and dimensions

$$\chi_D \cdot \chi_{D'} = \sum_j \chi_j = \sum_{\rho} m_{\rho} \chi^{(\rho)} \quad (2-27)$$

$$\dim D \cdot \dim D' = \sum_j \dim D_j = \sum_{\rho} m_{\rho} \dim D^{(\rho)} . \quad (2-28)$$

Example: the tensor product of two copies of the euclidean space \mathbb{R}^3 does not form an irreducible representation of the rotation group $\text{SO}(3)$. This space is generated by tensor products of vectors \vec{x} and \vec{y} and one may construct the scalar product $\vec{x} \cdot \vec{y}$ which is invariant under the group (trivial representation), a skew-symmetric rank 2 tensor

$$A_{ij} = x_i y_j - x_j y_i$$

which transforms as a dimension 3 irreducible representation (spin 1),⁴ and a symmetric traceless tensor

$$S_{ij} = x_i y_j + x_j y_i - \frac{2}{3} \delta_{ij} \vec{x} \cdot \vec{y}$$

which transforms as an irreducible representation of dimension 5 (spin 2); thus we may always decompose

$$x_i y_j = \frac{1}{3} \delta_{ij} \vec{x} \cdot \vec{y} + \frac{1}{2} A_{ij} + \frac{1}{2} S_{ij} ; \quad (2-29)$$

the total dimension is of course $9 = 3 \times 3 = 1 + 3 + 5$, and labelling in that simple case the representations by their dimension, we write

$$D^{(3)} \otimes D^{(3)} = D^{(1)} \oplus D^{(3)} \oplus D^{(5)} . \quad (2-30)$$

Or equivalently, in a “spin” notation

$$(1) \otimes (1) = (0) \oplus (1) \oplus (2)$$

in which one recognizes the familiar rules of “addition of angular momentum” (see Chap. 0)

$$(j) \otimes (j') = \bigoplus_{j''=|j-j'|}^{j+j'} (j'') . \quad (2-31)$$

⁴(such a tensor is “dual” to a vector: $A_{ij} = \epsilon_{ijk} z_k$, $z = x \times y$)

By iteration, one finds

$$D^{(3)} \otimes D^{(3)} \otimes D^{(3)} = D^{(1)} \oplus 3D^{(3)} \oplus 2D^{(5)} \oplus D^{(7)}, \quad (2-32)$$

with now multiplicities.

Invariants. A frequently encountered problem consists in counting the number of linearly independent invariants (under the action of a group G) in the tensor product of certain prescribed representations. This is an information contained in the decompositions into irreducible representations like (2-26, 2-30, 2-32), where the multiplicity of the identity representation provides this number of invariants in the product of the considered representations. Exercise : interpret in terms of classical geometric invariants the multiplicities $m_1 = 1, 1, 3$ of the identity representation that appear in tensor products $(1) \otimes (1)$, $(1) \otimes (1) \otimes (1)$, $(1) \otimes (1) \otimes (1) \otimes (1)$ of $SO(3)$. See also Problem II at the end of this chapter.

Clebsch-Gordan coefficients

Formula (2-25) describes how the representation matrices decompose into irreducible representations under a group transformation. It is also often important to know how vectors of the representations at hand decompose. Let $e_\alpha^{(\rho)}$, $\alpha = 1, \dots, \dim D^{(\rho)}$, be a basis of vectors of representation ρ . One wants to expand the product of two such basis vectors, that is $e_\alpha^{(\rho)} \otimes e_\beta^{(\sigma)}$, on some $e_\gamma^{(\tau)}$. As representation τ may appear m_τ times, one must introduce an extra index, $i = 1, \dots, m_\tau$. One writes

$$e_\alpha^{(\rho)} \otimes e_\beta^{(\sigma)} = \sum_{\tau, \gamma, i} C_{\rho, \alpha; \sigma, \beta | \tau_i, \gamma} e_\gamma^{(\tau_i)}. \quad (2-33)$$

or with notations borrowed from Quantum Mechanics

$$|\rho, \alpha; \sigma, \beta\rangle \equiv |\rho\alpha\rangle|\sigma\beta\rangle = \sum_{\tau, \gamma, i} \langle \tau_i \gamma | \rho, \alpha; \sigma, \beta \rangle |\tau_i \gamma\rangle. \quad (2-34)$$

The coefficients $C_{\rho, \alpha; \sigma, \beta | \tau_i, \gamma} = \langle \tau_i \gamma | \rho, \alpha; \sigma, \beta \rangle$ are the *Clebsch-Gordan coefficients*. In contrast with the multiplicities m_ρ in (2-26), they have no reason of being integers, as we saw in Chap. 0 on the case of the rotation group, nor even real in general. Suppose that we consider unitary representations and that the bases have been chosen orthonormal. Then C.-G. coefficients which represent a change of orthonormal basis in the space $E_1 \otimes E_2$ satisfy orthonormality and completeness conditions

$$\sum_{\tau, \gamma, i} \langle \tau_i \gamma | \rho, \alpha; \sigma, \beta \rangle \langle \tau_i \gamma | \rho, \alpha'; \sigma, \beta' \rangle^* = \delta_{\alpha, \alpha'} \delta_{\beta, \beta'} \quad (2-35)$$

$$\sum_{\alpha, \beta} \langle \tau_i \gamma | \rho, \alpha; \sigma, \beta \rangle \langle \tau'_j \gamma' | \rho, \alpha; \sigma, \beta \rangle^* = \delta_{\tau, \tau'} \delta_{\gamma, \gamma'} \delta_{i, j}. \quad (2-36)$$

This enables us to invert relation (2-34) into

$$|\tau_i \gamma\rangle = \sum_{\alpha, \beta} \langle \tau_i \gamma | \rho, \alpha; \sigma, \beta \rangle^* |\rho, \alpha; \sigma, \beta\rangle \quad (2-37)$$

and justifies the notation

$$\langle \rho, \alpha; \sigma, \beta | \tau_i \gamma \rangle = \langle \tau_i \gamma | \rho, \alpha; \sigma, \beta \rangle^* \quad (2-38)$$

$$| \tau_i \gamma \rangle = \sum_{\alpha, \beta} \langle \rho, \alpha; \sigma, \beta | \tau_i \gamma \rangle | \rho, \alpha; \sigma, \beta \rangle . \quad (2-39)$$

Finally, applying a group transformation on both sides of (2-34) and using these relations, one decomposes the product of matrices $\mathcal{D}^{(\rho)}$ and $\mathcal{D}^{(\sigma)}$ in a quite explicit way

$$\mathcal{D}_{\alpha\alpha'}^{(\rho)} \mathcal{D}_{\beta\beta'}^{(\sigma)} = \sum_{\tau, \gamma, \gamma', i} \langle \tau_i \gamma | \rho, \alpha; \sigma, \beta \rangle^* \langle \tau_i \gamma' | \rho, \alpha'; \sigma, \beta' \rangle \mathcal{D}_{\gamma\gamma'}^{(\tau_i)} . \quad (2-40)$$

We shall see below (§ 2.4.4) an application of these formulae to Wigner-Eckart theorem.

2.1.6 Decomposition into irreducible representations of a subgroup of a group representation

Let H be a subgroup of a group G , then any representation D of G may be restricted to H and yields a representation D' of the latter

$$\forall h \in H \quad D'(h) = D(h) . \quad (2-41)$$

This is a very common method to build representations of H , once those of G are known. In general, if D is irreducible, D' is not, and once again the question arises of its decomposition into irreducible representations. For example, given a finite subgroup of $SU(2)$, one wants to set up the (finite, as we see below) list of its irreducible representations, starting from those of $SU(2)$.

Another instance frequently encountered in physics: a symmetry group G is “broken” into a subgroup H ; how do the representations of G decompose into representations of H ? Examples: in solid state physics, the “point group” $G \subset SO(3)$ of symmetry (of rotations and reflexions) of a crystal is broken down to H by an external field; in particle physics, we shall encounter in Chap. 4 and 5 the instances of $SU(2) \subset SU(3)$; $U(1) \times SU(2) \times SU(3) \subset SU(5)$, etc.

2.2 Representations of Lie algebras

2.2.1 Definition. Universality

The notion of representation also applies to Lie algebras.

A representation of a Lie algebra \mathfrak{g} in a vector space E is by definition an homomorphism of \mathfrak{g} into the Lie algebra of linear operators on the space E , *i.e.* a map $X \in \mathfrak{g} \mapsto d(X) \in \text{End } E$ which respects linearity and Lie bracket $X, Y \in \mathfrak{g}$, $[X, Y] \mapsto d([X, Y]) = [d(X), d(Y)] \in \text{End } E$. A corollary of this definition is that in any representation of the algebra, the (representatives of) generators satisfy the same commutation relations. In other words, in appropriate bases,

the structure constants are the same in all representations. More precisely, if t_i is a basis of \mathfrak{g} , with $[t_i, t_j] = C_{ij}^k t_k$, and if $T_i = d(t_i)$ is its image by the representation d

$$[T_i, T_j] = [d(t_i), d(t_j)] = d([t_i, t_j]) = C_{ij}^k d(t_k) = C_{ij}^k T_k .$$

Thus calculations carried in some particular representation and *involving only commutation rules of the Lie algebra* remain valid in any representation. We have seen in Chap. 0, § 0.2.2, an illustration of this *universality property*. In contrast, Casimir operators take different values in different irreducible representations.

In parallel with the definitions of sect. 2.1.1, one defines the notions of faithful representation of a Lie algebra (its kernel $\ker d = \{X | d(X) = 0\}$ reduces to the element 0 of \mathfrak{g}), of reducible or irreducible representation (existence or not of an invariant subspace), etc.

2.2.2 Representations of a Lie group and of its Lie algebra

Any differentiable representation D of G into a space E gives a map d of the Lie algebra \mathfrak{g} into the algebra of operators on E . It is obtained by taking the infinitesimal form of $D(g)$, with $g(t) = I + tX$ (or $g = e^{tX}$)

$$d(X) := \left. \frac{d}{dt} \right|_{t=0} D(g(t)) , \tag{2-42}$$

or, for t infinitesimal,

$$D(e^{tX}) = e^{td(X)} . \tag{2-43}$$

Let us show that this map is indeed compatible with the Lie bracket, thus giving a representation of the Lie algebra. For this purpose, we repeat the discussion of chap. 1, § 1.3.4, to build the commutator in a natural way. Let $g(t) = e^{tX}$ and $h(u) = e^{uY}$ be two one-parameter subgroups, for t and u infinitesimally small and of same order. We have $e^{tX} e^{uY} e^{-tX} e^{-uY} = e^Z$ with $Z = ut[X, Y] + \dots$, whence

$$\begin{aligned} e^{d(Z)} = D(e^Z) &= D(e^{tX} e^{uY} e^{-tX} e^{-uY}) = D(e^{tX}) D(e^{uY}) D(e^{-tX}) D(e^{-uY}) \\ &= e^{td(X)} e^{ud(Y)} e^{-td(X)} e^{-ud(Y)} \\ &= e^{ut[d(X), d(Y)] + \dots} , \end{aligned} \tag{2-44}$$

and by identification of the leading terms, $d([X, Y]) = [d(X), d(Y)]$, qed.

◦ This connection between a representation of G and a representation of \mathfrak{g} applies in particular to a representation of G which plays a special role, the *adjoint representation* of G into its Lie algebra \mathfrak{g} . It is defined by the following action

$$X \in \mathfrak{g} \quad D^{\text{adj}}(g)(X) = gXg^{-1} , \tag{2-45}$$

which we denote $\text{Ad } g X$. (The right hand side of (2-45) must be understood either as resulting from the derivative at $t = 0$ of $g e^{tX} g^{-1}$, or, following the standpoint of these notes, as a matrix multiplication, since then the matrices g and X act in the same space.)

The adjoint representation of G gives rise to a representation of \mathfrak{g} in the space \mathfrak{g} , also called adjoint representation. It is obtained by taking the infinitesimal form of (2-45), formally

$g = I + tY$, or by considering the one-parameter subgroup generated by $Y \in \mathfrak{g}$, $g(t) = \exp tY$ and by calculating $\text{Ad } g(t)X = g(t)Xg^{-1}(t) = X + t[Y, X] + O(t^2)$ (cf. Chap. 1 (1.29)), whence

$$\left. \frac{d}{dt} \text{Ad } g(t)X \right|_{t=0} = [Y, X] = \text{ad } Y X . \quad (2-46)$$

where we recover (and justify) our notation ad of Chap. 1.

Exercise: show that matrices T_i defined by $(T_i)_j^k = -C_{ij}^k$ satisfy commutation relations of the Lie algebra as a consequence of the Jacobi identity, and thus form a basis of generators in the adjoint representation.

Remark. To a unitary representation of G corresponds a representation of \mathfrak{g} by antihermitian operators (or matrices). Physicists, who love Hermitian operators, usually include an “ i ” in front of the infinitesimal generators: for example $e^{-i\psi J}$, $[J_a, J_b] = i\epsilon_{abc}J_c$, etc.

◦ Conversely, a representation of a Lie algebra \mathfrak{g} generates a representation of the unique connected and simply connected group whose Lie algebra is \mathfrak{g} . In other words if $X \mapsto d(X)$ is a representation of the algebra, $e^X \mapsto e^{d(X)}$ is a representation of the group. Indeed, the BCH formula being “universal”, *i.e.* involving only linear combinations of brackets in the Lie algebra, and being thus insensitive to the representation of \mathfrak{g} , we have:

$$e^X e^Y = e^Z \mapsto e^{d(X)} e^{d(Y)} = e^{d(Z)} ,$$

showing that the homomorphism of Lie algebras integrates into a group homomorphism in the neighbourhood of the identity. One finally proves that such a local homomorphism of a connected and simply connected group G into a group G' (here, the linear group $\text{GL}(E)$) extends in a unique way into an infinitely differentiable homomorphism of the whole G into G' . To summarize, in order to find the (possibly unitary) representations of the group G it is sufficient to find the representations by (possibly antihermitian) operators of its Lie algebra.

This fundamental principle has already been illustrated in Chap. 0 on the concrete cases of $\text{SU}(2)$ and $\text{SL}(2, \mathbb{C})$.

2.3 Representations of compact Lie groups

In this section, we study the representations of compact groups on the field \mathbb{C} of complex numbers. Most of the results rely on the fact that one may integrate over the group with the Haar measure $d\mu(g)$. Occasionally, we will compare with the non compact case. It is thus useful to have in mind two archetypical cases: the compact group $\text{U}(1) = \{e^{ix}\}$ with $x \in \mathbb{R}/2\pi\mathbb{Z}$ (an angle modulo 2π), and the non compact group \mathbb{R} , the additive group of real numbers. The case of finite groups, very close to that of compact groups, will be briefly mentioned at the end.

2.3.1 Orthogonality and completeness

Let G be a compact group. We shall admit that its inequivalent irreducible representations are labelled by a discrete index, written in upper position: $D^{(\rho)}$. These representations are a

priori of finite or infinite dimension –in fact we shall see below that the dimension n_ρ of $D^{(\rho)}$ is in fact finite. In a finite or countable basis, the matrices $\mathcal{D}_{\alpha\beta}^{(\rho)}$ may be assumed to be unitary, according to the result of § 2.1.3. (In contrast, a generic representation of a non compact compact depends on a continuous parameter. And we shall see that its unitary representations are necessarily of infinite dimension.)

In our two cases of reference, the irreducible representations of $U(1)$ (hence of dimension 1 for this abelian group) are such that $D^{(k)}(x)D^{(k)}(x') = D^{(k)}(x+x')$, they are of the form $D^{(k)}(x) = e^{ikx}$ with $k \in \mathbb{Z}$, the latter condition to make the representation single valued when one changes the determination $x \rightarrow x + 2\pi n$. For $G = \mathbb{R}$, one may also take $x \mapsto e^{ikx}$, but nothing restricts $k \in \mathbb{C}$, except unitarity which imposes $k \in \mathbb{R}$.

Theorem: For a compact group, the matrices $\mathcal{D}_{\alpha\beta}^{(\rho)}$ satisfy the following orthogonality properties

$$\boxed{\int \frac{d\mu(g)}{v(G)} \mathcal{D}_{\alpha\beta}^{(\rho)}(g) \mathcal{D}_{\alpha'\beta'}^{(\rho')*}(g) = \frac{1}{n_\rho} \delta_{\rho\rho'} \delta_{\alpha\alpha'} \delta_{\beta\beta'}} \quad (2-47)$$

and their characters satisfy thus

$$\boxed{\int \frac{d\mu(g)}{v(G)} \chi^{(\rho)}(g) \chi^{(\rho')*}(g) = \delta_{\rho\rho'}} \quad (2-48)$$

In these formulae, $d\mu(g)$ denotes the Haar measure and $v(G) = \int d\mu(g)$ is the “volume of the group”.

Proof: Take M an arbitrary matrix of dimension $n_\rho \times n_{\rho'}$ and consider the matrix

$$V = \int d\mu(g') \mathcal{D}^{(\rho)}(g') M \mathcal{D}^{(\rho')\dagger}(g') \quad (2-49)$$

The left hand side of (2-47) is (up to a factor $v(G)$) the derivative with respect to $M_{\beta\beta'}$ of $V_{\alpha\alpha'}$. The representations being unitary, $\mathcal{D}^\dagger(g) = \mathcal{D}(g^{-1})$, it is easy, using the left invariance of the measure $d\mu(g') = d\mu(gg')$, to check that V satisfies

$$V \overline{\mathcal{D}^{(\rho')}}(g) = \mathcal{D}^{(\rho)}(g) V \quad (2-50)$$

for all $g \in G$. By Schur lemma, the matrix V is thus vanishing if representations ρ and ρ' are different, and a multiple of the identity if $\rho = \rho'$.

a) In the former case, choosing a matrix M whose only non vanishing element is $M_{\beta\beta'} = 1$ and identifying the matrix element $V_{\alpha\alpha'}$, one finds the orthogonality condition (2-47) with $\delta_{\rho\rho'} = 0$.

b) If $\rho = \rho'$, choose first $M_{11} = 1$, the other $M_{\beta\beta'}$ vanishing. One has $V = c_1 I$, where the coefficient c_1 is determined by taking the trace: $c_1 n_\rho = v(G) \mathcal{D}_{11}(I) = v(G)$, which proves that the dimension n_ρ is finite.

c) Repeating the argument with an arbitrary matrix M , one finds again $V = c_M I$ and one computes c_M by taking the trace: $c_M n_\rho = v(G) \text{tr} M$, which, upon differentiation wrt $M_{\beta\beta'}$, leads to the orthonormality (2-47), qed.

The proposition (2-48) then follows simply from the previous one by taking the trace on $\alpha = \beta$ and $\alpha' = \beta'$.

Let us stress two important consequences of that discussion:

- we just saw that any irreducible (and unitary) representation of a compact group is of finite dimension;
- the relation (2-48) implies that two irreducible representations $D^{(\rho)}$ and $D^{(\sigma)}$ are equivalent (in fact identical, according to our labelling convention) iff their characters are equal: $\chi^{(\rho)} = \chi^{(\sigma)} \iff \rho = \sigma$.

Case of a non compact group

A large part of the previous calculation still applies to a non compact group, provided it has a left invariant measure (which holds true for a wide class of groups, cf Chap. 1, end of § 1.2.4) and if the representation is in a prehilbertian separable space, hence with a discrete basis, and is square integrable: $\mathcal{D}_{\alpha\beta} \in L^2(G)$. Choosing M as in b), one finds again $\int d\mu(g) = c_1 \text{tr } I$. In the lhs, the integral over the group ("volume of the group" G) diverges. In the rhs, $\text{tr } I$, the dimension of the representation, is thus infinite.

More generally, one may assert

Any unitary square integrable representation of a non compact group is of infinite dimension.

Of course, the trivial representation $g \mapsto 1$ (which is not in $L^2(G)$) evades the argument.

Let us test these results on the two cases $U(1)$ and \mathbb{R} . For the unitary representation e^{ikx} of $U(1)$, the relation (2-47) (or (2-48), which makes no difference for these representations of dimension 1) expresses that

$$\int_0^{2\pi} \frac{dx}{2\pi} e^{ikx} e^{-ik'x} = \delta_{kk'} ,$$

as is well known. On the other hand on \mathbb{R} it would lead to

$$\int_{-\infty}^{\infty} dx e^{ikx} e^{-ik'x} = 2\pi\delta(k - k')$$

with a Dirac function. Of course this expression is meaningless for $k = k'$, the representation is not square integrable.

Completeness.

We return to compact groups. One may prove that the matrices $\mathcal{D}_{\alpha\beta}^{(\rho)}(g)$ also satisfy a completeness property

$$\boxed{\sum_{\rho, \alpha, \beta} n_{\rho} \mathcal{D}_{\alpha\beta}^{(\rho)}(g) \mathcal{D}_{\alpha\beta}^{(\rho)*}(g') = v(G) \delta(g, g') ,} \quad (2-51)$$

or stated differently

$$\sum_{\rho, \alpha, \beta} n_{\rho} \mathcal{D}_{\alpha\beta}^{(\rho)}(g) \mathcal{D}_{\beta\alpha}^{(\rho)\dagger}(g') = \sum_{\rho} n_{\rho} \chi^{(\rho)}(g.g'^{-1}) = v(G) \delta(g, g') , \quad (2 - 51)'$$

where $\delta(g, g')$ is the Dirac distribution adapted to the Haar measure, *i.e.* such that $\int d\mu(g') f(g') \delta(g, g') = f(g)$ for any sufficiently regular function f on G .

This completeness property is important: it tells us that any \mathbb{C} -valued function on the group, continuous or square integrable, may be expanded on the functions $\mathcal{D}_{\alpha\beta}^{(\rho)}(g)$

$$f(g) = \int d\mu(g') \delta(g, g') f(g') = \sum_{\rho, \alpha, \beta} n_{\rho} \mathcal{D}_{\alpha\beta}^{(\rho)}(g) \int \frac{d\mu(g')}{v(G)} \mathcal{D}_{\beta\alpha}^{(\rho)\dagger}(g') f(g') =: \sum_{\rho, \alpha, \beta} n_{\rho} \mathcal{D}_{\alpha\beta}^{(\rho)}(g) f_{\alpha\beta}^{(\rho)}. \quad (2-52)$$

This is the *Peter–Weyl theorem*, a non trivial statement that we admit⁵. A corollary then asserts that characters $\chi^{(\rho)}$ of a compact group form a complete system of *class functions*, *i.e.* invariant under $g \rightarrow hgh^{-1}$. In other words, any continuous (or L^2) class function can be expanded of irreducible characters.

Let us prove the latter assertion. If f is a continuous class function, $f(g) = f(hgh^{-1})$, we apply the Peter-Weyl theorem and examine the integral appearing in (2-52):

$$\begin{aligned} f_{\alpha\beta}^{(\rho)} &= \int \frac{d\mu(g')}{v(G)} f(g') \mathcal{D}_{\beta\alpha}^{(\rho)\dagger}(g') = \int \frac{d\mu(g')}{v(G)} f(hg'h^{-1}) \mathcal{D}_{\beta\alpha}^{(\rho)\dagger}(hg'h^{-1}) \quad \forall h \\ &= \int \frac{d\mu(h)}{v(G)} \frac{d\mu(g')}{v(G)} f(g') \mathcal{D}_{\beta\gamma}^{(\rho)\dagger}(h) \mathcal{D}_{\gamma\delta}^{(\rho)\dagger}(g') \mathcal{D}_{\delta\alpha}^{(\rho)\dagger}(h^{-1}) \\ &= \int \frac{d\mu(g')}{v(G)} f(g') \mathcal{D}_{\gamma\delta}^{(\rho)\dagger}(g') \frac{1}{n_{\rho}} \delta_{\alpha\beta} \delta_{\gamma\delta} \quad \text{by (2-47)} \\ &= \frac{1}{n_{\rho}} \int \frac{d\mu(g')}{v(G)} f(g') \chi^{(\rho)*}(g') \delta_{\alpha\beta} \end{aligned} \quad (2-53)$$

from which it follows that (2-52) reduces to an expansion on characters, *qed*.

Let us test these completeness relations again in the case of $U(1)$. They express that

$$\sum_{k=-\infty}^{\infty} e^{ikx} e^{-ikx'} = 2\pi \delta_P(x - x') \quad (2-54)$$

where $\delta_P(x - x') = \sum_{\ell=-\infty}^{\infty} \delta(x - x' - 2\pi\ell)$ is the periodic Dirac distribution (alias “Dirac’s comb”). Then (2-52) means that any 2π -periodic function (with adequate regularity conditions) may be represented by its Fourier *series*

$$f(x) = \sum_{k=-\infty}^{\infty} e^{ikx} f_k \quad f_k = \int_{-\pi}^{\pi} \frac{dx}{2\pi} f(x) e^{-ikx}. \quad (2-55)$$

For the non compact group \mathbb{R} , the completeness relation (which is still true in that case) amounts to a Fourier *transformation*

$$f(x) = \int_{-\infty}^{\infty} dk \tilde{f}(k) e^{ikx} \quad \tilde{f}(k) = \int_{-\infty}^{\infty} \frac{dx}{2\pi} f(x) e^{-ikx}. \quad (2-56)$$

The Peter–Weyl theorem for an arbitrary group is thus a generalization of Fourier decompositions.

The $SO(2)$ rotation group in the plane is isomorphic to the $U(1)$ group. If we look at real representations, their dimension is no longer equal to 1 but to 2 (except the trivial representation)

$$\mathcal{D}^{(k)}(\alpha) = \begin{pmatrix} \cos k\alpha & -\sin k\alpha \\ \sin k\alpha & \cos k\alpha \end{pmatrix}, \quad k \in \mathbb{N}^* \quad , \quad \chi^{(k)}(\alpha) = 2 \cos k\alpha \quad (2-57)$$

What are now the orthogonality and completeness relations?

⁵For a proof, see for example, T. Bröcker and T. tom Dieck, see bibliography at the end of this chapter

2.3.2 Consequences

For a compact group,

(i) any representation being completely reducible, its character reads

$$\chi = \sum_{\rho} m_{\rho} \chi^{(\rho)} \quad (2-58)$$

and multiplicities may be computed by

$$m_{\rho} = \int \frac{d\mu(g)}{v(G)} \chi(g) \chi^{(\rho)*}(g) . \quad (2-59)$$

One also has $\|\chi\|^2 := \int \frac{d\mu(g)}{v(G)} |\chi(g)|^2 = \sum_{\rho} m_{\rho}^2$, an integer greater or equal to 1. Thus a representation is irreducible iff its character satisfies the condition $\int \frac{d\mu(g)}{v(G)} |\chi(g)|^2 = 1$. And the computation of $\|\chi\|^2$ gives indications on the number of irreducible representations appearing in the decomposition of the representation of character χ , a very useful information in the contexts mentioned in § 2.1.5 and 2.1.6.

More generally, any class function may be expanded on irreducible characters (Peter-Weyl).

(ii) In a similar way one may determine multiplicities in the Clebsch-Gordan decomposition of a direct product of two representations by projecting the product of their characters on irreducible characters. Let us illustrate this on the product of two irreducible representations ρ et σ

$$D^{(\rho)} \otimes D^{(\sigma)} = \oplus_{\tau} m_{\tau} D^{(\tau)} \quad (2-60)$$

$$\chi^{(\rho)} \chi^{(\sigma)} = \sum_{\tau} m_{\tau} \chi^{(\tau)} \quad (2-61)$$

$$m_{\tau} = \int \frac{d\mu(g)}{v(G)} \chi^{(\rho)}(g) \chi^{(\sigma)}(g) \chi^{(\tau)*}(g) , \quad (2-62)$$

and hence the representation τ appears in the product $\rho \otimes \sigma$ with the same multiplicity as σ^* in $\rho \otimes \tau^*$.

Case of $SU(2)$

It is a good exercise to understand how the different properties discussed in this section are realized by representation matrices of $SU(2)$. This will be discussed in detail in TD and in App. E.

2.3.3 Case of finite groups

We discuss only briefly the case of finite groups. Theorems (2-47, 2-48, 2-51) and their consequences (2-58, 2-59, 2-60), which are based on the existence of an invariant measure, remain of course valid. It suffices to replace in these theorems the group volume $v(G)$ by the order $|G|$

(=number of elements) of G , and $\int d\mu(g)$ by $\sum_{g \in G}$:

$$\frac{1}{|G|} \sum_{g \in G} \mathcal{D}_{\alpha\beta}^{(\rho)}(g) \mathcal{D}_{\alpha'\beta'}^{(\rho')*}(g) = \frac{1}{n_\rho} \delta_{\rho\rho'} \delta_{\alpha\alpha'} \delta_{\beta\beta'} \quad (2-63)$$

$$\sum_{\rho, \alpha, \beta} \frac{n_\rho}{|G|} \mathcal{D}_{\alpha\beta}^{(\rho)}(g) \mathcal{D}_{\alpha\beta}^{(\rho)*}(g') = \delta_{g, g'} . \quad (2-64)$$

But representations of finite groups enjoy additional properties. Let us show that the dimensions of inequivalent irreducible representations verify

$$\sum_{\rho} n_\rho^2 = |G| . \quad (2-65)$$

This follows from the fact that the system of equations (2-63) expresses that the matrix $\mathcal{U}_{\rho, \alpha\beta; g} := \left(\frac{n_\rho}{|G|}\right)^{\frac{1}{2}} \mathcal{D}_{\alpha\beta}^{(\rho)}(g)$ of dimensions $\sum_{\rho} n_\rho^2 \times |G|$ satisfies $\mathcal{U}\mathcal{U}^\dagger = I$, $\mathcal{U}^\dagger\mathcal{U} = I$, which is possible only if it is a square matrix, qed.

Moreover

Proposition. *The number r of inequivalent irreducible representations is finite and is equal to the number m of classes \mathcal{C}_i in the group.*

Proof: Denoting $\chi_j^{(\rho)}$ the value of character $\chi^{(\rho)}$ in class \mathcal{C}_i , one may rewrite the orthogonality and completeness properties of characters as

$$\frac{1}{|G|} \sum_{i=1}^m |\mathcal{C}_i| \chi_i^{(\rho)} \chi_i^{(\rho')*} = \delta_{\rho\rho'} \quad (2-66a)$$

$$\frac{|\mathcal{C}_i|}{|G|} \sum_{\rho=1}^r \chi_i^{(\rho)} \chi_j^{(\rho)*} = \delta_{ij} . \quad (2-66b)$$

(Exercise : derive the second relation from (2-52) and (2-53), applied to a finite group.)

But once again, these relations mean that the matrix $\mathcal{K}_{\rho i} := \left(\frac{|\mathcal{C}_i|}{|G|}\right)^{\frac{1}{2}} \chi_i^{(\rho)}$ of dimensions $r \times m$ satisfies $\mathcal{K}\mathcal{K}^\dagger = I$, $\mathcal{K}^\dagger\mathcal{K} = I$, thus is a square (and unitary) matrix, $m = r$, qed.

The *character table* of a finite group is the square table made of the (real or complex) numbers $\chi_i^{(\rho)}$, $\rho = 1, \dots, r$, $i = 1, \dots, m = r$. Its rows and columns satisfy the orthogonality properties (2-66).

We illustrate it on the example of the group T , subgroup of the rotation group $\text{SO}(3)$ leaving invariant a regular tetrahedron. This group of order 12 has 4 conjugacy classes \mathcal{C}_i , that of the identity, that of the 3 rotations of π around an axis joining the middles of opposite edges, that of the 4 rotations of $2\pi/3$ around an axis passing through a vertex, and that of the 4 rotations of $-2\pi/3$, see Fig. 2.1 .

This group has 4 irreducible representations, and one easily checks using (2-65) that their dimensions can only be $n_\rho = 1, 1, 1$ and 3. The character table is thus a 4×4 table, of which one row is already known, that of the identity representation D_1 , and one column, that of dimensions n_ρ . The spin 1 representation of $\text{SO}(3)$ yields a dimension 3 representation of T whose character χ takes the values $\chi_i = 1 + 2 \cos \theta_i = (3, -1, 0, 0)$ in the four classes ; according to the criterion of §2.3.2, $\|\chi\|^2 = \sum_i \frac{|\mathcal{C}_i|}{|G|} |\chi_i|^2 = 1$ and this character is irreducible. This gives a second row (called D_4). The spin 2 representation of $\text{SO}(3)$ gives a representation of dimension 5 which is reducible (same criterion) into a sum of 3 irreps, and is orthogonal to D_1 . This is the sum of rows D_2 , D_3 et D_4 , in which $j = e^{2\pi i/3}$, with $j + j^2 = -1$.

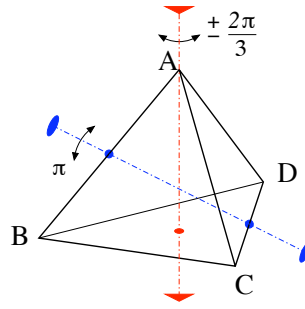


Figure 2.1: A tetrahedron, with two axes of rotation

↓ irreps. ρ \ Classes C_i →	$C(0)$	$C(\pi)$	$C(\frac{2\pi}{3})$	$C(-\frac{2\pi}{3})$
D_1	1	1	1	1
D_2	1	1	j	j^2
D_3	1	1	j^2	j
D_4	3	-1	0	0
$ C_i $	1	3	4	4

Check that relations (2-66) are satisfied. Explain why the group T is nothing else than the alternate group A_4 of even permutations of 4 objects.

2.3.4 Recap

For a *compact* group, any irreducible representation is of finite dimension and equivalent to unitary representation. Its matrix elements and characters satisfy orthogonality and completeness relations. The set of irreducible representations is *discrete*.

For a *finite* group, (a case very superficially treated in this course), the same orthogonality and completeness properties are satisfied. And one has additional properties, for example the number of inequivalent irreducible representations is *finite*, and equal to the number of conjugacy classes of the group.

For a *non compact* group, the unitary representations are generally of infinite dimension. (On the other hand there may exist non unitary finite dimensional representations, see for instance $SL(2, \mathbb{C})$). The set of irreducible representations is indexed by discrete and *continuous* parameters.

2.4 Projective representations. Wigner theorem.

2.4.1 Definition

A projective representation of a group G is a linear representation up to a phase of that group (here we restrict ourselves to unitary representations). For $g_1, g_2 \in G$, one has

$$U(g_1)U(g_2) = e^{i\zeta(g_1, g_2)}U(g_1g_2) . \quad (2-67)$$

One may always choose $U(e) = I$, and thus $\forall g \quad \zeta(e, g) = \zeta(g, e) = 0$. One may also redefine $U(g) \rightarrow U'(g) = e^{i\alpha(g)}U(g)$, which changes

$$\zeta(g_1, g_2) \rightarrow \zeta'(g_1, g_2) = \zeta(g_1, g_2) + \alpha(g_1) + \alpha(g_2) - \alpha(g_1g_2) . \quad (2-68)$$

The function $\zeta(g_1, g_2)$ of $G \times G$ in \mathbb{R} is what is called a *2-cochain*. It is *closed* (and it is thus called *2-cocycle*) because of the associativity property:

$$\forall g_1, g_2, g_3 \quad (\partial\zeta)(g_1, g_2, g_3) := \zeta(g_1, g_2) + \zeta(g_1g_2, g_3) - \zeta(g_2, g_3) - \zeta(g_1, g_2g_3) = 0 \quad (2-69)$$

(check it). In general, for a n -cochain $\varphi(g_1, \dots, g_n)$, one defines the operator ∂ which takes n -cochains to $n + 1$ -cochains:

$$(\partial\varphi)(g_1, \dots, g_{n+1}) = \sum_{i=1}^n (-1)^{i+1} \varphi(g_1, g_2, \dots, (g_i g_{i+1}), \dots, g_{n+1}) - \varphi(g_2, \dots, g_{n+1}) + (-1)^n \varphi(g_1, \dots, g_n) .$$

For a 1-cochain $\alpha(g)$, $\partial\alpha(g_1, g_2) = \alpha(g_1g_2) - \alpha(g_1) - \alpha(g_2)$, and hence (2-68) reads $\zeta' = \zeta - \partial\alpha$.

Check that $\partial^2 = 0$.

The questions whether representation $U(g)$ is intrinsically projective, or may be brought back to an ordinary representation by a change of phase amounts to knowing if the cocycle ζ is *trivial*, *i.e.* if there exists $\alpha(g)$ such that in (2-68), $\zeta' = 0$. In other words, is the 2-cocycle ζ , which is closed ($\partial\zeta = 0$) by (2-69), also *exact*, *i.e.* of the form $\zeta = \partial\alpha$? This is a typical problem of *cohomology*. Cohomology of Lie groups is a broad and much studied subject, ... on which we won't dwell in these lectures.

One may summarize a fairly long and complex discussion (sketched below in § 2.4.5) by saying that for a semi-simple group G , such as $SO(n)$, the origin of the projective representations is to be found in the non simple-connectivity of G . Indeed, in the case of a non simply connected group G , the unitary representations of \tilde{G} , its universal covering, give representations up to a phase of G . For example, one recovers that the projective representations of $SO(3)$ (up to a sign) are representations of $SU(2)$. This is also the case of the Lorentz group $O(1,3)$, the universal covering of which is $SL(2, \mathbb{C})$.

Before we proceed, it is legitimate to ask the question: why are projective representations of interest for the physicist? The reason is that transformations of a quantum system make use of them, as we shall now see.

2.4.2 Wigner theorem

Consider a quantum system, the (pure) states of which are represented by rays⁶ of a Hilbert space \mathcal{H} , and in which the observables are self-adjoint operators on \mathcal{H} . Suppose there exists a transformation g of the system (states *and* observables) which leave unchanged the observable quantities $|\langle \phi | A | \psi \rangle|^2$, *i.e.*

$$|\psi\rangle \rightarrow |{}^g\psi\rangle, \quad A \rightarrow {}^gA \quad \text{such that} \quad |\langle \phi | A | \psi \rangle| = |\langle {}^g\phi | {}^gA | {}^g\psi \rangle|. \quad (2-70)$$

One then proves the following theorem

⁶ray = vector up to scalar, up to a phase if normalized

Wigner theorem: *If a bijection between rays and between auto-adjoint operators of a Hilbert space \mathcal{H} preserves the modules of scalar products*

$$|\langle \phi|A|\psi \rangle| = |\langle {}^g\phi|{}^gA|{}^g\psi \rangle| , \quad (2-71)$$

then this bijection is realized by an operator $U(g)$, linear or antilinear, unitary on \mathcal{H} , and unique up to a phase, i.e.

$$|{}^g\psi \rangle = U(g)|\phi \rangle , \quad {}^gA = U(g)AU^\dagger(g) ; \quad U(g)U^\dagger(g) = U(g)^\dagger U(g) = I . \quad (2-72)$$

Recall first what is meant by antilinear operator. Such an operator satisfies

$$U(\lambda|\phi \rangle + \mu|\psi \rangle) = \lambda^*U|\phi \rangle + \mu^*U|\psi \rangle \quad (2-73)$$

and its adjoint is defined by

$$\langle \phi|U^\dagger|\psi \rangle = \langle U\phi|\psi \rangle^* = \langle \psi|U\phi \rangle , \quad (2-74)$$

so as to be consistent with linearity

$$\langle \lambda\phi|U^\dagger|\psi \rangle = \lambda^*\langle \phi|U^\dagger|\psi \rangle . \quad (2-75)$$

If it is also unitary,

$$\langle \psi|\phi \rangle^* = \langle \phi|\psi \rangle = \langle \phi|U^\dagger U|\psi \rangle = \langle U\phi|U\psi \rangle^* , \quad (2-76)$$

hence $\langle U\phi|U\psi \rangle = \langle \psi|\phi \rangle$.

The proof of the theorem is a bit cumbersome. It consists in showing that given an orthonormal basis $|\psi_k \rangle$ in \mathcal{H} , one may find representatives $|{}^g\psi_k \rangle$ of the transformed rays such that a representative of the transformed ray of $\sum c_k|\psi_k \rangle$ is $\sum c'_k|{}^g\psi_k \rangle$ with either **all** the $c'_k = c_k$, or **all** the $c'_k = c_k^*$. Stated differently, the action $|\psi \rangle \rightarrow |{}^g\psi \rangle$ is on the whole \mathcal{H} either linear, or antilinear.

Once the transformation of states by the operator $U(g)$ is known, one determines that of observables: ${}^gA = U(g)AU^\dagger(g)$ so as to have

$$\langle {}^g\phi|{}^gA|{}^g\psi \rangle = \langle U\phi|UAU^\dagger|U\psi \rangle \quad (2-77)$$

$$= \langle \phi|U^\dagger UAU^\dagger U|\psi \rangle^\# \quad (2-78)$$

$$= \langle \phi|A|\psi \rangle^\# \quad (2-79)$$

with $\# =$ nothing or $*$ depending on whether U is linear or antilinear.

The antilinear case is not of academic interest. One encounters it in the study of *time reversal*.

The T operation leaves unchanged the position operator \mathbf{x} , but changes the sign of velocities, hence of the momentum vector \mathbf{p}

$$\mathbf{x}' = U(T)\mathbf{x}U^\dagger(T) = \mathbf{x} \quad (2-80)$$

$$\mathbf{p}' = U(T)\mathbf{p}U^\dagger(T) = -\mathbf{p} . \quad (2-81)$$

The canonical commutation relations are consistent with T only if $U(T)$ is antilinear

$$[x'_j, p'_k] = -[x_j, p_k] = -i\hbar\delta_{jk} \quad (2-82)$$

$$= U(T)[x_j, p_k]U^\dagger(T) = U(T)i\hbar\delta_{jk}U^\dagger(T) \quad (2-83)$$

Another argument: $U(T)$ commutes with time translations, the generator of which is the Hamiltonian: $U(T)i\hbar H U^\dagger(T) = -i\hbar H$ (since $t \rightarrow -t$). If U were linear, one would conclude that $UHU^\dagger = -H$, something embarrassing if the spectrum of H is bounded from below, $\text{Spec}(H) \geq E_{min}$, as in any decent physical system!

The transformations of a quantum system, *i.e.* the bijections of Wigner theorem, form a group G : if g_1 and g_2 are two such bijections, their composition g_1g_2 is another one, and so is g_1^{-1} etc. By virtue of the unicity up to a phase of $U(g)$ in the theorem, the operators $U(g)$ (that will be assumed linear in the following) thus form a representation up to a phase, *i.e.* a projective representation of G .

An important point of terminology

Up to this point, we have been discussing *transformations* of a quantum system without any assumption on its possible *invariance* under these transformations, *i.e.* on the way they affect (or not) its dynamics. These transformations may be considered from an active standpoint: the original system is compared with the transformed system, or from a passive standpoint: the same system is examined in two different coordinate systems (two observers) obtained from one another by the transformation.

2.4.3 Invariances of a quantum system

Suppose now that under the action of some group of transformations G , the system is invariant, in the sense that its dynamics, controlled by its Hamiltonian H , is unchanged. Let us write

$$H = U(g)HU^\dagger(g)$$

or alternatively

$$[H, U(g)] = 0. \quad (2-84)$$

An *invariance* (or *symmetry*) of a quantum system under the action of a group G is thus defined as the existence of a unitary projective (linear or antilinear) representation of that group in the space of states, that commutes with the Hamiltonian.

- This situation implies the existence of *conservation laws*. To see that, note that any observable function of the $U(g)$ commutes with H , and is thus a conserved quantity

$$i\hbar \frac{\partial \mathcal{F}(U(g))}{\partial t} = [\mathcal{F}(U(g)), H] = 0 \quad (2-85)$$

and each of its eigenvalues is a “good quantum number”: if the system is in an eigenspace \mathcal{V} of \mathcal{F} at time t , it stays in \mathcal{V} in its time evolution. If G is a Lie group, take g an infinitesimal transformation and denote by T the infinitesimal generators in the representation under study,

$$U(g) = I - i \delta\alpha^j T_j$$

(where one chose self-adjoint T to have U unitary), the T_j are observables that commute with H , hence conserved quantities, but in general not simultaneously measurable.

Examples.

Translation group $\longrightarrow P_\mu$ energy–momentum; rotation group $\longrightarrow M_{\mu\nu}$ angular momentum. Note also that these operators T_i which realise in the quantum theory the infinitesimal operations of the group G form a representation of the Lie algebra \mathfrak{g} . One may thus state that they satisfy the commutation relations

$$[T_i, T_j] = iC_{ij}^k T_k \quad (2-86)$$

(with an “ i ” because one chose to consider Hermitian operators). The maximal number of these operators that may be simultaneously diagonalised, hence of these conserved quantities that may be fixed, depends on the structure of \mathfrak{g} and of these commutation relations.

• On the other hand, the assumption of invariance made above has another consequence, of frequent and important application. If the space of states \mathcal{H} which “carries” a representation of a group G is decomposed into irreducible representations, in each space $E^{(\rho)}$, assumed first to be of multiplicity 1, the Hamiltonian is a multiple of the identity operator, by Schur’s lemma. One has thus a complete information on the nature of the spectrum: eigenspace $E^{(\rho)}$ and multiplicity of the eigenvalue \mathcal{E}_ρ of H equal to $\dim E^{(\rho)}$ ⁷. If some representation spaces $E^{(\rho)}$ appear with a multiplicity m_ρ larger than 1, one has still to diagonalise H in the sum of these spaces $\oplus_i E^{(\rho,i)}$, which is certainly easier than the original diagonalisation problem in the initial space \mathcal{H} . We shall see below that the Wigner-Eckart theorem allows us to simplify further the complexity of this last step. Group theory has thus considerably simplified our task, although it does not give the *values* of the eigenvalues \mathcal{E}_ρ .

2.4.4 Transformations of observables. Wigner–Eckart theorem

According to (2-72), the transformation of an operator on \mathcal{H} obeys: $A \rightarrow U(g)AU(g)^\dagger$. Suppose we are given a set of such operators, A_α , $\alpha = 1, 2, \dots$, transforming linearly among themselves, *i.e.* forming a representation:

$$A_\alpha \rightarrow U(g)A_\alpha U(g)^\dagger = \sum_{\alpha'} A_{\alpha'} \mathcal{D}_{\alpha'\alpha}(g) . \quad (2-87)$$

If the representation D is irreducible, the operators A_α form what is called an irreducible operator (or “tensor”).

For example, in atomic physics, the angular momentum \vec{J} and the electric dipole moment $\sum_i q_i \vec{r}_i$ are operators transforming like vectors under rotations.

Using the notations of section 2.2, suppose that the A_α transform by the irreducible representation $D^{(\rho)}$ and apply them on states $|\sigma\beta\rangle$ transforming according to the irreducible representation $D^{(\sigma)}$. The resulting state transforms as

$$U(g)A_\alpha|\sigma\beta\rangle = U(g)A_\alpha U(g)^\dagger U(g)|\sigma\beta\rangle = \mathcal{D}_{\alpha'\alpha}^{(\rho)}(g) \mathcal{D}_{\beta'\beta}^{(\sigma)}(g) A_{\alpha'} |\sigma\beta'\rangle \quad (2-88)$$

⁷It may happen that the multiplicity of some eigenvalue \mathcal{E}_ρ of H is higher than m_ρ , either because of the existence of a symmetry group larger than G , or because some representations come in complex conjugate pairs, or for some “accidental” reason.

that is, according to the tensor product of representations $D^{(\rho)}$ and $D^{(\sigma)}$. Following (2-89), one may decompose on irreducible representations

$$\mathcal{D}_{\alpha'\alpha}^{(\rho)}(g)\mathcal{D}_{\beta'\beta}^{(\sigma)}(g) = \sum_{\tau,\gamma,\gamma',i} \langle \tau_i\gamma|\rho, \alpha; \sigma, \beta \rangle \langle \tau_i\gamma'|\rho, \alpha'; \sigma, \beta' \rangle^* \mathcal{D}_{\gamma'\gamma}^{(\tau)}(g). \quad (2-89)$$

Suppose now that the group G is compact (or finite). The representation matrices satisfy the orthogonality property (2-47). One may thus write

$$\begin{aligned} \langle \tau\gamma|A_\alpha|\sigma\beta \rangle &= \langle \tau\gamma|U(g)^\dagger U(g)A_\alpha|\sigma\beta \rangle && \forall g \in G \\ &= \int \frac{d\mu(g)}{v(G)} \langle \tau\gamma|U(g)^\dagger U(g)A_\alpha|\sigma\beta \rangle \\ &= \int \frac{d\mu(g)}{v(G)} \sum_{\alpha',\beta',\gamma'} \mathcal{D}_{\gamma'\gamma}^{(\tau)*}(g) \langle \tau\gamma'|A_{\alpha'}|\sigma\beta' \rangle \mathcal{D}_{\alpha'\alpha}^{(\rho)}(g) \mathcal{D}_{\beta'\beta}^{(\sigma)}(g) \\ &= \frac{1}{n_\tau} \sum_{\alpha',\beta',\gamma',i} \langle \tau_i\gamma|\rho, \alpha; \sigma, \beta \rangle \langle \tau_i\gamma'|\rho, \alpha'; \sigma, \beta' \rangle^* \langle \tau_i\gamma'|A_{\alpha'}|\sigma\beta' \rangle. \end{aligned} \quad (2-90)$$

Introduce the notation

$$\langle \tau \| A \| \sigma \rangle_i := \frac{1}{n_\tau} \sum_{\alpha',\beta',\gamma'} \langle \tau_i\gamma'|\rho, \alpha'; \sigma, \beta' \rangle^* \langle \tau_i\gamma'|A_{\alpha'}|\sigma\beta' \rangle. \quad (2-91)$$

It follows that (Wigner–Eckart theorem):

$$\langle \tau\gamma|A_\alpha|\sigma\beta \rangle = \sum_{i=1}^{m_\tau} \langle \tau \| A \| \sigma \rangle_i \langle \tau_i\gamma|\rho, \alpha; \sigma, \beta \rangle \quad (2-92)$$

in which the “reduced matrix elements” $\langle \cdot \| A \| \cdot \rangle_i$ are independent of α, β, γ . The matrix element of the lhs in (2-92) vanishes if the Clebsch-Gordan coefficient is zero, (in particular if the representation τ does not appear in the product of ρ and σ). This theorem has many consequences in atomic and nuclear physics, where it gives rise to “selection rules”. See for example in Appendix E.3 the case of the electric multipole moment operators.

This theorem enables us also to simplify the diagonalisation problem of the Hamiltonian H mentioned at the end of § 2.4.3, when a representation space appears with a multiplicity m_ρ . Labelling by an index $i = 1, \dots, m_\rho$ the various copies of representation ρ , one has thanks to (2-92)

$$\langle \rho\alpha i|H|\rho\alpha' i' \rangle = \delta_{\alpha\alpha'} \langle \rho i \| H \| \rho i' \rangle \quad (2-93)$$

and the problem boils down to the diagonalisation of a $m_\rho \times m_\rho$ matrix.

Exercise. For the group $\text{SO}(3)$, let K_1^m be the components of an irreducible vector operator (for example, the electric dipole moment of Appendix E.3). Using Wigner-Eckart theorem show that

$$\langle j, m_1|K_1^m|j, m_2 \rangle = \langle j, m_1|J^m|j, m_2 \rangle \frac{\langle \vec{J} \cdot \vec{K} \rangle}{j(j+1)} \quad (2-94)$$

where $\langle \vec{J} \cdot \vec{K} \rangle$ denotes the expectation value of $\vec{J} \cdot \vec{K}$ in state j . In other terms, one may replace \vec{K} by its projection $\vec{J} \frac{\langle \vec{J} \cdot \vec{K} \rangle}{j(j+1)}$.

2.4.5 Infinitesimal form of a projective representation. Central extension

If G is a Lie group of Lie algebra \mathfrak{g} , let t_a be a basis of \mathfrak{g}

$$[t_a, t_b] = C_{ab}{}^c t_c .$$

In a projective representation (2-67), let us examine the composition of two infinitesimal transformations of the form $I + \alpha t_a$ and $I + \beta t_b$. As $\zeta(I, g) = \zeta(g, I) = 0$, $\zeta(I + \alpha t_a, I + \beta t_b)$ is of order $\alpha\beta$

$$i\zeta(I + \alpha t_a, I + \beta t_b) = \alpha\beta z_{ab} . \quad (2-95)$$

The t_a are represented by T_a , and by expanding to second order, we find

$$e^{-i\zeta(I+\alpha t_a, I+\beta t_b)} U(e^{\alpha t_a}) U(e^{\beta t_b}) = U(e^{\alpha t_a} e^{\beta t_b}) = U\left(e^{(\alpha t_a + \beta t_b)} e^{\frac{1}{2}\alpha\beta [t_a, t_b]}\right)$$

and thus, with $U(e^{\alpha t_a}) = e^{\alpha T_a}$ etc,

$$\alpha\beta \left(-z_{ab} I + \frac{1}{2}[T_a, T_b] - \frac{1}{2}C_{ab}{}^c T_c \right) = 0$$

(which proves that z_{ab} must be antisymmetric in a, b). One thus finds that the commutation relations of T are modified by a *central term* (i.e. commuting with all the other generators)

$$[T_a, T_b] = C_{ab}{}^c T_c + 2z_{ab} I .$$

The existence of projective representations may thus imply the realization of a *central extension* of the Lie algebra. One calls that way the new Lie algebra generated by the T_a and by one or several new generator(s) C_{ab} commuting with all the T_a (and among themselves)

$$[T_a, T_b] = C_{ab}{}^c T_c + C_{ab} \quad [C_{ab}, T_c] = 0 \quad [C_{ab}, C_{cd}] = 0 . \quad (2-96)$$

(In an irreducible representation of the algebra, Schur's lemma ensures that $C_{ab} = c_{ab} I$.) The triviality (or non-triviality) of the cocycle ζ translates in infinitesimal form into the possibility (or impossibility) of getting rid of the central term by a redefinition of the T

$$T_a \rightarrow \tilde{T}_a = T_a + X_a \quad [\tilde{T}_a, \tilde{T}_b] = C_{ab}{}^c \tilde{T}_c , \quad (2-97)$$

in a way consistent with the constraints on the $C_{ab}{}^c$ and C_{ab} coming from the Jacobi identity.

Exercise. Write the constraint that the Jacobi identity puts on the constants $C_{ab}{}^c$ et C_{ab} . Show that $C_{ab} = C_{ab}{}^c D_c$ gives a solution and that a redefinition such as (2-97) is then possible.

One proves (Bargmann) that for a connected Lie group G , the cocycles are trivial if

1. there exists no non-trivial central extension of \mathfrak{g} ;
2. G is simply connected.

As for point 1), a theorem of Bargmann tells us that there is no non-trivial central extension for any semi-simple group, like the classical groups $SU(n)$, $SO(n)$, $Sp(2n)$. It is thus point 2) which is relevant.

If the group G is not simply connected, one studies the (say unitary) representations of its universal covering \tilde{G} , which are representations up to a phase of G (the group $\pi_1(G) = \tilde{G}/G$ is represented on $U(1)$). This is the case of the groups $SO(n)$ and their universal covering $Spin(n)$, (for example $SO(3)$), or of the Lorentz group $O(1,3)$, as recalled above.

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A short bibliography (cont'd)

In addition to references already given in the Introduction and in Chap. 1,

General representation theory

[Ki] A.A. Kirillov, *Elements of the theory of representations*, Springer.

[Kn] A. Knapp, *Representation Theory of semi-simple groups*, Princeton U. Pr.

[FH] W. Fulton and J. Harris, *Representation Theory*, Springer.

For a proof of Peter-Weyl theorem, see for example

[BrD] T. Bröcker and T. tom Dieck, *Representations of compact Lie groups*, Springer.

For a proof of Wigner theorem, see E. Wigner, [Wi], or A. Messiah, [M] vol. 2, p 774, or S. Weinberg, [Wf] chap 2, app A.

On projective representations, see

[Ba] V. Bargmann, *Ann. Math.* **59** (1954) 1-46, or

S. Weinberg [Wf] Chap 2.7.

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Appendix D. ‘Tensors, you said tensors?’

The word “tensor” covers several related but not quite identical concepts. The aim of this appendix is to (try to) clarify these matters...

D.1. Algebraic definition

Given two vector spaces E et F , their tensor product is by definition the vector space $E \otimes F$ generated by the pairs (x, y) , $x \in E$, $y \in F$, denoted $x \otimes y$. An element of $E \otimes F$ thus reads

$$z = \sum_{\alpha} x^{(\alpha)} \otimes y^{(\alpha)} \tag{D-1}$$

with a finite sum over vectors $x^{(\alpha)} \in E$, $y^{(\alpha)} \in F$ (a possible scalar coefficient λ_{α} has been absorbed into a redefinition of the vector $x^{(\alpha)}$).

If A , resp. B , is a linear operator acting in E , resp. F , $A \otimes B$ is the linear operator acting in $E \otimes F$ according to

$$A \otimes B(x \otimes y) = Ax \otimes By \tag{D-2}$$

$$A \otimes B \sum_{\alpha} (x^{(\alpha)} \otimes y^{(\alpha)}) = \sum_{\alpha} Ax^{(\alpha)} \otimes By^{(\alpha)} \tag{D-3}$$

In particular if E et F have two bases e_i and f_j , $z = x \otimes y = \sum_{i,j} x^i y^j e_i f_j$, the basis $E \otimes F$ and the components of z are labelled by pairs of indices (i, j) , and $A \otimes B$ is described in that basis by a matrix which is read off

$$(A \otimes B)z = \sum_{i,i',j,j'} A_{ii'} B_{jj'} x^{i'} y^{j'} e_i f_j =: (A \otimes B)_{ii',jj'} z^{i'j'} e_i \otimes f_j \tag{D-4}$$

thus

$$(A \otimes B)_{ij;i'j'} = A_{ii'}B_{jj'} , \quad (\text{D-5})$$

a formula which is sometimes taken as a definition of tensor product of two matrices.

D.2. Group action

If a group G has representations D and D' in two vector spaces E and F , $x \in E \mapsto D(g)x = e_i \mathcal{D}_{ij} x^j$, and likewise for $y \in F$, the tensor product representation $D \otimes D'$ in $E \otimes F$ is defined by

$$D(g) \otimes D'(g)(x \otimes y) = D(g)x \otimes D'(g)y \quad (\text{D-6})$$

in accord with (D-2). The matrix of $D \otimes D'$ in a basis $e_i \otimes f_j$ is $\mathcal{D}_{ii'} \mathcal{D}'_{jj'}$.

Another way of saying it is: if x “transforms by the representation D ” and y by D' , under the action of $g \in G$, *i.e.* $x' = D(g)x$, $y' = D'(g)y$, $x \otimes y \mapsto x' \otimes y'$, with

$$(x' \otimes y')^{ij} = x^i y^j = \mathcal{D}_{ii'} \mathcal{D}'_{jj'} x^{i'} y^{j'} , \quad (\text{D-7})$$

another formula sometimes taken as a definition of a tensor (under the action of G).

The previous construction of rank 2 tensors z^{ij} may be iterated to make tensor products $E_1 \otimes E_2 \otimes \cdots \otimes E_p$ and rank p tensors $z^{i_1 \cdots i_p}$. This is what we did in Chap. 0, § 0.3.3, in the construction of the representations of $SU(2)$ by symmetrized tensor products of the spin $\frac{1}{2}$ representation, or in § 0.6.4 for those of $SL(2, \mathbb{C})$, by symmetrized tensor products of the two representations with pointed or unpointed indices, $(0, \frac{1}{2})$ and $(\frac{1}{2}, 0)$.

Appendix E. More on representation matrices of $SU(2)$

We return to the representation matrices \mathcal{D}^j of $SU(2)$ defined and explicitly constructed in §0.3.2 and 0.3.3.

E.1. Orthogonality, completeness, characters

All unitary representations of $SU(2)$ have been constructed in Chap. 0. Following the discussion of §3, the matrix elements of \mathcal{D}^j satisfy orthogonality and completeness properties, which make use of the invariant measure on $SU(2)$ introduced in Chap. 1 (§ 1.2.4 and App. C)

$$(2j+1) \int \frac{d\mu(U)}{2\pi^2} \mathcal{D}_{mn}^j(U) \mathcal{D}_{m'n'}^{j'*}(U) = \delta_{jj'} \delta_{mm'} \delta_{nn'} \quad (\text{E-1})$$

$$\sum_{jmn} (2j+1) \mathcal{D}_{mn}^j(U) \mathcal{D}_{mn}^{j'*}(U') = 2\pi^2 \delta(U, U') .$$

The “delta function” $\delta(U, U')$ appearing in the rhs of (E-1) is the one adapted to the measure $d\mu(U)$, such that $\int d\mu(U') \delta(U, U') f(U') = f(U)$; in Euler angles parametrization α, β, γ for example,

$$\delta(U, U') = 8\delta(\alpha - \alpha') \delta(\cos \beta - \cos \beta') \delta(\gamma - \gamma') , \quad (\text{E-2})$$

(see Appendix C of Chap. 1). The meaning of equation (E-1) is that functions $\mathcal{D}_{mn}^j(U)$ form a complete basis in the space of functions (continuous or square integrable) on the group $SU(2)$ (Peter–Weyl theorem).

Characters of representations of $SU(2)$ follow from the previous expressions

$$\begin{aligned} \chi_j(U) = \chi_j(\psi) &= \text{tr } \mathcal{D}^j(\mathbf{n}, \psi) = \sum_{m=-j}^j e^{im\psi} \\ &= \frac{\sin\left(\frac{2j+1}{2}\psi\right)}{\sin\frac{\psi}{2}}. \end{aligned} \quad (\text{E-3})$$

Note that these expressions are polynomials (so-called Chebyshev polynomials of 2nd kind) of the variable $2 \cos \frac{\psi}{2}$ (see exercise D at the end of this chapter). In particular

$$\chi_0(\psi) = 1 \quad \chi_{\frac{1}{2}}(\psi) = 2 \cos \frac{\psi}{2} \quad \chi_1(\psi) = 1 + 2 \cos \psi \quad \text{etc.} \quad (\text{E-4})$$

One may then verify all the expected properties

unitarity and reality	$\chi_j(U^{-1}) = \chi_j^*(U) = \chi_j(U)$	
parity and periodicity	$\chi_j(-U) = \chi_j(2\pi + \psi) = (-1)^{2j} \chi_j(U)$	(E-5)
orthogonality	$\int_0^{2\pi} d\psi \sin^2 \frac{\psi}{2} \chi_j(\psi) \chi_{j'}(\psi) = \pi \delta_{jj'}$	
completeness	$\sum_{j=0, \frac{1}{2}, \dots} \chi_j(\psi) \chi_j(\psi') = \frac{\pi}{\sin^2 \frac{\psi}{2}} \delta(\psi - \psi') = \frac{\pi}{2 \sin \frac{\psi}{2}} \delta\left(\cos \frac{\psi}{2} - \cos \frac{\psi'}{2}\right)$	

The latter expresses that characters form a complete basis of class functions, *i.e.* of even 2π -periodic functions of $\frac{1}{2}\psi$. This is a variant of the Fourier expansion.

Does the multiplicity formula (2-60) lead to the well known formulae (2-31)?

E.2. Special functions. Spherical harmonics

We are by now familiar with the idea that infinitesimal generators act in each representation as differential operators. This is true in particular in the present case of $SU(2)$: the generators J_i appear as differential operators with respect to parameters of the rotation, compare with the case of a one-parameter subgroup $\exp -iJ\psi$ for which $J = i\partial/\partial\psi$. This gives rise to differential equations satisfied by the $\mathcal{D}_{m'm}^j$ and exposes their relation with “special functions” of mathematical physics.

We also noticed that the construction of the *Wigner D matrices* in § 0.3.3 applies not only to $SU(2)$ matrices but also to arbitrary matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in the linear group $GL(2, \mathbb{C})$. Equation (70) of Chap. 0 still holds true

$$P_{jm}(\xi', \eta') = \sum_{m'} P_{jm'}(\xi, \eta) \mathcal{D}_{m'm}^j(A). \quad (\text{0-70})$$

The combination $(a\xi + c\eta)^{j+m} (b\xi + d\eta)^{j-m}$ clearly satisfies

$$\left(\frac{\partial^2}{\partial a \partial d} - \frac{\partial^2}{\partial b \partial c} \right) (a\xi + c\eta)^{j+m} (b\xi + d\eta)^{j-m} = 0 \quad (\text{E-6})$$

and because of the independance of the $P_{jm}(\xi, \eta)$, the $\mathcal{D}_{m'm}^j(A)$ satisfy the same equation. If we now impose that $d = a^*$, $c = -b^*$, but $\rho^2 = |a|^2 + |b|^2$ is kept arbitrary, the matrices A satisfy $AA^\dagger = \rho^2 I$, $\det A = \rho^2$, hence $A = \rho U$, $U \in \text{SU}(2)$, and (E-6) leads to

$$\Delta_4 \mathcal{D}_{m'm}^j(A) = 4 \left(\frac{\partial^2}{\partial a \partial a^*} + \frac{\partial^2}{\partial b \partial b^*} \right) \mathcal{D}_{m'm}^j(A) = 0 \quad (\text{E-7})$$

where Δ_4 is the Laplacian in the space \mathbb{R}^4 with variables u_0, \mathbf{u} , and $a = u_0 + iu_3$, $b = u_1 + iu_2$. In polar coordinates

$$\Delta_4 = \frac{\partial^2}{\partial \rho^2} + \frac{3}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \Delta_{S^3} \quad (\text{E-8})$$

where the last term Δ_{S^3} , Laplacian on the unit sphere S^3 , acts only on “angular variables” $U \in \text{SU}(2)$. The functions \mathcal{D}^j being homogeneous of degree $2j$ in a, b, c, d hence in ρ , one finally gets

$$-\frac{1}{4} \Delta_{S^3} \mathcal{D}_{m'm}^j(U) = j(j+1) \mathcal{D}_{m'm}^j(U) . \quad (\text{E-9})$$

For example, using the parametrization by Euler angles, one finds

$$\left\{ \frac{1}{\sin \beta} \frac{\partial}{\partial \beta} \sin \beta \frac{\partial}{\partial \beta} + \frac{1}{\sin^2 \beta} \left[\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \gamma^2} - 2 \cos \beta \frac{\partial^2}{\partial \alpha \partial \gamma} \right] + j(j+1) \right\} \mathcal{D}^j(\alpha, \beta, \gamma)_{m'm} = 0 . \quad (\text{E-10})$$

For $m = 0$ (hence j necessarily integer), the dependence on γ disappears (see (00.3.14)). Choose for example $\gamma = 0$ and perform a change of notations $(j, m') \rightarrow (l, m)$ and $(\beta, \alpha) \rightarrow (\theta, \phi)$, so as to recover classical notations. The equation reduces to

$$\left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} + l(l+1) \right] \mathcal{D}_{m0}^l(\phi, \theta, 0) = 0 . \quad (\text{E-11})$$

The differential operator made of the first two terms is the Laplacian Δ_{S^2} on the unit sphere S^2 . Equation (E-11) thus defines *spherical harmonics* $Y_l^m(\theta, \phi)$ as eigenvectors of the Laplacian Δ_{S^2} . The correct normalisation is

$$\left[\frac{2l+1}{4\pi} \right]^{\frac{1}{2}} \mathcal{D}_{m0}^l(\phi, \theta, 0) = Y_l^{m*}(\theta, \phi) . \quad (\text{E-12})$$

Introduce also the *Legendre polynomials and functions* $P_l(u)$ and $P_l^m(u)$, which are defined for integer l and $u \in [-1, 1]$ by

$$P_l(u) = \frac{1}{2^l l!} \frac{d^l}{du^l} (u^2 - 1)^l \quad (\text{E-13})$$

$$P_l^m(u) = (1 - u^2)^{\frac{1}{2}m} \frac{d^m}{du^m} P_l(u) \quad \text{for } 0 \leq m \leq l . \quad (\text{E-14})$$

The Legendre polynomials $P_l(u)$ are orthogonal polynomials on the interval $[-1, 1]$ with the weight 1: $\int_{-1}^1 P_l(u) P_{l'}(u) du = \frac{2}{2l+1} \delta_{ll'}$. The first P_l read

$$P_0 = 1 \quad P_1 = u \quad P_2 = \frac{1}{2}(3u^2 - 1) \quad P_3 = \frac{1}{2}(5u^3 - 3u) , \dots \quad (\text{E-15})$$

while $P_l^0 = P_l$, $P_l^1 = (1-u^2)^{\frac{1}{2}}P_l$, etc. The spherical harmonics are related to Legendre functions $P_l^m(\cos \theta)$ (for $m \geq 0$) by

$$Y_l^m(\theta, \phi) = (-1)^m \left[\frac{(2l+1)(l-m)}{4\pi(l+m)} \right]^{\frac{1}{2}} P_l^m(\cos \theta) e^{im\phi} \quad (\text{E-16})$$

and thus

$$\mathcal{D}_{m0}^l(0, \theta, 0) = d_{m0}^l(\theta) = (-1)^m \left[\frac{(l-m)}{(l+m)} \right]^{\frac{1}{2}} P_l^m(\cos \theta) = \left(\frac{4\pi}{2l+1} \right)^{\frac{1}{2}} Y_l^{m*}(\theta, 0). \quad (\text{E-17})$$

In particular, $d_{00}^l(\theta) = P_l(\cos \theta)$. In general, $d_{m'm}^l(\theta)$ is related to the Jacobi polynomial

$$P_l^{(\alpha, \beta)}(u) = \frac{(-1)^l}{2^l l!} (1-u)^{-\alpha} (1+u)^{-\beta} \frac{d^l}{du^l} [(1-u)^{\alpha+l} (1+u)^{\beta+l}] \quad (\text{E-18})$$

by

$$d_{m'm}^j(\theta) = \left[\frac{(j+m')!(j-m')!}{(j+m)!(j-m)!} \right]^{\frac{1}{2}} \left(\cos \frac{\theta}{2} \right)^{m+m'} \left(\sin \frac{\theta}{2} \right)^{m-m'} P_{j-m'}^{(m'-m, m'+m)}(\cos \theta). \quad (\text{E-19})$$

Jacobi and Legendre polynomials pertain to the general theory of orthogonal polynomials, for which one shows that they satisfy 3-term linear recursion relations, and also differential equations. For instance, Jacobi polynomials are orthogonal for the measure

$$\int_{-1}^1 du (1-u)^\alpha (1+u)^\beta P_j^{(\alpha, \beta)}(u) P_{j'}^{(\alpha, \beta)}(u) = \delta_{jj'} \frac{2^{\alpha+\beta+1} \Gamma(l+\alpha+1) \Gamma(l+\beta+1)}{(2l+\alpha+\beta+1) l! \Gamma(l+\alpha+\beta+1)} \quad (\text{E-20})$$

and satisfy the recursion relation

$$\begin{aligned} & 2(l+1)(l+\alpha+\beta+1)(2l+\alpha+\beta) P_{l+1}^{(\alpha, \beta)}(u) \\ &= (2l+\alpha+\beta+1)[(2l+\alpha+\beta)(2l+\alpha+\beta+2)u + \alpha^2 - \beta^2] P_l^{(\alpha, \beta)}(u) - 2(l+\alpha)(l+\beta)(2l+\alpha+\beta+2) P_{l-1}^{(\alpha, \beta)}(u). \end{aligned} \quad (\text{E-21})$$

The Jacobi polynomial $P_l^{(\alpha, \beta)}(u)$ is a solution of the differential equation

$$\{(1-u^2) \frac{d^2}{du^2} + [\beta - \alpha - (2 + \alpha + \beta)u] \frac{d}{du} + l(l + \alpha + \beta + 1)\} P_l^{(\alpha, \beta)}(u) = 0. \quad (\text{E-22})$$

The Legendre polynomials correspond to the case $\alpha = \beta = 0$. These relations appear here as related to those of the \mathcal{D}^j . This is a general feature: many “special functions” (Bessel, etc) are related to representation matrices of groups. Group theory thus gives a geometric perspective to results of classical analysis.

Return to spherical harmonics and their properties.

(i) They satisfy the differential equations

$$(\Delta_{S^2} + l(l+1)) Y_l^m = 0 \quad (\text{E-23})$$

$$J_z Y_l^m = -i \frac{\partial}{\partial \phi} Y_l^m = m Y_l^m \quad (\text{E-24})$$

and may be written as

$$Y_l^m(\theta, \phi) = \frac{(-1)^l}{2^l l!} \sqrt{\frac{(2l+1)(l+m)!}{4\pi(l-m)!}} e^{im\phi} \sin^{-m} \theta \left(\frac{d}{d \cos \theta} \right)^{l-m} \sin^{2l} \theta. \quad (\text{E-25})$$

(ii) They are normalized to 1 on the unit sphere and more generally satisfy orthogonality and completeness properties

$$\int d\Omega Y_l^{m*} Y_{l'}^{m'} = \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta Y_l^{m*} Y_{l'}^{m'} = \delta_{ll'} \delta_{mm'} \quad (\text{E-26})$$

$$\sum_{l=0}^{\infty} \sum_{m=-l}^l Y_l^{m*}(\theta, \phi) Y_l^m(\theta', \phi') = \delta(\Omega - \Omega') = \frac{\delta(\theta - \theta') \delta(\phi - \phi')}{\sin\theta} \quad (\text{E-27})$$

$$= \delta(\cos\theta - \cos\theta') \delta(\phi - \phi') \quad (\text{E-28})$$

(iii) One may consider $Y_l^m(\theta, \phi)$ as a function of the unit vector \mathbf{n} with polar angles θ, ϕ . If the vector \mathbf{n} is transformed into \mathbf{n}' by the rotation R , one has

$$Y_l^m(\mathbf{n}') = Y_l^{m'}(\mathbf{n}) \mathcal{D}^l(R)_{m'm} \quad (\text{E-29})$$

which expresses that the Y_l^m transform as vectors of the spin l representation.

(iv) One checks on the above expression the symmetry properties in m

$$Y_l^{m*}(\theta, \phi) = (-1)^m Y_l^{-m}(\theta, \phi) \quad (\text{E-30})$$

and parity

$$Y_l^m(\pi - \theta, \phi + \pi) = (-1)^l Y_l^m(\theta, \phi) . \quad (\text{E-31})$$

Note that for $\theta = 0$, $Y_l^m(0, \phi)$ vanishes except for $m = 0$, see (E-13, E-16).

(v) Spherical harmonics satisfy also recursion formulae of two types: those coming from the action of J_{\pm} , differential operators acting as in (119)

$$e^{\pm i\phi} \left[\pm \frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\phi} \right] Y_l^m = \sqrt{l(l+1) - m(m \pm 1)} Y_l^{m \pm 1} \quad (\text{E-32})$$

and those coming from the tensor product with the vector representation

$$\sqrt{2l+1} \cos\theta Y_l^m = \left(\frac{(l+m)(l-m)}{2l-1} \right)^{\frac{1}{2}} Y_{l-1}^m + \left(\frac{(l+m+1)(l-m+1)}{2l+3} \right)^{\frac{1}{2}} Y_{l+1}^m . \quad (\text{E-33})$$

More generally, one has a product formula

$$Y_l^m(\theta, \phi) Y_{l'}^{m'}(\theta, \phi) = \sum_L \langle lm; l'm' | L, m+m' \rangle \left[\frac{(2l+1)(2l'+1)}{4\pi(2L+1)} \right]^{\frac{1}{2}} Y_L^{m+m'}(\theta, \phi) . \quad (\text{E-34})$$

(vi) Finally let us quote the very useful “addition theorem”

$$\frac{2l+1}{4\pi} P_l(\cos\theta) = \sum_{m=-l}^l Y_l^m(\mathbf{n}) Y_l^{m*}(\mathbf{n}') \quad (\text{E-35})$$

where θ denotes the angle between directions \mathbf{n} and \mathbf{n}' . This may be proved by showing that the rhs satisfies the same differential equation as the P_l (see exercise 1 below).

Exercises.

1. Prove that the Legendre polynomial P_l verifies

$$(\Delta_{S^2} + l(l+1)) P_l(\mathbf{n}, \mathbf{n}') = 0$$

as a function of \mathbf{n} or of \mathbf{n}' , as well as $(\mathbf{J} + \mathbf{J}') P_l = 0$ where \mathbf{J} and \mathbf{J}' are generators of rotations of \mathbf{n} and \mathbf{n}' respectively. Conclude that there exists an expansion on spherical harmonics given by the addition theorem of (E-35) (Remember that $P_l(1) = 1$).

2. Prove that a generating function of Legendre polynomials is

$$\frac{1}{\sqrt{1-2ut+t^2}} = \sum_{l=0}^{\infty} t^l P_l(u). \quad (\text{E-36})$$

Hint: show that the differential equation of the P_l (a particular case of (E-22) for $\alpha = \beta = 0$) is indeed satisfied and that the P_l appearing in that formula are polynomials in u . Derive from it the identity (assuming $r' < r$),

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{l=0}^{\infty} \frac{r'^l}{r^{l+1}} P_l(\cos \theta) = \sum_{l,m} \frac{4\pi}{2l+1} \frac{r'^l}{r^{l+1}} Y_l^{m*}(\mathbf{n}) Y_l^m(\mathbf{n}'). \quad (\text{E-37})$$

The expression of the first Y_l^m may be useful

$$\begin{aligned} Y_0^0 &= \frac{1}{\sqrt{4\pi}} \\ Y_1^0 &= \sqrt{\frac{3}{4\pi}} \cos \theta & Y_1^{\pm 1} &= \mp \sqrt{\frac{3}{8\pi}} \sin \theta e^{\pm i\phi} \\ Y_2^0 &= \sqrt{\frac{5}{16\pi}} (3 \cos^3 \theta - 1) & Y_2^{\pm 1} &= \mp \sqrt{\frac{15}{8\pi}} \cos \theta \sin \theta e^{\pm i\phi} & Y_2^{\pm 2} &= \sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{\pm 2i\phi}. \end{aligned} \quad (\text{E-38})$$

E.3. Physical applications

E.3.1. Multipole moments

Consider the electric potential created by a static charge distribution $\rho(\vec{r})$

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{d^3r' \rho(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

and expand it on spherical harmonics following (E-37). One finds

$$\phi(\vec{r}) = \frac{1}{\epsilon_0} \sum_{l,m} \frac{1}{2l+1} \frac{Y_l^{m*}(\mathbf{n})}{r^{l+1}} Q_{lm} \quad (\text{E-39})$$

where the Q_{lm} , defined by

$$Q_{lm} = \int d^3r' \rho(\vec{r}') r'^l Y_l^m(\mathbf{n}') \quad (\text{E-40})$$

are the *multipole moments* of the charge distribution ρ . For example, if $\rho(\vec{r}) = \rho(r)$ is invariant by rotation, only Q_{00} is non vanishing and is equal to the total charge (up to a factor $1/\sqrt{4\pi}$)

$$Q_{00} = \frac{Q}{\sqrt{4\pi}} = \sqrt{4\pi} \int r^2 dr \rho(r) \quad \phi(r) = \frac{Q}{4\pi\epsilon_0 r}.$$

For an arbitrary $\rho(\vec{r})$, the three components of Q_{1m} reconstruct the dipole moment $\int d^3r' \rho(\vec{r}') \vec{r}'$. More generally, under rotations, the Q_{lm} are the components of a tensor operator transforming according to the spin l representation and (see. (E-31), of parity $(-1)^l$).

In Quantum Mechanics, les Q_{lm} become operators in the Hilbert space of the theory. One may apply the Wigner-Eckart theorem and conclude that

$$\langle j_1, m_1 | Q_{lm} | j_2, m_2 \rangle = \langle j_1 || Q_l || j_2 \rangle \langle j_1, m_1 | l, m; j_2, m_2 \rangle$$

with a reduced matrix element which is independent of the m . In particular, if $j_1 = j_2 = j$, the expectation value of Q_l is non zero only for $l \leq 2j$.

E.3.2. Eigenstates of the angular momentum in Quantum Mechanics

Spherical harmonics may be interpreted as wave functions in coordinates θ, ϕ of the eigenstates of the angular momentum $\vec{L} = \hbar \vec{J} = \hbar \vec{r} \times \vec{\nabla}$

$$Y_l^m(\theta, \phi) = \langle \theta, \phi | l, m \rangle$$

in analogy with

$$\frac{1}{(2\pi)^{3/2}} e^{i\vec{x} \cdot \vec{p}} = \langle \vec{x} | \vec{p} \rangle .$$

(We take $\hbar = 1$.) In particular, suppose that in a scattering process described by a rotation invariant Hamiltonian, a state of initial momentum \vec{p}_i along the z -axis, (*i.e.* $\theta = \phi = 0$), interacts with a scattering center and comes out in a state of momentum \vec{p}_f , with $|p_i| = |p_f| = p$, along the direction $\mathbf{n} = (\theta, \phi)$. One writes the scattering amplitude

$$\begin{aligned} \langle p, \theta, \phi | \mathcal{T} | p, 0, 0 \rangle &= \sum_{l'mm'} Y_l^m(\theta, \phi) \langle p, l, m | \mathcal{T} | p, l', m' \rangle Y_{l'}^{m'*}(0, 0) \\ &= \sum_{lm} Y_l^m(\theta, \phi) \langle p, l, m | \mathcal{T} | p, l, m \rangle Y_l^{m*}(0, 0) \\ &= \sum_l \frac{2l+1}{4\pi} \mathcal{T}_l(p) P_l(\cos \theta) \end{aligned} \quad (\text{E-41})$$

using once again the addition formula and $\langle plm | \mathcal{T} | pl'm' \rangle = \delta_{ll'} \delta_{mm'} \mathcal{T}_l(p)$ expressing rotation invariance. This is the very useful *partial wave expansion* of the scattering amplitude.

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Exercises for chapter 2

A. Unitary representations of a simple group

Let G be a *simple* non abelian group, and D be unitary representation of G .

1. Show that $\det D$ is a representation of dimension 1 of the group, and a homomorphism of the group into the group $U(1)$.
2. What can be said about the kernel K of this homomorphism? Show that any “commutator” $g_1 g_2 g_1^{-1} g_2^{-1}$ belongs to K and thus that K cannot be trivial.
3. Conclude that the representation is unimodular (of determinant 1).
4. Can we apply that argument to $SO(3)$? to $SU(2)$?

B. Adjoint representation

1. Show that if the Lie algebra \mathfrak{g} of a Lie group G is simple, the adjoint representation of G is irreducible.
2. Show that if \mathfrak{g} is semi-simple, its adjoint representation is faithful: $\ker \text{ad} = 0$.

C. Tensor product $D \otimes D^*$

Let G be a compact group and $D^{(\rho)}$ its irreducible representations. Denote $D^{(1)}$ the identity representation, $D^{(\bar{\rho})}$ the conjugate representation of $D^{(\rho)}$.

What is the multiplicity of $D^{(1)}$ in the decomposition of $D^{(\rho)} \otimes D^{(\bar{\rho})}$ into irreducible representations?

D. Chebyshev polynomials

Consider the expression

$$U_l = \frac{\sin(l+1)\theta}{\sin \theta}, \tag{2-113}$$

where l is an integer ≥ 0 .

1. By an elementary trigonometric calculation, express $U_{l-1} + U_{l+1}$ in terms of U_l , with an l independent coefficient.
2. Conclude that U_l is a polynomial in $z = 2 \cos \theta$ of degree l , which we denote $U_l(z)$.
3. What is the group theoretic interpretation of the result in 1. ?
4. With the minimum of additional computations, what can be said about

$$\frac{2}{\pi} \int_{-1}^1 dz (1-z^2)^{\frac{1}{2}} U_l(z) U_{l'}(z)$$

and

$$\frac{2}{\pi} \int_{-1}^1 dz (1-z^2)^{\frac{1}{2}} U_l(z) U_{l'}(z) U_{l''}(z) ?$$

The $U_l(z)$ are the Chebyshev polynomials (Tchebichev in the French transcription) of 2nd kind. They are orthogonal (the first relation in 4.) and satisfy a 3-term recursion relation (question 1.), which are two general properties of orthogonal polynomials.

E. Spherical Harmonics

Show that the integral

$$\int d\Omega Y_{l_1}^{m_1}(\theta, \phi) Y_{l_2}^{m_2}(\theta, \phi) Y_{l_3}^{m_3}(\theta, \phi)$$

is proportional to the Clebsch-Gordan coefficient $(-1)^{m_3} \langle l_1, m_1; l_2, m_2 | l_3, -m_3 \rangle$, with an m independent factor to be determined.

Problem I. Decomposition of an amplitude

Consider two real unitary representations (ρ) and (σ) of a simple compact Lie group G of dimension d . Denote $|\rho, \alpha\rangle$, resp. $|\sigma, \beta\rangle$, two bases of these representations, and $T_{\alpha\alpha'}^{(\rho)a}$, resp. $T_{\beta\beta'}^{(\sigma)a}$, $a = 1, \dots, d$, the representation matrices in a basis of the Lie algebra. Explain why this basis may be assumed to be orthonormal wrt the Killing form. These matrices are taken to be real *skew-symmetric* and thus satisfy $\text{tr } T^a T^b = -\delta_{ab}$. Consider now the quantity

$$X_{\alpha\beta; \alpha'\beta'} := \sum_{a=1}^d T_{\alpha\alpha'}^{(\rho)a} T_{\beta\beta'}^{(\sigma)a} . \quad (2-114)$$

To simplify things, we assume that all irreducible representations appearing in the tensor product of representations (ρ) et (σ) are real and with multiplicity 1. Let $|\tau\gamma\rangle$ be a basis of such a representation. The (real) Clebsch-Gordan coefficients are written as matrices

$$(\mathcal{M}^{(\tau\gamma)})_{\alpha\beta} = \langle \tau\gamma | \rho\alpha; \sigma\beta \rangle . \quad (2-115)$$

1. Recall why these coefficients satisfy orthogonality and completeness relations and write them.
2. Show that it follows that

$$X_{\alpha\beta; \alpha'\beta'} = - \sum_{\tau\gamma} (\mathcal{M}^{(\tau\gamma)})_{\alpha\beta} (T^{(\rho)a} \mathcal{M}^{(\tau\gamma)} T^{(\sigma)a})_{\alpha'\beta'} . \quad (2-116)$$

3. Acting with the infinitesimal generator T^a on the two sides of the relation

$$|\rho\alpha; \sigma\beta\rangle = \sum_{\tau, \gamma} (\mathcal{M}^{(\tau\gamma)})_{\alpha\beta} |\tau\gamma\rangle \quad (2-117)$$

show that one gets

$$\sum_{\gamma'} T_{\gamma\gamma'}^{(\tau)a} (\mathcal{M}^{(\tau\gamma')})_{\alpha\beta} = \sum_{\alpha'} (\mathcal{M}^{(\tau\gamma)})_{\alpha'\beta} (T^{(\rho)a})_{\alpha'\alpha} + \sum_{m'_2} (\mathcal{M}^{(\tau\gamma)})_{\alpha\beta'} (T^{(\sigma)a})_{\beta'\beta} \quad (2-118)$$

or, in terms of matrices of dimensions $\dim(\rho) \times \dim(\sigma)$

$$\sum_{\gamma'} T_{\gamma\gamma'}^{(\tau)a} \mathcal{M}^{(\tau\gamma')} = -T^{(\rho)a} \mathcal{M}^{(\tau\gamma)} + \mathcal{M}^{(\tau\gamma)} T^{(\sigma)a} . \quad (2-119)$$

4. Using repeatedly this relation (2-119) in (2-116), show that one finds

$$X_{\alpha\beta;\alpha'\beta'} = \frac{1}{2} \sum_{\tau\gamma} (C_\rho + C_\sigma - C_\tau) (\mathcal{M}^{(\tau\gamma)})_{\alpha\beta} (\mathcal{M}^{(\tau\gamma)})_{\alpha'\beta'} \quad (2-120)$$

where the C are Casimir operators, for example

$$C_\rho = - \sum_a (T^{(\rho)a})^2 . \quad (2-121)$$

5. Why can one say that “large representations” τ tend to make the coefficient $(C_\rho + C_\sigma - C_\tau)$ increasingly negative? (One may take the example of $SU(2)$ with ρ and σ two spin j ($j \in \mathbb{N}$) representations).
6. Can you propose a field theory in which the coefficient $X_{\alpha\beta;\alpha'\beta'}$ would appear in a two-body scattering amplitude (in the tree approximation)? What is the consequence of the property derived in 5) on that amplitude?

Problem II. Tensor product in $SU(2)$

1. Let $R_{\frac{1}{2}}$ denote the spin $\frac{1}{2}$ representation of $SU(2)$; we want to compute the multiplicity n_r of the identity representation in the decomposition into irreducible representations of the tensor product of r copies of $R_{\frac{1}{2}}$.
- Interpret n_r in terms of the number of linearly independent invariants, multilinear in ξ_1, \dots, ξ_r , where the ξ_i are spinors transforming under the representation $R_{\frac{1}{2}}$.
 - By convention $n_0 = 1$. With no calculation, what are n_1 and n_2 ?
 - Show that n_r may be expressed with an integral involving characters $\chi_j(\psi)$ of $SU(2)$. (Do not attempt to compute this integral explicitly for arbitrary r .)
 - Check that this formula gives the values of n_1 et n_2 found in b).
 - We shall now show that the n_r may also be obtained by the following graphical and recursive method. On the graph of Fig. 2.2, attach $n_0 = 1$ to the leftmost vertex, then to each vertex S , attach the sum $\alpha = \beta + \gamma$ of numbers on vertices immediately on the left of S .
 - Show that the n_r are the numbers located on the horizontal axis. What is the interpretation of the horizontal and vertical axes?
 - Compute with this method the value of n_4 and n_6 .
2. One wants to repeat this computation for the spin 1 representation R_1 , and hence to determine the number N_r of times where the identity representation appears in the tensor product of r copies of R_1 .

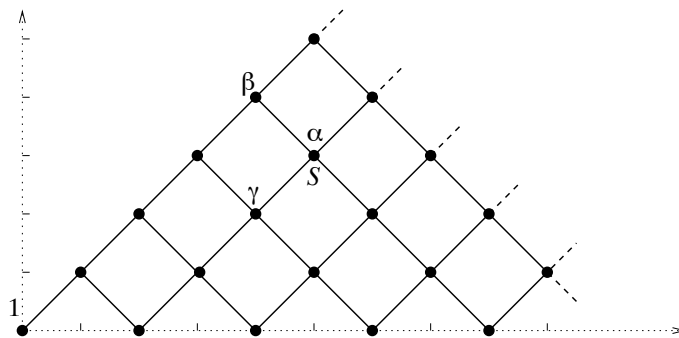


Figure 2.2: A graphical construction of the n_r

- How should the graph of fig 2.2 be modified to yield the N_r ?
- Compute N_2 , N_3 et N_4 by this method.
- What do these numbers represent in terms of vectors V_1, \dots, V_r transforming under the representation R_1 ?

Problem III. Real, complex and quaternionic representations

Preliminary question

Given a vector space E of dimension d , one denotes $E \otimes E$ or $E^{\otimes 2}$ the space of rank 2 tensors and $(E \otimes E)_S$, resp. $(E \otimes E)_A$, the space of symmetric, resp. antisymmetric, rank 2 tensors, also called (anti)symmetrized tensor product. What is the dimension of spaces $E \otimes E$, $(E \otimes E)_S$, $(E \otimes E)_A$?

A. Real and quaternionic representations

- Consider a compact group G . If $D(g)$ is a representation of G , show that $D^{-1T}(g)$ is also a representation, called the *contragredient representation*.
- Recall briefly why one may assume with no loss of generality that the representations of G are unitary, which we assume in the following.
Show that the contragredient representation is then identical to the complex conjugate one.
- Suppose that the unitary representation D is (unitarily) equivalent to its contragredient (or conjugate) representation. Show that there exists a unitary matrix S such that

$$D = SD^{-1T}S^{-1} \quad (2-122)$$

- Show that (2-122) implies that the bilinear form S is invariant.
Is this form degenerate?

5. Using (2-122) show that

$$DSS^{-1T} = SS^{-1T}D . \quad (2-123)$$

6. Show that if D is irreducible, $S = \lambda S^T$, with $\lambda^2 = 1$.

7. Conclude that the invariant form S is either symmetric or antisymmetric.

In the former case (S symmetric), the representation is called *real*, in the latter (S antisymmetric), it is called *pseudoreal* (or *quaternionic*). One may prove that in the former case, there exists a basis on \mathbb{R} in which the representation matrices are real, and that no such basis exists in the latter case.

8. Do you know an example of the second case?

B. Frobenius–Schur indicator

1. Let G be a finite or compact Lie group. Its irreducible representations are labelled by an index ρ and one denotes $\chi^{(\rho)}(g)$ their character. Let $\chi(g)$ be the character of some arbitrary representation, reducible or not.

(a) For any function F on the finite G , one denotes $\langle F \rangle$ its group average

$$\langle F \rangle = \frac{1}{|G|} \sum_{g \in G} F(g) . \quad (2-124)$$

How to extend that definition to the case of a compact Lie group (and a continuous function F)?

(b) - Recall why $\langle \chi \rangle$ is an integer and what it means.

- If $\bar{\rho}$ denotes the conjugate representation of the irreducible representation ρ , recall why $\langle \chi^{(\rho)} \chi^{(\bar{\rho})} \rangle = 1$ and what it implies on the decomposition of $\rho \otimes \bar{\rho}$ into irreducible representations.

(c) Show that an irreducible representation ρ is equivalent to $\bar{\rho}$ iff

$$\left\langle (\chi^{(\rho)}(g))^2 \right\rangle = 1 .$$

Evaluate this expression if ρ is not equivalent to $\bar{\rho}$.

2. We now consider a representation $D^{(\rho)}$ acting in a space E , and its tensor square $D^{(\rho) \otimes 2} \equiv D^{(\rho)} \otimes D^{(\rho)}$, which acts on rank 2 tensors of $E \otimes E$.

(a) Write explicitly the action of $D^{(\rho) \otimes 2}$ on a tensor $t = \{t^{ij}\}$,

$$t^{ij} \mapsto t'^{ij} = \dots$$

- (b) Show that any rank 2 tensor, $t = \{t^{ij}\}$, is the sum of a symmetric tensor t_S and of an antisymmetric one t_A , transforming under independent representations. Write explicitly the transformation matrices, paying due care to the symmetry properties of the tensors under consideration.
- (c) Show that the characters of the representations of symmetric and antisymmetric tensors are respectively

$$\chi^{(\rho \otimes \rho)_S^A}(g) = \frac{1}{2} ((\chi^{(\rho)}(g))^2 \pm \chi^{(\rho)}(g^2)) . \quad (2-125)$$

- (d) What is the value of these characters for $g = e$, the identity in the group? Could this result have been anticipated?
3. One then defines the Frobenius–Schur indicator of the irreducible representation ρ by

$$\text{ind}(\rho) = \langle \chi^{(\rho)}(g^2) \rangle . \quad (2-126)$$

- (a) Using the results of 2., show that one may write

$$\text{ind}(\rho) = \langle \chi^{(\rho \otimes \rho)_S} \rangle - \langle \chi^{(\rho \otimes \rho)_A} \rangle .$$

- (b) Using the results of 1., show that

$$\langle (\chi^{(\rho)}(g))^2 \rangle = \langle \chi^{(\rho \otimes \rho)_S} \rangle + \langle \chi^{(\rho \otimes \rho)_A} \rangle$$

takes the value 0 or 1, depending on the case: discuss.

- (c) - Show that $\langle \chi^{(\rho \otimes \rho)_S} \rangle$ and $\langle \chi^{(\rho \otimes \rho)_A} \rangle$ are non negative integers and give a certain multiplicity to be discussed.
 - Finally show that the Frobenius–Schur indicator of (2-126) can take only the three values 0 et ± 1 according to cases to be discussed.
- (d) What is the relation between this discussion and that of part A?

4. \star We now restrict to the case of a finite group G . For any $h \in G$, we define $Q(h) := \sum_{\rho} \text{ind}(\rho) \chi^{(\rho)}(h)$. Prove the
Theorem $Q(h) = \#\{g \in G | g^2 = h\}$