Chapter 1

Groups. Lie groups and Lie algebras

1.1 Generalities on groups

1.1.1 Definitions and first examples

Let us consider a group G, with an operation denoted $.$, $\times$ or $+$ depending on the case, a neutral element (or "identity") e (or 1 or I or 0), and an inverse g^{-1} (or $-a$). If the operation is commutative, the group is called *abelian*. If the groupe is finite, *i.e.* has a finite number of elements, we call that number the *order* of the group and denote it by $|G|$. In these lectures we will be mainly interested in infinite groups, discrete or continuous.

Examples (that the physicist may encounter ...)

- Finite groups
  - the cyclic group $\mathbb{Z}_p$ of order p, considered geometrically as the invariance rotation group of a circle with p marked equidistant points, or as the multiplicative group of p-th roots of the unity, $\{e^{2i\pi q/p}\}$, $q = 0, 1, \cdots, p-1$, or as the additive group of integers modulo p;
  - the groups of rotation invariance and the groups of rotation *and* reflexion invariance of regular solids or of regular lattices, of great importance in solid state physics and crystallography;
  - the permutation group S_n of n objects, called also the symmetric group, of order $n!$;
  - the homotopy groups, to be encountered soon, are other examples, and there are many others.
- Discrete infinite groups.
  The simplest example is the additive group $\mathbb{Z}$. Let us also mention the translation groups of regular lattices.

  Also the groups generated by reflexions in a finite number of hyperplanes of the Euclidean space $\mathbb{R}^n$, that are finite or infinite depending on the *arrangement* of these hyperplanes, see Weyl groups in Chap. 4.

Another important example is the modular group $PSL(2,\mathbb{Z})$ of matrices $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ with integer coefficients, of determinant 1, $ad-bc=1$, with matrices A and $-A$ identified. Given a 2-dimensional lattice in the complex plane generated by two complex numbers ω_1 and ω_2 of non real ratio (why ?), this group describes the changes of basis $(\omega_1,\omega_2)^T \to (\omega_1',\omega_2')^T = A(\omega_1,\omega_2)^T$ that leave invariant the area of the elementary cell ($\Im(\omega_2\omega_1^*)=\Im(\omega_2'\omega_1'^*)$) and their effect on $\tau=\omega_2/\omega_1$: $\tau\to(a\tau+b)/(c\tau+d)$. This group plays an important role in mathematics in the study of elliptic functions, modular forms, etc, and in physics, in string theory and conformal field theory …

- Continuous groups. We shall be dealing only with matrix groups of finite dimension, *i.e.* subgroups of the linear groups $\mathrm{GL}(n,\mathbb{R})$ ou $\mathrm{GL}(n,\mathbb{C})$, for some n. In particular
  - $\mathrm{U}(n)$, the group of complex unitary matrices, $UU^\dagger=I$, which is the invariance group of the sesquilinear form $(x,y)=\sum x^{*i}y^i$;
  - $\mathrm{SU}(n)$ its unimodular subgroup, of unitary matrices of determinant $\det U=1$;
  - $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ are orthogonal groups of invariance of the symmetric bilinear form $\sum_{i=1}^n x_iy_i$. Matrices of $\mathrm{SO}(n)$ have determinant 1 ;
  - $\mathrm{U}(p,q)$, $\mathrm{SU}(p,q)$, resp. $\mathrm{O}(p,q)$, $\mathrm{SO}(p,q)$, invariance groups of a sesquilinear, resp. bilinear form, of signature $((+)^p,(-)^q)$ (*e.g.* the Lorentz group O(3,1)).

    Most often one considers groups $\mathrm{O}(n,\mathbb{R})$, $\mathrm{SO}(n,\mathbb{R})$ of matrices with real coefficients, but groups $\mathrm{O}(n,\mathbb{C})$, $\mathrm{SO}(n,\mathbb{C})$ of invariance of the same bilinear form over the complex numbers may also play a role.
  - $\mathrm{Sp}(2n,\mathbb{R})$: Let Z be the matrix $2n\times 2n$ made of a diagonal of n blocks $i\sigma_2$: $Z=\mathrm{diag}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$, and consider the bilinear *skew-symmetric* form

    $$(X,Y)=X^TZY=\sum_{i=1}^n(x_{2i-1}y_{2i}-y_{2i-1}x_{2i})\ . \tag{1.1}$$

    The symplectic group $\mathrm{Sp}(2n,\mathbb{R})$ is the group of real $2n\times 2n$ matrices B that preserve that form: $B^TZB=Z$. That form appears naturally in Hamiltonian mechanics with the symplectic 2-form $\omega=\sum_{i=1}^n dp_i\wedge dq_i=\frac12 Z_{ij}d\xi_i\wedge d\xi_j$ in the coordinates $\xi=(p_1,q_1,p_2,\cdots,q_n)$; ω is invariant by action of $\mathrm{Sp}(2n,\mathbb{R})$ on ξ. For $n=1$, verify that Sp(2,$\mathbb{R}$)=SL(2,$\mathbb{R}$).

    One may also consider the complex symplectic group $\mathrm{Sp}(2n,\mathbb{C})$. A related group, often denoted $\mathrm{Sp}(n)$ but that I shall denote $\mathrm{USp}(n)$ to avoid confusion with the previous ones, the *unitary symplectic group*, is the invariance group of a Hermitian quaternionic form, $\mathrm{USp}(n)=\mathrm{U}(2n)\cap\mathrm{Sp}(2n,\mathbb{C})$ See Appendix A.
  - the group of displacements in $\mathbb{R}^3$, and groups obtained by adjoining dilatations, and then inversions with respect to a point;
  - the group of *conformal transformations*, *i.e.* angle preserving, in $\mathbb{R}^n$ (see Problem at the end of this chapter).
  - the Galilean group of transformations $\mathbf{x}'=R\mathbf{x}+\mathbf{v}t+\mathbf{x}_0$, $t'=t+t_0$,
  - the Poincaré group, in which translations are adjoined to the Lorentz group O(3,1),
  - etc etc.

1.1.2 Conjugacy classes of a group

In a group G we define the following equivalence relation:

$$a \sim b \ \text{ iff } \ \exists\, g \in G \ : \quad a = g.b.g^{-1} \tag{1.2}$$

and the elements a et b are said to be *conjugate*.

The equivalence classes (*conjugacy classes*) that follow provide a partition of G, since any element belongs to a unique class. For a finite group, the different classes generally have different orders (or cardinalities). For instance, the class of the neutral element e has a unique element, e itself.

We have already noted (in Chap. 0) that in the rotation group $SO(3)$, a conjugacy class is characterized by the rotation angle ψ (around some unitary vector $\mathbf{n}$). But this notion is also familiar in the group $\mathrm{U}(n)$, where a class is characterized by an unordered n-tuple of eigenvalues $(e^{i\alpha_1}, \dots, e^{i\alpha_n})$. This notion of class plays an important role in the discussion of representations of groups and will be abundantly illustrated in the following.

What are the conjugacy classes in the symmetric group S_n ? One proves easily that any permutation σ of S_n decomposes into a product of cycles (cyclic permutations) on distinct elements. (To show that, construct the cycle $(1, \sigma(1), \sigma^2(1), \cdots)$; then, once back to 1, construct another cycle starting from a number not yet reached, etc.). Finally if σ is made of p_1 cycles of length 1, p_2 of length 2, etc, with $\sum ip_i = n$, one writes $\sigma \in [1^{p_1}2^{p_2}\cdots]$, and one may prove that this decomposition into cycles characterizes the conjugacy classes : two permutations are conjugate iff they have the same decomposition into cycles.

1.1.3 Subgroups

The notion of subgroup H, subset of a group G itself endowed with a group structure, is familiar. The subgroup is *proper* if it is not identical to G. If H is a subgroup, for any $a \in G$, the set $a^{-1}.H.a$ of elements of the form $a^{-1}.h.a$, $h \in H$ forms also a subgroup, called *conjugate subgroup* to H.

Examples of particular subgroups are provided by :

- the center Z :

  In a group G, the *center* is the subset Z of elements that commute with all other elements of G:

  $$Z = \{a \mid \forall g \in G, a.g = g.a\} \tag{1.3}$$

  Z is a subgroup G, and is proper if G is nonabelian. Examples: the center of the group $GL(2, \mathbb{R})$ of regular 2×2 matrices is the set of matrices multiple of I; the center of $\mathrm{SU}(2)$ is the group $\mathbb{Z}_2$ of matrices $\pm I$ (check by direct calculation).

- the centralizer of an element a :
  The *centralizer* (or *commutant*) of a given element a of G is the set of elements of G that commute with a.

  $$Z_a = \{g \in G | a.g = g.a\} \tag{1.4}$$

The commutant Z_a is never empty: it contains at least the subgroup generated by a. The center Z is the intersection of all commutants. Example: in the group $GL(2,\mathbb{R})$, the commutant of the Pauli matrix $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the abelian group of matrices of the form $a\mathbf{I} + b\sigma_1$, $a^2 - b^2 \neq 0$.

- More generally, given a subset S of a group G, its *centralizer* $Z(S)$ and its *normalizer* $N(S)$ are the subgroups commuting respectively individually with any element of S or globally with S as a whole

$$\begin{aligned} Z(S) = & \quad \{y : \forall s \in S \quad y.s = s.y\} & (1.5) \\ N(S) = & \quad \{x : x^{-1}.S.x = S\} \,. & (1.6) \end{aligned}$$

1.1.4 Homomorphism of a group G into a group G'

An *homomorphism* of a group G into a group G' is a map ρ of G into G' which respects the composition law:

$$\forall g, h \in G, \qquad \rho(g.h) = \rho(g).\rho(h) \tag{1.7}$$

In particular, ρ maps the neutral element of G onto that of G', and the inverse of g onto that of $g' = \rho(g)$: $\rho(g^{-1}) = (\rho(g))^{-1}$.

An example of homomorphism that we shall study in great detail is that of a linear representation of a group, whose definition has been given in Chap. 00 and that we return to in Chap. 2.

The *kernel* of the homomorphism, denoted $\ker\rho$, is the set of *preimages* of the neutral element in G': $\ker\rho = \{x \in G : \rho(x) = e'\}$. It is a subgroup of G.

For example, the parity (or signature) of a permutation of S_n defines an homomorphism from S_n into $\mathbb{Z}_2$. Its kernel is made of even permutations: this is *alternating group* A_n of order $n!/2$.

1.1.5 Cosets with respect to a subgroup

Consider a subgroup H of a group G. We define the following relation between elements of G :

$$g \sim g' \iff g.{g'}^{-1} \in H, \tag{1.8}$$

which may also be rewritten as

$$g \sim g' \iff \exists h \in H \ : \ g = h.g' \quad \text{ou encore} \quad g \in H.g' \,. \tag{1.9}$$

This is an equivalence relation (check !), called the *right equivalence.* One defines in a similar way the *left equivalence* by

$$g \sim_{\mathrm{L}} g' \iff g^{-1}.g' \in H \Leftrightarrow g \in g'.H. \tag{1.10}$$

This relation (say, right) defines equivalence classes that give a partition of G; if g_j is a representative of class j, that class, called *right-coset*, may be denoted $H.g_j$. The elements of H form by themselves a coset. The set of (say right) cosets is denoted G/H and called the (right) coset "space". If H is of finite order $|H|$, all cosets have $|H|$ elements, and if G is itself of finite order $|G|$, it is partitioned into $|G|/|H|$ classes, and one obtains the Lagrange theorem as a corollary : the order $|H|$ of any subgroup H divides that of G, and the ratio $|G|/|H|$ is the order (=cardinality) of the coset space G/H.

The left equivalence gives rise in general to a different partition. For example, the group S_3 has a $\mathbb{Z}_2$ subgroup generated by the permutation of the two elements 1 et 2. Exercise: check that the left and right cosets do not coincide.

1.1.6 Invariant subgroups

Consider a group G with a subgroup H. H is an *invariant subgroup* (one also says *normal*) if one of the following equivalent properties holds true

- $\forall g \in G$, $\forall h \in H$, $ghg^{-1} \in H$.
- left and right cosets coincide;
- H is equal to all its conjugates $\forall g \in G, \quad gHg^{-1} = H$.

Exercise: check the equivalence of these three assertions.

The important property to remember is the following:

• *If H is an invariant subgroup G, the coset space G/H may be given a group structure, and is called the quotient group.*

Note that in general one cannot consider the quotient group G/H as a subgroup of G.

Let us sketch the proof. If $g_1 \sim g'_1$ and $g_2 \sim g'_2$, $\exists h_1, h_2 \in H \quad : \quad g_1 = h_1.g'_1,\ g_2 = g'_2.h_2$, hence $g_1.g_2 = h_1.(g'_1.g'_2).h_2$ *i.e.* $g_1.g_2 \sim g'_1.g'_2$ et $g_1^{-1} = {g'}_1^{-1}.h_1^{-1} \sim {g'}_1^{-1}$. The equivalence relation is thus compatible with the composition and inverse operations, and if $[g_1]$ and $[g_2]$ denote two cosets, one defines $[g_1].[g_2] = [g_1.g_2]$ where on the right hand side (rhs), one takes any representative g_1 of the coset $[g_1]$ and g_2 of $[g_2]$; and likewise for the inverse. Thus the group structure passes to the coset space. The coset made by H is the neutral element in the quotient group.

Example of an invariant subgroup: The *kernel* of an homomorphism ρ of G into G' is an invariant subgroup: show that the quotient group is isomorphic to the image $\rho(G) \subset G'$ of G by ρ.

1.1.7 Simple, semi-simple groups

A group is *simple* if it has no non-trivial invariant subgroup (non trivial, *i.e.* different from $\{e\}$ and from G itself). A group is *semi-simple* if it has no non-trivial *abelian* invariant subgroup. Any simple group is obviously semi-simple.

This notion is important in representation theory and in the classification of groups.

Examples : The rotation group in two dimensions is not simple, and not even semi-simple (why?). The group SO(3) is simple (non trivial proof, see below, section 1.2.2). The group SU(2) is neither simple nor semi-simple, as it contains the invariant subgroup $\{I, -I\}$. The group S_n is not simple, for $n > 2$ (why?).

1.2 Continuous groups. Topological properties. Lie groups.

A continuous group (one also says a *topological group*) is a topological space (hence endowed with a basis of neighbourhoods that allows us to define notions of continuity etc[1]) with a group structure, such that the composition and inverse operations $(g,h)\mapsto g.h$ et $g\mapsto g^{-1}$ are continuous functions. In other words, if g' is nearby g (in the sense of the topology of G), and h' nearby h, then $g'.h'$ is nearby $g.h$ and ${g'}^{-1}$ is nearby g^{-1}.

The matrix groups presented at the beginning of this chapter all belong to this class of topological groups, but there are also groups of "infinite dimension" like the group of diffeomorphisms invoked in General Relativity, or of gauge transformations in gauge theories.

Let us first study some topological properties of these continuous groups.

1.2.1 Connectivity

A group may be connected or not. If G is not connected, the *connected component of the identity* (*i.e.* of the neutral element) is an invariant subgroup.

One may be interested in the connectivity in the general topological sense (a topological space E is connected if its only subspaces that are both open and closed are E and $\emptyset$), but we shall be mainly concerned by the *arc connectivity*: for any pair of points, there exists a continuous path in the space (here the group) that joins them. Show that the connected component of the identity is an invariant subgroup for both definitions. Ref. [K-S, Po].

Examples. O(3) is disconnected and the connected component of the identity is SO(3); for the Lorentz group $\mathcal{L}$=O(3,1), the connected component of I is its proper orthochronous subgroup $\mathcal{L}_+^\uparrow$, (see Chap. 0), the other "sheets" then result from the application on it of parity P, of time reversal T and their product $PT\dots$

1.2.2 Simple connectivity. Homotopy group. Universal covering

This notion should not be mistaken for the previous one.

As it does not apply only to groups, consider first an arbitrary topological space E. Let us consider closed paths drawn in the space (or "loops"), with a fixed end-point x_0, *i.e.* continuous maps $x(t)$ from $[0,1]$ into E such that $x(0)=x(1)=x_0$. Given two such closed paths $x_1(.)$ et $x_2(.)$ from x_0 to x_0, can one deform them continuously into one another? In other words, is there a continuous function $f(t,\xi)$ of two variables $t,\xi\in[0,1]$, taking its values in the space E, such that

$$\begin{array}{rrclcl}
\forall\xi\in[0,1] & f(0,\xi) & = & f(1,\xi)=x_0 & : & \text{closed paths} \\
\forall t\in[0,1] & f(t,0) & = & x_1(t) \qquad f(t,1)=x_2(t) & : & \text{interpolation}\ .
\end{array} \tag{1.11}$$

If this is the case, one says that the paths x_1 and x_2 are *homotopic* (this is an equivalence relation between paths), or equivalently that they belong to the same *homotopy class*, see fig. 1.1.

---

[1] See Appendix B for a reminder of some points of vocabulary…

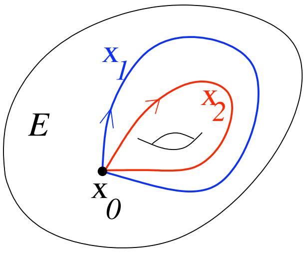


Figure 1.1: The paths x_1 et x_2 are homotopic. But none of them is homotopic to the "trivial" path that stays at x_0. The space is not simply connected.

One may also *compose* paths: If $x_1(.)$ and $x_2(.)$ are two paths from x_0 to x_0, their product $x_2 \circ x_1$ also goes from x_0 to x_0 by following first x_1 and then x_2. The inverse path of $x_1(.)$ for that composition is the same path but followed in the reverse direction: $x_1^{-1}(t) := x_1(1-t)$. Both the composition and the inverse are compatible with homotopy: if $x_1 \sim x_1'$ and $x_2 \sim x_2'$, then $x_2 \circ x_1 \sim x_2' \circ x_1'$ and $x_1^{-1} \sim {x_1'}^{-1}$. These operations thus pass to classes, giving the set of homotopy classes a group structure: this is *homotopy group* $\pi_1(E, x_0)$. Hence, a representative of the identity class is given by the "trivial" path, $x(t) = x_0$, $\forall t$. One finally shows that in a connected space, homotopy groups relative to different end-points x_0 are isomorphic; if E is a connected group, see below, one may take for example the base point x_0 to be the identity $x_0 = e$. One may thus talk of **the** homotopy group (or *fundamental group*) $\pi_1(E)$. For more details, see for example [Po], [DNF].

If all paths from x_0 to x_0 may be continuously contracted into the trivial path $\{x_0\}$, $\pi_1(E)$ is trivial, and E is said to be *simply connected*. In the opposite case, one may prove and we shall admit that one may construct a space $\tilde{E}$, called *the universal covering space* of E, such that $\tilde{E}$ is simply connected and that **locally**, E and $\tilde{E}$ are *homeomorphic*. This means that there exists a continuous and surjective mapping p from $\tilde{E}$ to E such that any point x in $\tilde{E}$ has a neighborhood V_x and that $V_x \mapsto p(V_x)$ is a homeomorphism, *i.e.* a bicontinuous bijection [2]. The universal covering space $\tilde{E}$ of E is unique (up to a homeomorphism).

Let us now restrict ourselves to the case where $E = G$, a topological group. Then one shows that its covering $\widetilde{G}$ is also a group, the *universal covering group*, and moreover, that the map p is a homomorphism of $\widetilde{G}$ into G. Its kernel which is an invariant subgroup of $\widetilde{G}$, is proved to be isomorphic to the homotopy group $\pi_1(G)$ ([Po], sect. 51). The quotient group is isomorphic to G

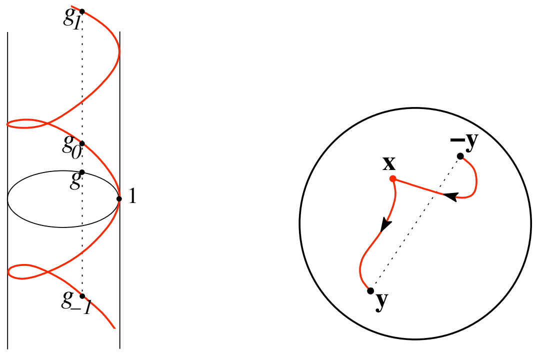
$$\widetilde{G}/\pi_1(G) \simeq G \ , \tag{1.12}$$

(according to a general property of the quotient group by the kernel of an homomorphism, cf. sect. 1.1.6).

One may construct the universal covering group $\widetilde{G}$ by considering paths that join the identity e to a point g, and their equivalence classes under continuous deformation **with fixed ends**. $\widetilde{G}$ is the set of these equivalence classes. It is a group for the multiplication of paths defined as follows: if two paths $g_1(t)$ and $g_2(t)$ join e to g_1 and to g_2 respectively, the path $g_1(t).g_2(t)$ joins e to $g_1.g_2$. This composition law is compatible with equivalence

---

[2] "bicontinuous" means that the map and its inverse are both continuous.



(a) (b)

Figure 1.2: (a) The group U(1), identified with the circle and its universal covering group $\mathbb{R}$, identified with the helix. An element $g \in$ U(1) is lifted to points $\cdots, g_{-1}, g_0, g_1, \cdots$ on the helix. (b) In the ball B^3 representing SO(3), the points $\mathbf{y}$ and $-\mathbf{y}$ of the surface are identified. A path going from $\mathbf{x}$ to $\mathbf{x}$ via $\mathbf{y}$ and $-\mathbf{y}$ is thus closed but non contractible: SO(3) is non simply connected.

and gives $\widetilde{G}$ a group structure and one shows that $\widetilde{G}$ is simply connected (cf. [Po] sect. 51). The projection p of $\widetilde{G}$ into G associates with any class of paths their common end-point. One may verify that this is indeed a local homeomorphism and a group homomorphism, and that its kernel is the homotopy group $\pi_1(G)$.

Example : The group G =U(1) of complex numbers of modulus 1, seen as the unit circle S^1, is non simply connected: a path from the identity 1 to 1 may wind an arbitrary number of times around the circle and this (positive or negative) winding number characterizes the different homotopy classes: the homotopy group is $\pi_1(\mathrm{U}(1)) = \mathbb{Z}$. The group $\widetilde{G}$ is nothing else than the additive group $\mathbb{R}$ and may be visualised as a helix above U(1). The quotient is $\mathbb{R}/\mathbb{Z} \simeq$ U(1), which must be interpreted as the fact that a point of U(1), *i.e.* an angle, is a real number modulo an integer multiple of 2π. One may also say that $\pi_1(S^1) = \mathbb{Z}$. More generally one may convince oneself that for spheres, $\pi_1(S^n)$ is trivial (all loops are contractible) as soon as $n > 1$.

Another fundamental example: The rotation group SO(3) is not simply connected, as foreseen in Chap. 0. To see this fact, represent the rotation $R_{\mathbf{n}}(\psi)$ by the point $\mathbf{x} = \tan\frac{\psi}{4}\mathbf{n}$ of an auxiliary space $\mathbb{R}^3$; all these points are in the ball B^3 of radius 1, with the identity rotation at the center and rotations of angle π on the sphere $S^2 = \partial B^3$, but because of $R_{\mathbf{n}}(\pi) = R_{-\mathbf{n}}(\pi)$, (see Chap 0, sect. 1.1), diametrically opposed points must be identified. It follows that there exists in SO(3) closed loops that are non contractible: a path from $\mathbf{x}$ to $\mathbf{x}$ passing through two diametrically opposed points on the sphere S^2 must be considered as closed but is not contractible (Fig. 1.2.b). There exist two classes of non homotopic closed loops from $\mathbf{x}$ to $\mathbf{x}$ and the group SO(3) is doubly connected. Its homotopy group is $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}_2$. In fact, we already know the universal covering group of SO(3): it is the group SU(2), which has been shown to be homeomorphic to the sphere S^3, hence is simply connected, and for which there exists a homomorphism mapping it to SO(3), according to $\pm U_{\mathbf{n}}(\psi) = \pm(\cos\frac{\psi}{2} - i\sin\frac{\psi}{2}\sigma.\mathbf{n}) \mapsto R_{\mathbf{n}}(\psi)$, see Chap. 0, sect. 1.2.

This property of SO(3) to be non simply connected may be illustrated by various home experiments, the precise interpretation of which may not be obvious, such as "Dirac's belt" and "Feynman's plate", see `http://gregegan.customer.netspace.net.au/APPLETS/21/21.html` and `http://www.math.utah.edu/~palais/links.html` for nice animations, and V. Stojanoska et O. Stoytchev, *Mathematical Magazine*, **81**, 2008, 345-357, for a detailed discussion involving the *braid group.*

The same visualisation of rotations by the interior of the unit ball also permits to understand the above assertion that the group SO(3) is simple. Suppose it is not, and let $R = R_{\mathbf{n}}(\psi)$ be an element of an invariant subgroup of SO(3), which also contains all the conjugates of R (by definition of an invariant subgroup). These conjugates are represented by points of the sphere of radius $\tan\psi/4$. The invariant subgroup containing $R_{\mathbf{n}}(\psi)$ and points that are arbitrarily close to its inverse $R_{-\mathbf{n}}(\psi)$ contains also points that are arbitrarily close to the identity, which by conjugation, fill a small ball in the vicinity of the identity. It remains to show that the products of such elements fill all the bowl, *i.e.* that the invariant subgroup may only be SO(3) itself; this is in fact true for any connected Lie group, as we shall see below.

Other examples: classical groups. One may prove that

- the groups SU(n) are all simply connected, for any n, whereas $\pi_1(\mathrm{U}(n)) = \mathbb{Z}$;
- for the group SO(2)≅ U(1), we have seen that $\pi_1(\mathrm{SO}(2))= \mathbb{Z}$;
- for any $n > 2$, SO(n) is doubly connected, $\pi_1(\mathrm{SO}(n))= \mathbb{Z}_2$, and its covering group is called Spin(n). Hence Spin(3)=SU(2).

The notion of homotopy, *i.e.* of continuous deformation, that we have just applied to loops, *i.e.* to maps of S^1 into a manifold $\mathcal{V}$ (a group G here), may be extended to maps of a sphere S^n into $\mathcal{V}$. Even though the composition of such maps is less easy to visualise, it may be defined and is again compatible with homotopy, leading to the definition of the homotopy group $\pi_n(\mathcal{V})$. For example $\pi_n(S^n) = \mathbb{Z}$. See [DNF] for more details and the determination of these groups π_n. This notion is important in physics to describe topological defects, solitons, instantons, monopoles, etc. See Fig. 1.3 for *vortex* and *anti-vortex* configurations of unit vectors in 2 dimensions, of respective winding number (or *vorticity*) ±1.

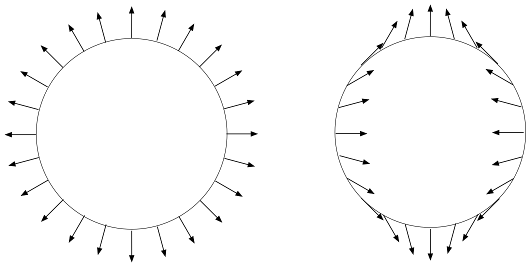


Figure 1.3: Two configurations of unit vectors realizing homotopically non trivial mapping $S^1 \to S^1$. Those are respectively the *vortex* and *anti-vortex* of the XY model of statistical mechanics, see for example `http://www.ibiblio.org/e-notes/Perc/xy.htm` for more details and nice figures.

1.2.3 Compact and non compact groups

If the domain $\mathcal{D}$ in which the parameters of the group G take their values is compact, G is said to be a *compact group.*

Recall some of the many equivalent characterizations of a compact space E. From any infinite sequence one may extract a converging subsequence. Given a covering of E by a set of open sets U_i, E may be covered by a *finite* number of them. Any continuous function on E is bounded, etc. For a subset $\mathcal{D}$ of $\mathbb{R}^d$, being compact is equivalent to being closed and bounded.

Examples. The unitary groups U(n) and their subgroups SU(n), O(n), SO(n), USp($n/2$) (n even), are compact. The groups SL($n, \mathbb{R}$) or SL($n, \mathbb{C}$), Sp($n, \mathbb{R}$) or Sp($n, \mathbb{C}$), the translation group in $\mathbb{R}^n$, the Galilean group, the Lorentz and Poincaré groups are not, why ?

1.2.4 Invariant measure

When dealing with a *finite* group, one often considers sums over all elements of the group and makes use of the "rearrangement lemma", in which one writes

$$\sum_{g\in G} f(g'g) = \sum_{h=g'g\in G} f(g'g) = \sum_{g\in G} f(g) \ ,$$

(left invariance), the same thing with $g'g$ changed into gg' (right invariance), and also

$$\sum_{g\in G} f(g^{-1}) = \sum_{g^{-1}\in G} f(g^{-1}) = \sum_{g\in G} f(g) \ .$$

Can one do similar operations in continuous groups, the finite sum being replaced by an integral, which converges and enjoys the same invariances ? This requires the existence of an integration measure, with left and right invariance, and invariance under inversion:

$$d\mu(g) = d\mu(g'.g) = d\mu(g.g') = d\mu(g^{-1})$$

such that $\int d\mu(g) f(g)$ be finite for any continuous function f on the group.

One may prove (and we admit) that

- *if the group is compact, such a measure exists and is unique up to a normalization.*

This is the *Haar measure.*

For example, in the unitary group U(n), one may construct explicitly the Haar measure, using the method proposed in chap. 0, Appendix 0: one first defines a metric on U(n) by writing $ds^2 = \mathrm{tr}\ dU.dU^\dagger$ in any parametrization; this metric is invariant under $U \to UU'$ or $U \to U'U$ and by $U \to U^{-1} = U^\dagger$; the measure $d\mu(U)$ that follows has the same properties. See Appendix C for the explicit calculation for SU(2) and U(n), and more details in the TD.

Conversely if the group is non compact, left and right measures may still exist, they may even coincide, (for non compact abelian or semi-simple groups) but their integral over the group diverges.

Thus, if G is *locally compact,* (*i.e.* any point has a basis of compact neighbourhoods), one proves that there exists a left invariant measure, unique up to a multiplicative constant. There exists also a right invariant measure, but they may not coincide. For example, take

$$G = \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \Big| x, y \in \mathbb{R}, y > 0 \right\}$$

one easily checks that $d\mu_L(g) = y^{-2} dxdy\,, \quad d\mu_R(g) = y^{-1} dxdy$ are left and right invariant measures, respectively, and that their integrals diverge. See [Bu].

1.2.5 Lie groups

Imposing more structure on a continuous group leads us in a natural way to the notion of *Lie groups*.

According to the usual definition, a Lie group is a topological group which is also a differentiable manifold and such that the composition and inverse operations $G \times G \to G$ and $G \to G$ are infinitely differentiable functions. One sometimes also requests them to be analytic real functions, *i.e.* functions for which the Taylor series converges to the function. That the two definitions coexist in the literature is a hint that the weakest (infinite differentiability) implies the strongest. In fact, according to a remarkable theorem (Montgomery et Zippen,1955), much weaker hypotheses suffice to ensure the property of Lie group.
A topological connected group which is locally homeomorphic to $\mathbb{R}^d$, for some *finite* d, is a Lie group. In other words, the existence of a *finite* number of local coordinates, together with the properties of being a topological group (continuity of the group operations), are sufficient to imply the analyticity properties ! [3] This shows that the structure of Lie group is quite powerful and rigid. There exist, however, infinite dimensional Lie groups.

To avoid a mathematical discussion unnecessary for our purpose, we shall restrict ourselves to continuous groups of finite size matrices. In such a group, the matrix elements of $g \in G$ depend continuously on real parameters $(\xi^1, \xi^2, \cdots \xi^d) \in \mathcal{D} \subset \mathbb{R}^d$, and in the group operations $g(\xi'') = g(\xi').g(\xi)$, and $g(\xi)^{-1} = g(\xi'')$, the ξ''^i are continuous (in fact analytic) functions of the ξ^j (and ξ'^j). Such a group is called a *Lie group*, and d is its *dimension*.

More precisely, in the spirit of differential geometry, one has in general to introduce several domains $\mathcal{D}_j$, with continuous (in fact analytic) transition functions between coordinate charts, etc.

Examples : all the matrix groups presented in §1.1 are Lie groups. Check that the dimension of $\mathrm{U}(n)$ is n^2, that of $\mathrm{SU}(n)$ is n^2-1, that $\mathrm{O}(n)$ or $\mathrm{SO}(n)$ is $n(n-1)/2$. What is the dimension of $\mathrm{Sp}(2n,\mathbb{R})$? of the Galilean group in $\mathbb{R}^3$? of the Lorentz and Poincaré groups ?

Show that $\dim(\mathrm{Sp}(2n,\mathbb{R}))=\dim(\mathrm{USp}(n))=\dim(\mathrm{SO}(2n+1))$, and we shall see below in chap. 3 that this is not an accident.

The study of a Lie group and of its representations involves two steps: first a local study of its tangent space in the vicinity of the identity (its Lie algebra), and then a global study of its topology, *i.e.* an information not provided by the local study.

1.3 Local study of a Lie group. Lie algebra

1.3.1 Algebras and Lie algebras de Lie. Definitions

Let us first recall the definition of an algebra.
An *algebra* is a vector space over a field (for physicists, $\mathbb{R}$ or $\mathbb{C}$), endowed with a product denoted $X * Y$, (not necessarily associative), bilinear in X and Y

$$(\lambda_1 X_1 + \lambda_2 X_2) * Y = \lambda_1 X_1 * Y + \lambda_2 X_2 * Y \qquad (1.13)$$

$$X * (\mu_1 Y_1 + \mu_2 Y_2) = \mu_1 X * Y_1 + \mu_2 X * Y_2 \ . \qquad (1.14)$$

---

[3] For an elementary example of such a phenomenon, consider a function f of one real variable, satisfying $f(x)f(y) = f(x+y)$. Under the only assumption of continuity, show that $f(x) = \exp kx$, hence that it is analytic !

Examples: the set $M(n,\mathbb{R})$ or $M(n,\mathbb{C})$ of $n\times n$ matrices with real, resp. complex coefficients, is an associative algebra for the usual matrix product. The set of vectors of $\mathbb{R}^3$ is a (non associative !) algebra for the vector product (denoted $\wedge$ in the French literature, and $\times$ in the anglo-saxon one) .

A *Lie algebra* is an algebra in which the product denoted $[X,Y]$ has the additional properties of being antisymmetric and of satisfying the Jacobi identity

$$[X,Y]=-[Y,X] \tag{1.15}$$

$$[X_1,[X_2,X_3]]+[X_2,[X_3,X_1]]+[X_3,[X_1,X_2]]=0\ . \tag{1.16}$$

Examples : Any associative algebra for a product denoted $*$, in particular any matrix algebra, is a Lie algebra for a product, the *Lie bracket*, defined by the commutator

$$[X,Y]=X*Y-Y*X\ .$$

The bilinearity and antisymmetry properties are obvious, and verifying the Jacobi identity takes one line. Another example: the space $\mathbb{R}^3$ with the above-mentionned vector product is in fact a Lie algebra, with the Jacobi identity following from the "double vector product" formula, $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})=(\mathbf{u}.\mathbf{w})\mathbf{v}-(\mathbf{u}.\mathbf{v})\mathbf{w}$.

1.3.2 Tangent space in a Lie group

Consider a Lie group G and a *one-parameter subgroup* $g(t)$, where t is a real parameter taking values in a neighborhood of 0, with $g(0)=e$; in other words, $g(t)$ is a curve in G, assumed to be differentiable, and passing through the identity, and one assumes that (for t near 0)

$$g(t_1)g(t_2)=g(t_1+t_2)\qquad g^{-1}(t)=g(-t)\ . \tag{1.17}$$

The composition law in this subgroup locally amounts to the addition of parameters t; thus, locally, this one-parameter subgroup is isomorphic to the abelian group $\mathbb{R}$. It is then natural to differentiate

$$g(t+\delta t)=g(t)g(\delta t)\quad\Leftrightarrow\quad g^{-1}(t)g(t+\delta t)=g(\delta t)\ . \tag{1.18}$$

As we have chosen to restrict ourselves to matrix groups, (with $e\equiv I$, the identity matrix), we may write the linear tangent map in the form

$$g(\delta t)=I+\delta tX+\cdots$$

which defines a vector X in the tangent space. One may also write

$$X=\frac{d}{dt}g(t)\Big|_{t=0}\ , \tag{1.19}$$

this is the velocity at $t=0$ (or at $g=e$) along the curve. Equation (1.18) thus reads

$$g'(t)=g(t)X\ . \tag{1.20}$$

As usual in differentiable geometry, (see Appendix B.3), the tangent space T_eG at e to the group G, which we denote from now on $\mathfrak{g}$, is the vector space generated by the tangent vectors to all one-parameter subgroups (*i.e.* all velocity vectors at $t = 0$). If coordinates ξ^α of G have been chosen in the vicinity of e ($\equiv I$), a tangent vector is a differential operator $X = X^\alpha \frac{\partial}{\partial \xi^\alpha}$. The dimension (as a vector space) of this tangent space is equal to the dimension of the (group) manifold defined above as the number of (real) parameters $\dim \mathfrak{g} = \dim G$.

In the case of a group $G \subset \mathrm{GL}(n, \mathbb{R})$ to which we are restricting ourselves, $X \in \mathfrak{g} \subset M(n, \mathbb{R})$, the set of real $n \times n$ matrices, and one may carry out all calculations in that algebra. In particular, one may integrate (1.20) as

$$g(t) = \exp tX = \sum_{n=0} \frac{t^n}{n!} X^n \ , \tag{1.21}$$

a converging sum. (In fact, the assumption that the group is a matrix group may be relaxed, provided one makes sense of the map exp from $\mathfrak{g}$ to G, a map that enjoys some of the usual properties of the exponential, see Appendix B.4.)

1.3.3 Relations between the tangent space $\mathfrak{g}$ and the group G

1. If G is the linear group $\mathrm{GL}(n, \mathbb{R})$, $\mathfrak{g}$ is the algebra of real $n \times n$ matrices, denoted $M(n, \mathbb{R})$. If G is the group of unitary matrices $\mathrm{U}(n)$, $\mathfrak{g}$ is the space of anti-Hermitian $n \times n$ matrices. Moreover they are traceless if $G = \mathrm{SU}(n)$. Likewise, for the orthogonal $\mathrm{O}(n)$ group, $\mathfrak{g}$ is made of skew-symmetric, hence traceless, matrices.
For the symplectic group $G = \mathrm{USp}(n)$, $\mathfrak{g}$ is generated by "anti-selfdual" quaternionic matrices, see Appendix A.
For each of these cases, check that the characteristic property (anti-Hermitian, skew-symmetric, traceless, ...) is preserved by the commutator, thus making $\mathfrak{g}$ a Lie algebra.

2. The exponential map plays an important role in the reconstruction of the Lie group G from its tangent space $\mathfrak{g}$. One may prove, and we admit, that

- the map $X \in \mathfrak{g} \mapsto e^X \in G$ is bijective in the neighborhood of the identity;
- it is surjective (= every element in G is reached) if G is connected and compact;
- it is injective (any $g \in G$ has only one antecedent) only if G is simply connected. An example of non-injectivity is provided by $G = \mathrm{U}(1)$, for which $\mathfrak{g} = i\mathbb{R}$ and all the $i(x + 2\pi k)$, $k \in \mathbb{Z}$ have the same image by exp. The converse is in general wrong: for example in SU(2) which is simply connected, if $\mathbf{n}$ is a unit vector, $e^{i\pi \mathbf{n}.\boldsymbol{\sigma}} = -I$, hence all elements $\pi i \mathbf{n}.\boldsymbol{\sigma}$ de $\mathfrak{g} = \mathrm{su}(2)$ have the same image!

⋆ Example of a non-compact group for which the exp map is non surjective: $G = \mathrm{SL}(2, \mathbb{R})$, for which $\mathfrak{g} = \mathrm{sl}(2, \mathbb{R})$, the set of real traceless matrices. For any $A \in \mathfrak{g}$, hence traceless, use its characteristic equation to show that $\mathrm{tr}\ A^{2n+1} = 0$, $\mathrm{tr}\ A^{2n} = 2(-\det A)^n$, hence $\mathrm{tr}\ e^A = 2\cosh\sqrt{-\det A} \geq -2$. There exist in G, however, matrices of trace < -2, for instance $\mathrm{diag}\,(-2, -\frac{1}{2})$.

⋆ For a non compact group, the exp map may still be useful. One may prove that any element of a matrix group may be written as the product of a finite number of exponentials of elements in its Lie algebra. [Cornwell p 151].

⋆ Observe that one still has $\det e^X = e^{\operatorname{tr} X}$, a property easily established if X belongs to the set of diagonalizable matrices. As the latter are dense in $M(d,\mathbb{R})$, the property holds true in general.

1.3.4 The tangent space as a Lie algebra

Let us now show that the tangent space $\mathfrak{g}$ of G at $e \equiv I$ has a Lie algebra structure. Given two one-parameter groups generated by two independent vectors X et Y of $\mathfrak{g}$, we measure their lack of commutativity by constructing their *commutator* (in a sense different from the usual one!) $g = e^{tX}e^{uY}e^{-tX}e^{-uY}$; for small $t \sim u$, that g is close to the identity, and may be written $g = \exp Z$, $Z \in \mathfrak{g}$. Compute Z to the first non trivial order

$$\begin{aligned} e^{tX}e^{uY}e^{-tX}e^{-uY} &= (I + tX + \tfrac{1}{2}t^2X^2)(I + uY + \tfrac{1}{2}u^2Y^2)(I - tX + \tfrac{1}{2}t^2X^2)(I - uY + \tfrac{1}{2}u^2Y^2) \\ &= I + (XY - YX)tu + O(t^3) \ . \end{aligned} \tag{1.22}$$

The computation has been carried out in the associative algebra of matrices, the neutral element being denoted I. All the neglected terms are of third order since $t \sim u$. To order 2, one thus sees the appearance of the commutator in the usual sense, $XY - YX$, *i.e.* the Lie bracket of matrices X and Y. In general, for an arbitrary Lie group, the bracket is defined by

$$e^{tX}e^{uY}e^{-tX}e^{-uY} = e^Z \quad , \quad Z = tu[X,Y] + O(t^3) \tag{1.23}$$

and one proves that this bracket has the properties (1.15) of a Lie bracket.

This fundamental result follows from a detailed discussion of the local form of the group operations in a Lie group ("Lie equations", see for example [OR]).

• Adjoint map in the Lie algebra $\mathfrak{g}$. Baker-Campbell-Hausdorff formula

Let us introduce a handy notation. For any $X \in \mathfrak{g}$, let $\operatorname{ad} X$ be the linear operator in the Lie algebra defined by

$$Y \mapsto (\operatorname{ad} X)Y := [X,Y] \ , \tag{1.24}$$

hence

$$(\operatorname{ad}^p X)Y = [X,[X,\cdots[X,Y]\cdots]]$$

with p brackets (commutators).

Given two elements X and Y in $\mathfrak{g}$, and e^X and e^Y the elements they generate in G, does there exist a $Z \in \mathfrak{g}$ such that $e^Xe^Y = e^Z$? The answer is yes, at least for X et Y small enough.

Note first that if $[X,Y] = 0$, the ordinary rules of computation apply and $Z = X + Y$. In general, the *Baker-Campbell-Hausdorff formula*, that we admit, gives an explicit expression of Z.

$$\begin{aligned} e^Xe^Y &= e^Z \\ Z &= X + \int_0^1 dt\psi(\exp \operatorname{ad} X\ \exp t \operatorname{ad} Y)Y \end{aligned} \tag{1.25}$$

where $\psi(.)$ is the function

$$\psi(u)=\frac{u\ln u}{u-1}=1+\frac{1}{2}(u-1)-\frac{1}{6}(u-1)^2+\cdots\ , \tag{1.26}$$

which is regular at $u=1$. The first terms in the expansion in powers of X and Y read explicitly

$$Z=X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}\Big([X,[X,Y]]+[Y,[Y,X]]\Big)+\cdots \tag{1.27}$$

This complicated formula has some useful particular cases. Hence if X et Y commute with $[X,Y]$, (1.25) simplifies to

$$e^Xe^Y=e^{X+Y+\frac{1}{2}[X,Y]}=e^{X+Y}e^{\frac{1}{2}[X,Y]}\ , \tag{1.28}$$

a formula that one may prove directly using the general identity

$$e^XYe^{-X}=\sum_0^\infty\frac{1}{n!}\mathrm{ad}^{\,n}X\,Y \tag{1.29}$$

(which is nothing else than the Taylor expansion at $t=0$ of $e^{tX}Ye^{-tX}$ evaluated at $t=1$), and writing and solving the differential equation satisfied by $f(t)=e^{tX}\,e^{tY}$, $f(0)=1$

$$f'(t)\quad=(X+e^{tX}Ye^{-tX})f(t) \tag{1.30}$$

$$=(X+Y+t[X,Y])f(t)\ . \tag{1.31}$$

On the other hand, to first order in Y, one may replace the argument of ψ in (1.25) by $\exp\mathrm{ad}\,X$ and then

$$Z=X+\sum_{n=0}^\infty\frac{B_n}{n!}(-1)^n\,(\mathrm{ad}\,X)^n\ Y+\mathrm{O}(Y^2) \tag{1.32}$$

where the B_n are the Bernoulli numbers: $\frac{t}{e^t-1}=\sum_0 B_n\frac{t^n}{n!}$, $B_0=1, B_2=\frac{1}{6}, B_4=-\frac{1}{30}$ and, beside $B_1=-\frac{1}{2}$, all B of odd index vanish. Still to first order in Y, one has also

$$e^{X+Y}=e^X+\int_0^1 dt\,e^{tX}Ye^{(1-t)X}+\mathrm{O}(Y^2)$$

which is obtained by writing and solving the differential equation satisfied by $F(t)=\exp t(X+Y).\exp -tX$.

The convergence of these expressions may be proven for X et Y small enough. Note that this BCH formula makes only use of the ad map in the Lie algebra, and not of the ordinary matrix multiplication in $\mathrm{GL}(d,\mathbb{R})$. This is what makes it a canonical and universal formula.

1.3.5 An explicit example: the Lie algebra of SO(n)

From the definition of elements of $\mathfrak{g}$ as tangent vectors in G at $e\equiv I$, or else from the construction of one-parameter subgroups associated with each $X\in\mathfrak{g}$, follows the interpretation of X as "infinitesimal generator" of the Lie group G. The actual determination of the Lie algebra of a given Lie group G may be done in several ways, depending on the way the group is defined or represented.

If one has an explicit parametrization of the elements of G in terms of d real parameters, infinitesimal generators are obtained by differentiation wrt these parameters. See in Chap. 0, the explicit cases of SO(3) and SU(2) treated in that way.

If the group has been defined as the invariance group of some quadratic form in variables x, one may derive an expression of the infinitesimal generators as differential operators in x. Let us illustrate it on the group $\mathrm{O}(n)$, the invariance group of the form $\sum_{i=1}^n x_i^2$ in $\mathbb{R}^n$. The most general linear transformation leaving that form invariant is $x \to x' = Ox$, with O orthogonal. In an infinitesimal form, $O = I + \omega$, and $\omega = -\omega^T$ is an arbitrary skew-symmetric real matrix. An infinitesimal transformation of the form $\delta x^i = \omega^i{}_j x^j$ may also be written

$$\delta x^i = \omega^i{}_j x^j = -\tfrac{1}{2}\omega^{kl} J_{kl} x^i \tag{1.33}$$

$$J_{kl} = x^k \partial_l - x^l \partial_k \qquad : \qquad J_{kl} x^i = x^k \delta_{il} - x^l \delta_{ik} \tag{1.34}$$

(note that we allow to raise and lower freely the indices, thanks to the signature $(+)^n$ of the metric). This yields an explicit representation of infinitesimal generators of the so(n) algebra as differential operators. It is then a simple matter to compute the commutation relations[4]

$$[J_{ij}, J_{kl}] = \delta_{il} J_{jk} - \delta_{ik} J_{jl} - \delta_{jl} J_{ik} + \delta_{jk} J_{il} \ . \tag{1.35}$$

(In other words, the only non-vanishing commutators are of the form $[J_{ij}, J_{ik}] = -J_{jk}$ for any triplet $i \neq j \neq k \neq i$, and those that follow by antisymmetry in the indices.)

One may proceed in a different way, by using a basis of matrices in the Lie algebra, regarded as the space of skew-symmetric $n \times n$ matrices. Such a basis is provided by matrices A_{ij} labelled by pairs of indices $1 \le i < j \le n$, with matrix elements

$$(A_{ij})_k{}^l = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} \ .$$

Hence the matrix A_{ij} has only two non vanishing (and opposite) elements, at the intersection of the i-th row and j-th column and vice versa. Check that these matrices A_{ij} have commutation relations given (1.35).

Exercise : repeat this discussion and the computation of commutation relations for the group $\mathrm{SO}(p,q)$ of invariance of the form $\sum_{i=1}^p x_i^2 - \sum_{i=p+1}^{p+q} x_i^2$. It is useful to introduce the metric tensor $g = \mathrm{diag}\,((+1)^p, (-1)^q)$.

1.3.6 An example of infinite dimension: the Virasoro algebra

In these notes, we are restricting our attention to Lie groups and algebras of finite dimension. Let us give here an example of infinite dimension. One considers diffeomorphisms $z \mapsto z' = f(z)$ where f is an analytic (holomorphic) function of its argument except maybe at 0 and at infinity. (One also speaks of the "diffeomorphisms of the circle".) This is obviously a group and an infinite dimensional manifold, which manifests itself in the algebra of infinitesimal diffeomorphisms $z \mapsto z' = z + \epsilon(z)$, generated by differential operators ℓ_n

$$\ell_n = -z^{n+1}\frac{\partial}{\partial z} \ , \qquad n \in \mathbb{Z} \tag{1.36}$$

which satisfy

$$[\ell_n, \ell_m] = (n-m)\ell_{n+m} \, . \tag{1.37}$$

This Lie algebra is the *Witt algebra*. A modified form of this algebra, with a *central extension* (see. Chap. 2), *i.e.* with an additional "central" generator c commuting with all generators, is called *Virasoro algebra* and appears naturally in physics. Calling L_n et c the generators of that algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n,-m} \qquad [c, L_n] = 0 \ . \tag{1.38}$$

---

[4] Note that wrt to the calculation carried out in the O(3,1) group in Chap. 0, § 0.6.2, we have changed our conventions and use here anti-Hermitian generators.

(The L_n may be thought of as quantum realizations of the operators ℓ_n, with the c term resulting from quantum effects…)

Check that the Jacobi identity is indeed satisfied by this algebra. One proves that this is the most general central extension of (1.37) that respects the Jacobi identity. Show that the subalgebra generated by $L_{\pm 1},\ L_0$ is not affected by the central term. What is the geometric interpretation of the corresponding transformations?

The Virasoro algebra plays a central role in the construction of conformal field theories in 2d and in their application to two-dimensional critical phenomena and to string theory. More details in [DFMS].

1.4 Relations between properties of $\mathfrak{g}$ and G

Let us examine how properties of G translate in $\mathfrak{g}$.

1.4.1 Simplicity, semi-simplicity

Let us define the infinitesimal version of the notion of invariant subgroup. An *ideal* (also sometimes called an invariant subalgebra) in a Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{I}$ of $\mathfrak{g}$ which is stable under multiplication (defined by the Lie bracket) by any element of $\mathfrak{g}$, *i.e.* such that $[\mathfrak{I},\mathfrak{g}]\subset\mathfrak{I}$. The ideal is called *abelian* si $[\mathfrak{I},\mathfrak{I}]=\{0\}$.

A Lie algebra $\mathfrak{g}$ is *simple* if $\mathfrak{g}$ has no other ideal than $\{0\}$. It is *semi-simple* if $\mathfrak{g}$ has no other abelian ideal than $\{0\}$.

Example. Consider the Lie algebra of SO(4), denoted so(4), see the formulae given in (1.35) for so(n). It is easy to check that the combinations

$$A_1 := \frac{1}{2}(J_{12}-J_{34}),\ A_2 = \frac{1}{2}(J_{13}+J_{24}),\ A_3 := \frac{1}{2}(J_{14}-J_{23})$$

commute with

$$B_1 := \frac{1}{2}(J_{12}+J_{34}),\ B_2 = \frac{1}{2}(-J_{13}+J_{24}),\ B_3 := \frac{1}{2}(J_{14}+J_{23})$$

and that

$$[A_i,A_j]=\epsilon_{ijk}A_k \qquad [B_i,B_j]=\epsilon_{ijk}B_k\ , \qquad [A_i,B_j]=0$$

where one sees two commuting copies of so(3). One writes so(4)=so(3)⊕ so(3). Obviously the algebra so(4) is not simple, but it is semi-simple.

Notice the difference between this case of so(4) and the case of the algebra so(3,1) studied in Chap. 0, § 0.6.2. There, the indefinite signature forced us to complexify the algebra to "decouple" the two copies of the algebra so(3).

One has the following relations

$$G \text{ simple} \Longrightarrow \mathfrak{g} \text{ simple}$$
$$G \text{ semi-simple} \Longrightarrow \mathfrak{g} \text{ semi-simple}$$

but the converse is not true ! Several different Lie groups may have the same Lie algebra, *e.g.* SO(3) which is simple, and SU(2) which is not semi-simple, as seen above in §1.1.7. [5]

---

[5] Beware! Some authors call "simple" any Lie group, whose Lie algebra is simple. This amounts to making a distinction between the concepts of simple group and simple Lie group. The latter is such that it has no non trivial invariant *Lie group*. Thus the Lie group SU(2) is a simple Lie group but not a simple group, as it has an invariant subgroup which is not of Lie type…

1.4.2 Compacity. Complexification

A semi-simple Lie algebra is said to be compact if it is the Lie algebra of a compact Lie group. At first sight, this definition looks non intrinsic to the algebra and seems to depend on the Lie group from which it derives. We shall see below that a condition (Cartan criterion) allows to remove this dependance.

At this stage one should examine the issue of complexification. Several distinct groups may have different Lie algebras, that become isomorphic when the parameters are complexified. For instance, the groups O(3) et O(2,1), the first compact, the second non compact, have Lie algebras

$$\mathrm{o}(3)\ \begin{cases} X_1 = z\partial_y - y\partial_z & \\ X_2 = x\partial_z - z\partial_x & [X_1,X_2] = y\partial_x - x\partial_y = X_3 \text{ etc} \\ X_3 = y\partial_x - x\partial_y & \end{cases}$$

$$\mathrm{o}(2,1)\ \begin{cases} \widetilde{X}_1 = z\partial_y + y\partial_z & [\widetilde{X}_1,\widetilde{X}_2] = y\partial_x - x\partial_y = \widetilde{X}_3 \\ \widetilde{X}_2 = x\partial_z + z\partial_x & [\widetilde{X}_2,\widetilde{X}_3] = -z\partial_y - y\partial_z = -\widetilde{X}_1 \\ \widetilde{X}_3 = y\partial_x - x\partial_y & [\widetilde{X}_3,\widetilde{X}_1] = -x\partial_z - z\partial_x = -\widetilde{X}_2 \end{cases} \tag{1.39}$$

that non isomorphic on the real numbers, but $i\widetilde{X}_1$, $i\widetilde{X}_2$ et $-\widetilde{X}_3$ verify the o(3) algebra. The algebras o(3) and o(2,1) are said to have the same complexified form $\mathfrak{g}_c$, or else, to be two real forms of $\mathfrak{g}_c$, but only one of them, namely o(3), (or so(3)=su(2)), of this complexified form is compact. This complexified form is the sl(2,$\mathbb{C}$) algebra, of which sl(2,$\mathbb{R}$) is another non compact real form. (See Exercise B and TD).

The algebras so(4) and so(3,1) studied above and in Chap. 0 provide another example of two algebras, which are two non-isomorphic real forms of the same complexified form. Another example is provided by sp($2n,\mathbb{R}$) et usp(n). (See Appendix A).

More generally, one may prove ([FH] p. 130) that

- *any semi-simple complex Lie algebra has a unique real compact form.*

To summarize, local topological properties of the Lie group are transcribed in the Lie algebra. The Lie algebra, however, is unable to capture global topological properties of the group.

1.4.3 Connectivity, simple-connectivity

– If G is non connected and G' is the subgroup of the connected component of the identity, the Lie algebras of G and G' coincide: $\mathfrak{g} = \mathfrak{g}'$.

– If G is non simply connected, let $\widetilde{G}$ be its universal covering group. G et $\widetilde{G}$ being locally isomorphic, they have the same Lie algebra. Examples: U(1) and $\mathbb{R}$; SO(3) and SU(2); SO(3,1) and SL$(2,\mathbb{C})$.

To summarize:

Given a Lie group G, we have constructed its Lie algebra. Conversely, a theorem by Cartan asserts that any Lie algebra is the Lie algebra of some Lie group [Ki, p.99]. More precisely, to every Lie algebra $\mathfrak{g}$ corresponds a unique connected and simply connected Lie group G, whose Lie algebra is $\mathfrak{g}$. Any other connected Lie group G' with the same Lie algebra $\mathfrak{g}$ has the form $G' = G/H$ with H a finite or discrete invariant subgroup of G. This agrees with what we saw above: if G is the covering group of G', $G' = G/\pi_1(G')$. For example U(1)=$\mathbb{R}/\mathbb{Z}$, SO(3)=SU(2)/$\mathbb{Z}_2$. If G' is non connected, the previous property applies to the connected component of the identity.

1.4.4 Structure constants. Killing form. Cartan criteria

Given a basis $\{t_\alpha\}$ in a d-dimensional Lie algebra $\mathfrak{g}$, any element X of $\mathfrak{g}$ reads $X = \sum_{\alpha=1}^d x^\alpha t_\alpha$. The *structure constants* of $\mathfrak{g}$ (in that basis), defined by

$$[t_\alpha, t_\beta] = C_{\alpha\beta}{}^{\gamma}\, t_\gamma \ , \tag{1.40}$$

are clearly antisymmetric in their two lower indices $C_{\alpha\beta}{}^{\gamma} = -C_{\beta\alpha}{}^{\gamma}$. Return to the linear operator $\mathrm{ad}\, X$ defined above in (1.24)

$$\mathrm{ad}\, X\, Z = [X, Z] = \sum x^\alpha z^\beta C_{\alpha\beta}{}^{\gamma} t_\gamma \ ,$$

and for $X, Y \in \mathfrak{g}$ consider the linear operator $\mathrm{ad}\, X\, \mathrm{ad}\, Y$ which acts in the Lie algebra according to

$$\mathrm{ad}\, X\, \mathrm{ad}\, Y\ Z = [X, [Y, Z]] = C_{\alpha\delta}{}^{\epsilon} C_{\beta\gamma}{}^{\delta} x^\alpha y^\beta z^\gamma t_\epsilon \ .$$

Exercises (easy !): show that the Jacobi identity is equivalent to the identity

$$\sum_\delta \left(C_{\alpha\delta}{}^{\epsilon} C_{\beta\gamma}{}^{\delta} + C_{\beta\delta}{}^{\epsilon} C_{\gamma\alpha}{}^{\delta} + C_{\gamma\delta}{}^{\epsilon} C_{\alpha\beta}{}^{\delta} \right) = 0 \tag{1.41}$$

(note the structure : a cyclic permutation on the three indices α, β, γ with ϵ fixed and summation over the repeated δ) ; and show that this identity may also be expressed as

$$[\mathrm{ad}\, X, \mathrm{ad}\, Y] Z = \mathrm{ad}\, [X, Y] Z \ . \tag{1.42}$$

Taking the trace of this linear operator $\mathrm{ad}\, X\, \mathrm{ad}\, Y$ defines the *Killing form*

$$(X, Y) := \mathrm{tr}\, (\mathrm{ad}\, X \mathrm{ad}\, Y) = \sum_{\gamma,\delta} C_{\alpha\delta}{}^{\gamma} C_{\beta\gamma}{}^{\delta} x^\alpha y^\beta =: g_{\alpha\beta} x^\alpha y^\beta \ , \tag{1.43}$$

a symmetric bilinear form (a scalar product) on vectors of the Lie algebra. The symmetric tensor $g_{\alpha\beta}$ is thus given by

$$g_{\alpha\beta} = \sum_{\gamma,\delta} C_{\alpha\delta}{}^{\gamma} C_{\beta\gamma}{}^{\delta} = \mathrm{tr}\, (\mathrm{ad}\, t_\alpha\, \mathrm{ad}\, t_\beta) \ .$$

(Symmetry in α, β is manifest on the 1st expression, it follows from the cyclicity of the trace in the 2nd.)

Note that this Killing form is invariant under the action of any $\mathrm{ad}\, Z$:

$$\forall X, Y, Z \in \mathfrak{g} \quad ([Z, X], Y) + (X, [Z, Y]) = 0 \tag{1.44}$$

(think of $\mathrm{ad}\,Z$ as an infinitesimal generator acting like a derivative, either on the first term, or on the second). Indeed the first term equals $\mathrm{tr}\,(\mathrm{ad}\,Z\mathrm{ad}\,X\mathrm{ad}\,Y - \mathrm{ad}\,X\mathrm{ad}\,Z\mathrm{ad}\,Y)$ while the second is $\mathrm{tr}\,(\mathrm{ad}\,X\mathrm{ad}\,Z\mathrm{ad}\,Y - \mathrm{ad}\,X\mathrm{ad}\,Y\mathrm{ad}\,Z)$, and they cancel thanks to the cyclicity of the trace. One may prove that in a simple Lie algebra, an invariant symmetric form is necessarily a multiple of the Killing form.

One may then use the tensor $g_{\alpha\beta}$ to lower the 3d label of ${C_{\alpha\beta}}^{\gamma}$, thus defining

$$C_{\alpha\beta\gamma} = {C_{\alpha\beta}}^{\delta} g_{\gamma\delta} = {C_{\alpha\beta}}^{\delta} {C_{\gamma\epsilon}}^{\kappa} {C_{\delta\kappa}}^{\epsilon} \,.$$

Let us then show that this $C_{\alpha\beta\gamma}$ is completely antisymmetric in α, β, γ. Given the already known antisymmetry in α, β, it suffices to show that $C_{\alpha\beta\gamma}$ is invariant by cyclic permutations. This follows from (1.44) which may be written in a more symmetric form as

$$(X, [Y, Z]) = (Y, [Z, X]) = (Z, [X, Y]) = C_{\alpha\beta\gamma} x^\alpha y^\beta z^\gamma = C_{\beta\gamma\alpha} y^\beta z^\gamma x^\alpha = C_{\gamma\alpha\beta} z^\gamma x^\alpha y^\beta \,, \qquad (1.45)$$

thus proving the announced property.

A quite remarkable theorem of E. Cartan states that:

- (i) *A Lie algebra is semi-simple iff the Killing form is non-degenerate, i.e.* $\det g \neq 0$.
- (ii) *A real semi-simple Lie algebra is compact iff the Killing form is negative definite.*

Those are the *Cartan criteria.*

In one way, property (i) is easy to prove. Suppose that $\mathfrak{g}$ is not semi-simple and let us show that $\det g = 0$. Let $\mathfrak{I}$ be an ideal of $\mathfrak{g}$, choose a basis of $\mathfrak{g}$ made of a basis of $\mathfrak{I}$, $\{t_i\}$, $i = 1, \cdots r$, complemented by t_a, $a = r+1, \cdots d$. For $1 \leq i, j \leq r$, compute $g_{ij} = \sum_{\alpha\beta} {C_{i\alpha}}^{\beta} {C_{j\beta}}^{\alpha}$. By definition of an ideal, α and β are themselves between 1 and r, $g_{ij} = \sum_{1\leq k,l\leq r} {C_{ik}}^{l} {C_{jl}}^{k}$. Hence the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{I}$ is the Killing form of $\mathfrak{I}$. If moreover the ideal is assumed to be abelian, $g_{ij} = 0$ and $g_{ia} = 0$ (Exercise: check that point!). The form is obviously degenerate ($\det g = 0$). The reciprocal, $\det g = 0 \Rightarrow \mathfrak{g}$ non semi-simple, is more delicate to prove.

Likewise, property (ii) is relatively easy to prove in the sense compacity $\Rightarrow$ definite negative form. Start from an arbitrary positive definite symmetric bilinear form; for example in a given basis $\{t_\alpha\}$, consider $\langle X, Y \rangle = \sum x^\alpha y^\beta$. For a compact group G, one can make this form invariant by averaging over G : $\varphi(X, Y) := \int \mathrm{d}\mu(g) \langle gXg^{-1}, gYg^{-1} \rangle$. It is invariant $\varphi(gXg^{-1}, gYg^{-1}) = \varphi(X, Y)$, or in infinitesimal form, $\varphi([Z, X], Y] + \varphi(X, [Z, Y]) = 0$, (cf (1.44)). It is also positive definite. Let e_α be a basis which diagonalises it, $\varphi(e_\alpha, e_\beta) = \delta_{\alpha\beta}$. Let us calculate in that basis the matrix of the $\mathrm{ad}\,X$ operator and show that it is antisymmetric, $(\mathrm{ad}\,X)_{\alpha\beta} = -(\mathrm{ad}\,X)_{\beta\alpha}$:

$$(\mathrm{ad}\,X)_{\alpha\beta} = \varphi(e_\alpha, [X, e_\beta]) = -\varphi(\mathrm{e}_\beta, [X, e_\alpha]) = -(\mathrm{ad}\,X)_{\beta\alpha} \,.$$

Hence the Killing form

$$(X, X) = \mathrm{tr}\,(\mathrm{ad}\,X\mathrm{ad}\,X) = \sum_{\alpha,\beta} (\mathrm{ad}\,X)_{\alpha\beta} (\mathrm{ad}\,X)_{\beta\alpha} = -\sum_{\alpha,\beta} ((\mathrm{ad}\,X)_{\alpha\beta})^2 \leq 0$$

is negative semi-definite, and if the algebra is semi-simple, it is negative definite, q.e.d.

Example. The case of SO(3) or SU(2) is familiar. The structure constants are given by the completely antisymmetric tensor $C_{\alpha\beta\gamma} = \epsilon_{\alpha\beta\gamma}$. The Killing form is $g_{\alpha\beta} = -2\delta_{\alpha\beta}$. Exercise : compute the Killing form for the algebra $so(2, 1)$, (see Exercise B).

A last important theorem (again by Cartan !) states that

- *Any semi-simple Lie algebra* $\mathfrak{g}$ *is a direct sum of simple Lie algebras* $\mathfrak{g}_i$

$$\mathfrak{g} = \oplus_i \mathfrak{g}_i \,.$$

This is a simple consequence of (1.45). Consider a semi-simple algebra $\mathfrak{g}$ with an ideal $\mathfrak{I}$ and call $\mathfrak{C}$ the complement of $\mathfrak{I}$ wrt the Killing form, *i.e.* $(\mathfrak{I},\mathfrak{C})=0$. By (1.45), $([\mathfrak{C},\mathfrak{I}],\mathfrak{I})=(\mathfrak{C},[\mathfrak{I},\mathfrak{I}])=(\mathfrak{C},\mathfrak{I})=0$ (since $\mathfrak{I}$ is a subalgebra), and $([\mathfrak{C},\mathfrak{I}],\mathfrak{C})=(\mathfrak{I},\mathfrak{C})=0$ (since $\mathfrak{I}$ is an ideal), hence $[\mathfrak{C},\mathfrak{I}]$, orthogonal to any element of $\mathfrak{g}$ for the non-degenerate Killing form, vanishes, $[\mathfrak{C},\mathfrak{I}]=0$, which means that $\mathfrak{g}=\mathfrak{I}\oplus\mathfrak{C}$. Iterating the argument on $\mathfrak{C}$, one gets the announced property.

Cartan made use of these properties to classify the simple complex and real Lie algebras. We return to this classification in Chap. 3.

1.4.5 Casimir operator(s)

With previous notations, given a semi-simple Lie algebra $\mathfrak{g}$, hence with an invertible Killing form, and a basis $\{t_\alpha\}$ of $\mathfrak{g}$, we define

$$C_2=\sum_{\alpha,\beta} g^{\alpha\beta}t_\alpha t_\beta \tag{1.46}$$

where $g^{\alpha\beta}$ is the inverse of $g_{\alpha\beta}$, *i.e.* $g_{\alpha\gamma}g^{\gamma\beta}=\delta_\alpha^\beta$.
Formally, this combinaison of the t's, which does not make use of the bracket, does not live in the Lie algebra but in its *universal enveloping algebra* $U\mathfrak{g}$, defined as the associative algebra of polynomials in elements of $\mathfrak{g}$. Here, since we restricted ourselves to $\mathfrak{g}\subset M(n,\mathbb{R})$, $U\mathfrak{g}$ may also be considered as a subalgebra of $M(n,\mathbb{R})$.

Let us now show that C_2 has a vanishing bracket (commutator) with any t_γ hence with any element of $\mathfrak{g}$. This is the quadratic *Casimir operator*.

$$\begin{aligned}
[C_2,t_\gamma] &= \sum_{\alpha,\beta} g^{\alpha\beta}[t_\alpha t_\beta,t_\gamma]\\
&= \sum_{\alpha,\beta} g^{\alpha\beta}\left(t_\alpha[t_\beta,t_\gamma]+[t_\alpha,t_\gamma]t_\beta\right)\\
&= \sum_{\alpha,\beta,\delta} g^{\alpha\beta}C_{\beta\gamma}{}^{\delta}(t_\alpha t_\delta+t_\delta t_\alpha)\\
&= \sum_{\alpha,\beta,\delta,\kappa} g^{\alpha\beta}g^{\delta\kappa}C_{\beta\gamma\kappa}(t_\alpha t_\delta+t_\delta t_\alpha)\ .
\end{aligned}\tag{1.47}$$

The term $\sum_{\beta\kappa} g^{\alpha\beta}g^{\delta\kappa}C_{\beta\gamma\kappa}$ is antisymmetric in $\alpha\leftrightarrow\delta$, while the term in parentheses is symmetric. The sum thus vanishes, q.e.d.

One shows that in a simple Lie algebra, (more precisely in its universal enveloping algebra), a quadratic expression in t that commutes with all the t's is proportional to the Casimir operator C_2. In other words, the quadratic Casimir operator is unique up to a factor.

Example. In the Lie algebra so(3)≅ su(2), the Casimir operator C_2 is (up to a sign) $\mathbf{J}^2$, which, as everybody knows, commutes with the infinitesimal generators J^i of the algebra. In a non simple algebra, there are as many quadratic operators as there are simple components, see for example the two Casimir operators $\mathbf{J}^2$ and $\mathbf{K}^2$ in the (complexified) so(3,1)≃ su(2)⊕ su(2) algebra of the Lorentz group (see Chap. 0 § 0.6.2); or P^2 and W^2 in the Poincaré algebra, see Chap. 0, § 0.6.5.

There may exist other, higher degree Casimir operators. Check that

$$C_r=g^{\alpha_1\alpha'_1}g^{\alpha_2\alpha'_2}\cdots g^{\alpha_r\alpha'_r}C_{\alpha_1\beta_1}{}^{\beta_2}C_{\alpha_2\beta_2}{}^{\beta_3}\cdots C_{\alpha_r\beta_r}{}^{\beta_1}t_{\alpha'_1}t_{\alpha'_2}\cdots t_{\alpha'_r}$$

has a vanishing bracket with any t_γ. What is that C_3 in su(2) ? See Bourbaki ([Bo], chap. **??**) for a discussion of these general Casimir operators. See also exercice C below.

If one remembers that infinitesimal generators (vectors of the Lie algebra) may be regarded as differential operators in the group coordinates, one realizes that the Casimir operators yield invariant (since commuting with the generators) differential operators. In particular, the quadratic Casimir operator corresponds to an invariant Laplacian on the group (see Chap. 0, § 0.2.3 for the case of SO(3)).

These Casimir operators will play an important role in the study of group representations.

A short bibliography

Mathematics books

[Bo] N. Bourbaki, *Groupes et Algèbres de Lie*, Chap. 1-9, Hermann 1960-1983.

[Bu] D. Bump, *Lie groups*, Series "Graduate Texts in Mathematics", vol. **225**, Springer 2004.

[Ch] C. Chevalley, *Theory of Lie groups*, Princeton University Press.

[D] J. Dieudonné, *Éléments d'analyse*, Gauthier-Villars, in particular volumes 5-7 (comprehensive but difficult!).

[DNF] Dubrovin, B. A., Fomenko, A. T., Novikov, S. P. *Modern geometry—methods and applications.* Part I. The geometry of surfaces, transformation groups, and fields. Graduate Texts in Mathematics, 93. Springer-Verlag, New York, 1992. Part II. The geometry and topology of manifolds. Graduate Texts in Mathematics, 104. Springer-Verlag, New York, 1985. (a third volume deals with homology...)

[Po] L.S. Pontryagin, *Topological Groups*, Gordon and Breach, 1966.

[W] H. Weyl, *Classical groups*, Princeton University Press

A recent book is close to the spirit of the present course :

[K-S] Y. Kosmann-Schwarzbach, *Groups and symmetries, From Finite Groups to Lie Groups*, Springer 2010.

Group theory for physicists

[Wi] E. Wigner, *Group Theory and its Applications to Quantum Mechanics.* Academ. Pr. 1959

[Co] J.F. Cornwell, *Group theory in physics. An introduction*, Academic Pr. contains much information but sometimes uses a terminology different from the rest of the literature...

[Gi] R. Gilmore, *Lie groups, Lie algebras and some of their applications*, Wiley

[Ha] M. Hamermesh, *Group theory and its applications to physical problems*, Addison-Wesley

[Itz] C. Itzykson, *Notes de cours pour l'Ecole de Physique Mathématique de l'Universit de Toulouse* (Saclay report (in French), September 1974)

[OR] L. O' Raifeartaigh, *Group structure of gauge theories*, Cambridge Univ. Pr. 1986.

See also lecture notes of group theory by and for physicists (in French), avalaible on the CCSD server `http://cel.ccsd.cnrs.fr/`, for example

J.-B. Z., *Introduction à la théorie des groupes et de leurs representations*, (Notes de cours au Magistère MIP 1994), which focuses mainly on finite groups.

Appendix A. Quaternion field and symplectic groups

A.1 Quaternions

The set of quaternions is the algebra over $\mathbb{C}$ generated by 4 elements, e_i, $i = 1, 2, 3$,

$$q = q^{(0)}1 + q^{(1)}e_1 + q^{(2)}e_2 + q^{(3)}e_3 \qquad q^{(\cdot)} \in \mathbb{C} \tag{A-1}$$

with multiplication $e_i^2 = e_1e_2e_3 = -1$, from which it follows that

$$e_1e_2 = -e_2e_1 = e_3$$

and cyclic permutations. One may represent the e_i in terms of Pauli matrices : $e_i \mapsto -i\sigma_i$.

The conjugate of q is the quaternion

$$\bar{q} = q^{(0)}1 - q^{(1)}e_1 - q^{(2)}e_2 - q^{(3)}e_3 \ . \tag{A-2}$$

not to be confused with its complex conjugate

$$q^* = q^{(0)*}1 + q^{(1)*}e_1 + q^{(2)*}e_2 + q^{(3)*}e_3 \ . \tag{A-3}$$

Note that $q\bar{q} := |q|^2 = (q^{(0)})^2 + (q^{(1)})^2 + (q^{(2)})^2 + (q^{(3)})^2$, the square norm of the quaternion, and hence $q^{-1} = \bar{q}/|q|^2$ if this norm is non-vanishing.

One may also define the Hermitian conjugate of q as

$$q^\dagger = \bar{q}^* = q^{(0)*}1 - q^{(1)*}e_1 - q^{(2)*}e_2 - q^{(3)*}e_3 \tag{A-4}$$

(in accordance with the fact that Pauli matrices are Hermitian).

Note that conjugation and Hermitian conjugation reverse the order of factors

$$\overline{(q_1q_2)} = \bar{q}_2\bar{q}_1 \qquad (q_1q_2)^\dagger = q_2^\dagger q_1^\dagger \ . \tag{A-5}$$

A *real quaternion* is a quaternion of the form (A-1) with $q^{(\mu)} \in \mathbb{R}$, hence identical with its complex conjugate.

The set of real quaternions forms a field, which is also a space of dimension 4 over $\mathbb{R}$. It is denoted $\mathbb{H}$ (from Hamilton).

A.2 Quaternionic matrices

Let us consider matrices Q with quaternionic elements $(Q)_{ij} = q_{ij}$, or $Q = (q_{ij})$. One may apply to Q the conjugations defined above. One may also transpose Q. The Hermitian conjugate of Q is

$$(Q^\dagger)_{ij} = q_{ji}^\dagger \ . \tag{A-6}$$

The *dual* Q^R of a quaternionic matrix Q is the matrix

$$(Q^R)_{ij} = \bar{q}_{ji} \ . \tag{A-7}$$

(It plays for quaternionic matrices the same role as Hermitian conjugates for complex matrices.) A quaternionic matrix is *self-dual* if

$$Q^R = Q = (q_{ij}) = (\bar{q}_{ji}) \ , \tag{A-8}$$

it is *real quaternionic* if

$$Q^R = Q^\dagger \quad \text{hence} \quad q_{ij} = q_{ij}^* \ , \tag{A-9}$$

i.e. if its elements are real quaternions.

A.3 Symplectic groups Sp($2n,\mathbb{R}$) and USp(n), and the Lie algebras sp($2n$) et usp(n)

Consider the $2n \times 2n$ matrix

$$S = \begin{pmatrix} 0 & \mathbf{1}_N \\ -\mathbf{1}_N & 0 \end{pmatrix} \tag{A-10}$$

and the associated "skew-symmetric" bilinear form

$$(X,Y) = X^T S Y = \sum_{i=1}^{n} (x_i y_{i+n} - y_i x_{i+n}) \ . \tag{A-11}$$

The symplectic group Sp($2n,\mathbb{R}$) is the group of real $2n \times 2n$ matrices that preserve that form

$$B^T S B = S \ . \tag{A-12}$$

In the basis where $X^T = (x_1, x_{n+1}, x_2, x_{n+2}, \cdots)$, the matrix $S = \mathrm{diag}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \mathrm{diag}\,(-e_2)$ in terms of quaternions, and the symplectic group is then generated by quaternionic $n \times n$ matrices Q satisfying $Q^R.Q = I$, (check !); the matrix B being real, however, the elements of Q are such that $q_{ij}^{(\alpha)}$ are real for $\alpha = 0, 2$ and purely imaginary for $\alpha = 1, 3$. This group is non compact. Its Lie algebra sp($2n,\mathbb{R}$) is generated by real matrices A such that $A^T S + SA = 0$. The dimension of that group or of its Lie algebra is $n(2n+1)$. For $n = 1$, Sp(2,$\mathbb{R}$)=SL(2,$\mathbb{R}$).

A related group is USp(n), generated by unitary real quaternionic $n \times n$ matrices $Q^R = Q^\dagger = Q^{-1}$. This is the invariance group of the quaternionic Hermitian form $\sum \bar{x}_i y_i$, $x, y \in \mathbb{H}^n$. It is compact since it is a subgroup of U($2n$). Its Lie algebra usp(n) is generated by antiselfdual real quaternionic matrices $A = -A^R = -A^\dagger$ (check !). Its dimension is again $n(2n+1)$. For $n = 1$, USp(1)=SU(2).

Expressing the condition on matrices A of sp($n,\mathbb{R}$) in terms of quaternions, one sees that the two algebras sp($2n,\mathbb{R}$) and usp(n) have the same complexified algebra, namely sp($2n,\mathbb{C}$). Only usp(n) is compact.

Appendix B. A short reminder of topology and differential geometry.

B.1 A lexicon of some concepts of topology used in these notes

Topological space : set E with a collection of *open subsets*, with the property that the union of open sets and the intersection of a finite number of them is an open subset, and that E and $\emptyset$ are open.

Closed subset of E : complement of an open subset of E.

Neighborhood of a point x : subset E that contains an open set containing x. Let $\mathcal{V}(x)$ be the set of neighborhoods of x.

A topological space is *separated* (or *Hausdorff*) if two distinct points have distinct neighborhoods. This will always be assumed in these notes.

Basis of neighborhoods $\mathcal{B}(x)$ *of a point* x : subset of $\mathcal{V}(x)$ such that any $V \in \mathcal{V}(x)$ contains a $W \in \mathcal{B}(x)$. (Intuitively, a basis is made of "enough" neighborhoods.)

Continuous function: a function f from topological space E to topological space F is called continuous if the inverse image of every open set in F is open in E.

Compact space E: topological (separated) space such that from any covering of E by open sets, one may extract a finite covering.

Consequences :
– any infinite sequence of points in E has an accumulation point in E;
– if E is compact and $f : E \mapsto F$ is continuous, $f(E)$ is compact;
– any continuous real function on a compact space E is bounded.
If E is a subspace of $\mathbb{R}^n$, E compact $\Leftrightarrow$ E closed and bounded (Borel-Lebesgue).
Locally compact space : (separated) space in which any point has at least one compact neighborhood. Examples : $\mathbb{R}$ is not compact but is locally compact ; $\mathbb{Q}$ is neither compact nor locally compact.

B.2 Notion of manifold

A *manifold* M of dimension n is a space which locally, in the vicinity of each point, "resembles" $\mathbb{R}^n$ or $\mathbb{C}^n$. Counter-examples are given by two secant lines, or by ——○. More precisely, there exists a collection of neighborhoods U_i covering M, with *charts* f_i, *i.e.* invertible and bicontinuous (homeomorphisms) functions between U_i and an open set of $\mathbb{R}^n$: $f_i(U_i) \subset \mathbb{R}^n$. Let m be a point of M, $m \in U_i$, and $f_i(m) = (x^1, x^2, \dots x^n)$ its image in $\mathbb{R}^n$: $(x^1, x^2, \dots x^n)$ are the *local coordinates* of m, which depend on the map. It is fundamental to know how to change the coordinate chart. The manifold is said to be differentiable of class C^k if fo r any pair of open sets U_i and U_j with a non-empty intersection, $f_j \circ f_i^{-1}$ which maps $f_i(U_i \cap U_j) \subset \mathbb{R}^n$ onto $f_j(U_i \cap U_j) \subset \mathbb{R}^n$ is of class C^k.

Example : the sphere S^2 is an analytic manifold of dimension 2. One may choose as two open sets the sphere with its North, resp. South, pole removed, with a map to $\mathbb{R}^2$ given by the stereographic projection (see Problem below) from that pole.

A *Riemann manifold* is a differentiable real manifold on the tangent vectors of which a positive definite inner product has been defined. If the inner product is only assumed to be a non degenerate form of signature $(+1)^p, (-1)^{n-p})$, the manifold is said to be *pseudo-Riemannian.* In local coordinates x^i, we have $X = X^i \frac{\partial}{\partial x^i}$, and the inner product and the squared length element are given by the metric tensor g

$$(X, Y) = g_{ij} X^i Y^j\,, \qquad ds^2 = g_{ij} dx^i dx^j\,. \tag{B-1}$$

B.3 Tangent space

In differential geometry, a tangent vector X to a manifold M at a point x_0 is a linear differential operator, of first order in the derivatives in x_0, acting on functions f on M. In local coordinates x^i,

$$X: \qquad f(x) \mapsto \sum_i X^i \left.\frac{\partial}{\partial x^i}\right|_{x_0} f(x)$$

and under a change of coordinates $\{x^i\} \to \{y^i\}$, these operators transform by the Jacobian matrix $\frac{\partial}{\partial y^j} = \sum_i \left.\frac{\partial x^i}{\partial y^j}\right|_{x_0} \frac{\partial}{\partial x^i}$ with the transformation of $X^i \to Y^j$ that follows from it.

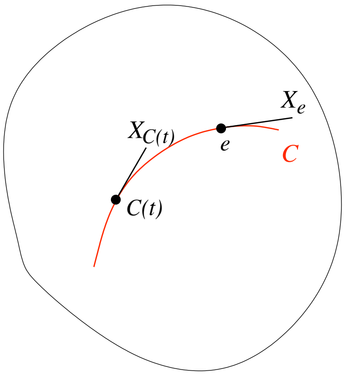


Figure 1.4: The field of tangent vectors to the curve $C(t)$ is a left-invariant vector field

Tangent vector to a curve : if a curve $C(t)$ passes through the point x_0 at $t = 0$, one may differentiate a function f along that curve

$$f \mapsto \left.\frac{\mathrm{d}f(C(t))}{\mathrm{d}t}\right|_{t=0} ,$$

which defines the tangent vector to the curve C at point x_0, also called velocity vector and denoted $C'(t)|_{t=0} = C'(0)$.

The tangent space to M at x_0, denoted $T_{x_0}M$, is the vector space generated by the velocity vectors of all curves passing through x_0. The space $T_{x_0}M$ has a basis made of $\frac{\partial}{\partial x^i}\big|_{x_0}$: it has the same dimension as M.

If a vector X_x tangent to M at x is defined for any x, this defines a *vector field* on the manifold M.

B.4 Lie group. Exponential map

Take a group G, e its identity. Let $C(t)$ be a curve passing through $C(0) = e$, and let $X_e = (C'(t))_{t=0}$ be its velocity vector at e. For $g \in G$, one defines the *left translate* $g.C(t)$ of C by g. Its velocity at g, $X_g = (g.C(t))'_{t=0}$, is called a *left translated vector* of X_e. The vector field $g \mapsto X_g$ is said to be left-invariant, it is the set of left translated vectors of X_e. The tangent space at e and the space of invariant vector fields are thus isomorphic, and are both denoted $\mathfrak{g}$.

Conversely, given a tangent vector X_e at e, let

$$C(t) = \exp tX_e \tag{B-2}$$

be the unique solution to the differential equation

$$C'(t) = X_{C(t)} \tag{B-3}$$

which expresses that the curve $C(t)$ is tangent at any of its points to the left-invariant vector field, that equation being supplemented by the initial condition that $C(0) = e$. (This first-order differential equation has a solution, determined up to a constant (in the group), and that constant is fixed uniquely by the initial condition.)

Let us now prove that the function exp defined by (B-2) satisfies property (1.17). Note that $C(t)$ satisfies (B-3), and so does $C(t+t')$. Thus $C(t+t') = k.C(t)$, (with k constant in the group), and that constant is fixed by taking $t = 0$, $C(t') = k$, hence $C(t+t') = C(t')C(t)$ and $C(-t) = C(t)^{-1}$, qed.

In the case of matrix groups considered in this course, the function exp is of course identical to the exponential function defined by its Taylor series (1.21).

Appendix C. Invariant measure on SU(2) and on U(n)

The group SU(2) being isomorphic to a sphere S^3 is compact and one may thus integrate a function on the group with a wide variety of measures $d\mu(g)$. The invariant measure, such that $d\mu(g.g_1) = d\mu(g_1.g) = d\mu(g^{-1}) = d\mu(g)$, is, on the other hand, unique up to a factor.

A possible way to determine that measure is to consider the transformation $U \to U' = U.V$ where U, V and hence U' are unitary of the form (0.10) (*i.e.* $U = u_0 I - \mathbf{u}.\boldsymbol{\sigma}$, $u \in S^3$ etc) ; if the condition $u_0^2 + \mathbf{u}^2 = 1$ is momentarily relaxed (but $v_0^2 + \mathbf{v}^2 = 1$ maintained), this defines a linear transformation $u \to u'$ which conserves the norm $\det U = u_0^2 + \mathbf{u}^2 = u_0'^2 + \mathbf{u}'^2 = \det U'$. This is thus a rotation of the space $\mathbb{R}^4$ which preserves the natural measure $\mathrm{d}^4 u\, \delta(u^2 - 1)$ on the unit sphere S^3 of equation $\det U = 1$. In other terms, that measure on the sphere S^3 gives a right invariant measure: $d\mu(U) = d\mu(U.V)$. One may prove in a similar way that it is left invariant: $d\mu(U) = d\mu(V.U)$. It is also invariant under $U \to U^{-1}$, since inversion in SU(2) amounts to the restriction to S^3 of the orthogonal transformation $u_0 \to u_0$, $\mathbf{u} \to -\mathbf{u}$ in $\mathbb{R}^4$, which preserves of course the natural measure on S^3 :

$$d\mu(U) = d\mu(UV) = d\mu(VU) = d\mu(U^{-1}) \ .$$

The explicit form of the measure depends on the chosen parametrization. If one uses the direction $\mathbf{n}$ (or its two polar angles θ et ϕ) and the rotation angle ψ, one finds

$$d\mu(U) = \frac{1}{2} \sin^2 \frac{\psi}{2} \sin\theta \, d\psi \, d\theta \, d\phi \tag{C-1}$$

normalized for SU(2) to

$$v(\mathrm{SU}(2)) = \int_{\mathrm{SU}(2)} d\mu(U) = \frac{1}{2} \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \int_0^{2\pi} d\psi \sin^2 \frac{\psi}{2} = 2\pi^2 \tag{C-2}$$

which is the "area" of the unit sphere S^3 and the volume of SU(2). For SO(3) where the angle ψ has a range restricted to $(0, \pi)$, one finds instead $v(\mathrm{SO}(3)) = \int_{\mathrm{SO}(3)} d\mu(g) = \pi^2$.

The expression in any other coordinate system, like the Euler angles, is then obtained by computing the adequate Jacobian

$$d\mu(U) = \frac{1}{8} \sin\beta \, d\alpha \, d\beta \, d\gamma \ . \tag{C-3}$$

(Note that $0 \le \gamma \le 4\pi$ for SU(2), whereas $0 \le \alpha \le 2\pi$ and $0 \le \beta \le \pi$).

Another method to derive these results appeals to the introduction of an invariant metric on the group; a square distance between two elements U et $U + dU$ is defined by $ds^2 = \frac{1}{2} \mathrm{tr}\, dU dU^\dagger$, it is invariant by $U \to UV$, $U \to VU$ ou $U \to U^{-1}$, and an invariant integration measure then follows (cf Chap. 0, App. 0). With the parametrization $(\mathbf{n} = (\theta, \phi), \psi)$, one finds

$$ds^2 = \frac{1}{2} \mathrm{tr}\ dU dU^\dagger = \left(d\frac{\psi}{2} \right)^2 + \sin^2 \frac{\psi}{2} \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \ , \tag{C-4}$$

which leads indeed to (C-1). In the parametrization by Euler angles,

$$U = e^{-i\alpha \frac{\sigma_3}{2}} e^{-i\beta \frac{\sigma_2}{2}} e^{-i\gamma \frac{\sigma_3}{2}} \tag{C-5}$$

whence

$$ds^2 = \frac{1}{2}\mathrm{tr}\, dU dU^\dagger = \frac{1}{4}\left(d\alpha^2 + 2d\alpha d\gamma\cos\beta + d\gamma^2 + d\beta^2\right) \tag{C-6}$$

and with $\sqrt{g} = \sin\beta$ one does recover (C-3) (check it !).

- **Case of U(n).**

Let us discuss rapidly the case of U(n). Any unitary matrix $U \in \mathrm{U}(n)$ may be diagonalized in the form

$$U = V\Lambda V^\dagger\ , \tag{C-7}$$

with $\Lambda = \mathrm{diag}\,(\lambda_1, \cdots, \lambda_n)$ and the λ_i are in fact of modulus one, $\lambda_j = e^{i\alpha_j}$. These λ_i may be regarded as "radial" variables, while V represents the "angular" variables. Note that V has to be restricted **not** to commute with the diagonal matrix Λ. If the latter is generic, with distinct eigenvalues λ_i, V lives in $\mathrm{U}(n)/\mathrm{U}(1)^n$. The natural metric, invariant under $U \mapsto U'U$ or $\mapsto UU'$, reads $\mathrm{tr}\,(\mathrm{d}U\mathrm{d}U^\dagger)$. But $\mathrm{d}U = V(\mathrm{d}\Lambda + [\mathrm{d}X, \Lambda])V^\dagger$, where $\mathrm{d}X := V^\dagger \mathrm{d}V$ is anti-Hermitian (and with no diagonal elements, why?). Thus $\mathrm{tr}\,(\mathrm{d}U\mathrm{d}U^\dagger) = \sum_i |\mathrm{d}\alpha_i|^2 + 2\sum_{i<j}|\mathrm{d}X_{ij}|^2|\lambda_i - \lambda_j|^2$ which defines the metric tensor $g_{\alpha\beta}$ in coordinates $\xi^\alpha = (\alpha_i, \Re X_{ij}, \Im X_{ij})$ and determines the integration measure

$$d\mu(U) = \sqrt{\det g}\prod \mathrm{d}\xi^\alpha = \mathrm{const.}\ |\Delta(e^{i\alpha})|^2 \prod \mathrm{d}\alpha_i d\mu(V)\ . \tag{C-8}$$

Here $\Delta(\lambda)$ is the *Vandermonde determinant*

$$\Delta(\lambda) := \prod_{i<j}(\lambda_i - \lambda_j) = \begin{vmatrix} \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \\ \vdots & & & \vdots \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ 1 & 1 & \cdots & 1 \end{vmatrix}\ . \tag{C-9}$$

The "radial" part of the integration measure is thus given by $|\Delta(e^{i\alpha})|^2 \prod \mathrm{d}\alpha_i$ up to a factor, or equivalently

$$d\mu(U) = \mathrm{const.}\prod_{i<j}\sin^2\left(\frac{\alpha_i - \alpha_j}{2}\right)\prod \mathrm{d}\alpha_i \times \text{angular part}\ . \tag{C-10}$$

Note that this radial part of the measure suffices if one has to integrate over the group a function of U which is invariant by $U \to VUV^\dagger$, $V \in \mathrm{U}(n)$. For example $\int d\mu(U)\,\mathrm{tr}\,P(U)$, with P a polynomial.

Exercises for chapter 1.

A. *Action of a group on a set*

A group G is said to act on a set E if there exists a homomorphism β of G into the group of bijections of E into itself.

1. Write explicitly the required conditions.

One then defines the *orbit* $O(x)$ of a point $x \in E$ as the set of images $\beta(g)x$ for all $g \in G$.

2. Show that belonging to the same orbit is an equivalence relation.

3. Example : action of $\mathrm{O}(n)$ on $\mathbb{R}^n$. What are the orbits?

4. A space is *homogeneous* if it has only one orbit. Show that a trivial example is given by the action of translations on $\mathbb{R}^n$. More generally, what can be said of the left action of G on itself, with $E = G$? Give other examples of homogeneous spaces for $G = \mathrm{O}(3)$ or $\mathcal{L}$ =O(3,1).

5. One also defines the *isotropy group* $S(x)$ of the element $x \in E$, (also called *stabilizer*, or, by physicists, *little group*): this is the subgroup of G leaving x invariant:

$$S(x) = \{g \in G | \beta(g)x = x\} \ . \tag{1-49}$$

Show that if x and y belong to the same orbit, their isotropy groups are conjugate. What is the isotropy group of a point $x \in \mathbb{R}^n$ under the action of $\mathrm{SO}(n)$? of a time-like vector p in Minkowski space under the action of the Lorentz group? Is $S(x)$ an invariant subgroup?

6. Show that there exists a bijection between points of the orbit $O(x)$ and the coset space $G/S(x)$. For a finite group G, deduce from it a relation between the orders (cardinalities) of G, $O(x)$ and $S(x)$. Is this set $G/S(x)$ homogeneous for the action of G ?

Chap. 2 will be devoted to the particular case where E is a vector space, with the linear transformations of $\mathrm{GL}(E)$ acting as bijections: one then speaks of representations of G in E.

B. *Lie groups and algebras of dimension 3.*

1. Recall the definition of the group SU(1,1). What is its dimension ?

2. Which equation defines its Lie algebra? What does that imply on the matrix elements of $X \in$ su(1,1) ? Prove that one may write a basis of su(1,1) in terms of 3 Pauli matrices and compute their commutation relations. Is this algebra isomorphic to the so(3) algebra?

3. One now considers the linear group SL(2,$\mathbb{R}$). What is its definition ? How is its Lie algebra defined? Give a basis in terms of Pauli matrices.

4. Prove the isomorphism of the two algebras su(1,1) et sl(2,$\mathbb{R}$).

5. Same questions with the algebra so(2,1) : definition, dimension, commutation relations, isomorphism with one of the previous algebras?

6. Using the Cartan criteria, discuss the semi-simplicity and the compacity of these various algebras. What is their relationship with su(2)?

(For the geometric relationship between the groups SU(1,1), SL(2,$\mathbb{R}$) et SO(1,2)), see §13 and §24, vol. 1 in [DNF].

C. *Casimir operators in* u(n).

1. Prove that the n^2 matrices $t_{(ij)}$ of size $n \times n$, $1 \leq, i, j \leq n$, with elements $(t_{(ij)})_{ab} = \delta_{ia}\delta_{jb}$ form a basis of the algebra u(n). Compute their commutation relations and the structure constants of the algebra.

2. Compute the Killing form in that basis and check that the properties related to Cartan criteria are satisfied.

3. Show that the elements in the envelopping algebra $C^{(r)} = \sum_{1\leq i_1,i_2,\cdots i_r \leq n} t_{(i_1 i_2)} t_{(i_2 i_3)} \cdots t_{(i_r i_1)}$ commute with all $t_{(ij)}$ and are thus Casimir operators of degree r.

4. How to modify this discussion for the su(n) algebra? ([Bu], chap 10).

Problem : Conformal transformations

I-1. We recall that in a (classical) local, translation invariant field theory, one may define a *stress-energy tensor* $\Theta_{\mu\nu}(x)$ such that

- under an infinitesimal change of coordinates $x^\mu \to x'^\mu = x^\mu + a^\mu(x)$, the action has a variation
$$\delta S = \int d^d x \, (\partial_\mu a_\nu) \, \Theta^{\mu\nu}(x) \ ; \tag{1-50}$$
- $\Theta_{\mu\nu}$ is conserved: $\partial_\mu \Theta^{\mu\nu}(x) = 0$;
- we assume that $\Theta_{\mu\nu}$ is symmetric in μ , ν.

Prove that if Θ is traceless, $\Theta_\mu^{\ \mu} = 0$, the action is also invariant under dilatations, $x^\mu \to x'^\mu = (1 + \delta\lambda)x^\mu$.

2. In a Riemannian or pseudo-Riemannian manifold of dimension d, with a metric tensor $g_{\mu\nu}(x)$ of signature $\{(+1)^p, (-1)^{d-p}\}$, a *conformal transformation* is a coordinate transformation $x^\mu \to x'^\mu$ which is a *local dilatation* of lengths

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu \to ds'^2 = g_{\mu\nu}(x') dx'^\mu dx'^\nu = \alpha(x) ds^2 \tag{1-51}$$

a) Write the infinitesimal form of that condition, when $x^\mu \to x'^\mu = x^\mu + a^\mu(x)$. (Hint : One may relate the dilatation parameter $1 + \delta\alpha$ to a^μ by taking some adequate trace.)
b) Prove that for an Euclidean or pseudo-Euclidean space of metric $g_{\mu\nu} = \mathrm{diag}\,\{(+1)^p, (-1)^{d-p}\}$, that condition may be recast as

$$\partial_\mu a_\nu + \partial_\nu a_\mu = \frac{2}{d} g_{\mu\nu} \partial_\rho a^\rho \ . \tag{1-52}$$

3. Prove, using (1-50,1-52), that under the conditions of 1. and 2.b, any field theory invariant under translations, rotations and dilatations is also invariant under conformal transformations.

4. We now study consequences of (1-52). We set $D := \frac{1}{d}\partial_\rho a^\rho$.

- a) Differentiating (1-52) with respect to x^ν , prove that
$$\partial^2 a_\mu = (2 - d)\partial_\mu D. \tag{1-53}$$

- b) Differentiating (1-53) w.r.t. x^μ, prove that in dimension $d > 1$, D is a harmonic function : $\partial^2 D = 0$.

- c) We assume in the following that $d \geq 2$. Differentiating (1-53) w.r.t. x^ν, symmetrizing it in μ and ν and using (1-52), prove that if $d > 2$, then $\partial_\mu \partial_\nu D = 0$. Show that it implies the existence of a constant scalar h and of a constant vector k such that $D = k_\mu x^\mu + h$.

- d) Differentiating (1-52) w.r.t. x^σ and antisymmetrizing it in ν and σ, prove that

$$\partial_\mu(\partial_\sigma a_\nu - \partial_\nu a_\sigma) = 2(g_{\mu\nu} k_\sigma - g_{\mu\sigma} k_\nu) = \partial_\mu(2k_\sigma x_\nu - 2k_\nu x_\sigma). \tag{1-54}$$

- e) Show that it implies the existence of a constant skew-symmetric tensor $l_{\sigma\nu}$ such that

$$\partial_\sigma a_\nu - \partial_\nu a_\sigma = (2k_\sigma x_\nu - 2k_\nu x_\sigma) + 2l_{\sigma\nu}\,, \tag{1-55}$$

  which, together with (1-52), gives

$$\partial_\sigma a_\nu = x_\nu k_\sigma - x_\sigma k_\nu + l_{\sigma\nu} + g_{\nu\sigma} k_\rho x^\rho + h g_{\nu\sigma}\ .$$

- f) Conclude that the general expression of an infinitesimal conformal transformation in dimension $d > 2$ reads

$$a_\nu = k_\sigma x^\sigma x_\nu - \frac{1}{2} x_\sigma x^\sigma k_\nu + l_{\sigma\nu} x^\sigma + h x_\nu + c_\nu \tag{1-56}$$

  with c a constant vector[6]. On how many independent real parameters does such a transformation depend in dimension d ?

II-1. One learns in geometry that in the (pseudo-)Euclidean space of dimension $d > 2$, conformal transformations are generated by translations, rotations, dilatations and "special conformal transformations", obtained by composition of an inversion $x^\mu \to x^\mu/x^2$, a translation and again an inversion. Write the finite and the infinitesimal forms of special conformal transformations, and show that this result is in agreement with (1-56), which justifies the previous assertion.

2. Write the expression of infinitesimal generators P_μ of translations, $J_{\mu\nu}$ of rotations, D of dilatations and K_μ of special transformations, as differential operators in x.

3. Write with the minimum of calculations the commutation relations of these generators (Hint : use already known results on the generators P_μ and $J_{\mu\nu}$, and make use of homogeneity and of the definition of special conformal transformations to reduce the only non-trivial computation to that of $[K_\mu, P_\nu]$). Check that these commutation "close" on this set of generators P, J, D et K.

4. What is the dimension of the conformal group in the Euclidean space $\mathbb{R}^d$?

III-1. To understand better the nature of the conformal group, one now maps the space $\mathbb{R}^d$, completed by the point at infinity and endowed with its metric $\mathbf{x}^2 = x_1^2 + \cdots + x_d^2$, on the

---

[6] This pretty argument is due to Michel Bauer.

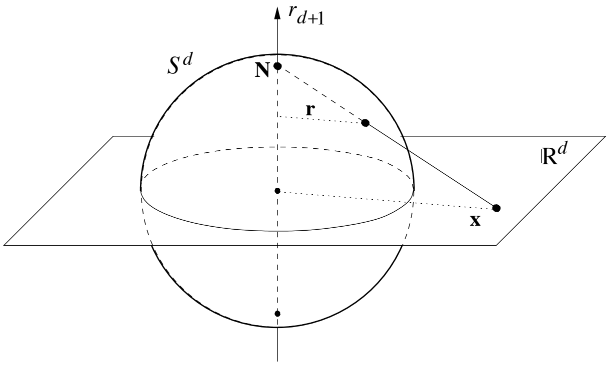


Figure 1.5: Stereographic projection from the North pole

sphere S^d. This sphere is defined by the equation $\mathbf{r}^2 + r_{d+1}^2 = 1$ in the space $\mathbb{R}^{d+1}$, and the mapping is the stereographic projection from the "North pole" $\mathbf{r} = 0,\ r_{d+1} = 1$ (see fig. 1.5). Prove that

$$\mathbf{r} = \frac{2\mathbf{x}}{\mathbf{x}^2+1} \qquad r_{d+1} = \frac{\mathbf{x}^2-1}{\mathbf{x}^2+1}\ .$$

What is the image of the point at infinity ? What is the effect of the inversion in $\mathbb{R}^d$ on the point $r = (\mathbf{r}, r_{d+1}) \in S^d$?

2. The previous sphere is in turn regarded as the section of the light-cone $\mathcal{C}$ in Minkowski space $\mathcal{M}_{d+1,1}$ of equation $z_0^2 - \mathbf{z}^2 - z_{d+1}^2 = 0$ by the hyperplane $z_0 = 1$. Prove that this establishes a one-to-one correspondance between points of $\mathbb{R}^d \cup \{\infty\}$ and *rays* of the light-cone (*i.e.* vectors up a dilatation) and that the expression of $\mathbf{x} \in \mathbb{R}^d$ as a function of $z = (z_0, \mathbf{z}, z_{d+1}) \in \mathcal{C}$ is

$$\mathbf{x} = \frac{\mathbf{z}}{z_0 - z_{d+1}}\ .$$

3. We now want to prove that the action of the conformal group in $\mathbb{R}^d$ follows from *linear* transformations in $\mathcal{M}_{d+1,1}$ that preserve the light-cone. Without any calculation, show that these transformations must then belong to the Lorentz group in $\mathcal{M}_{d+1,1}$, that is $O(d+1,1)$.

a) What are the linear transformations of z corresponding to rotations of $\mathbf{x}$ in $\mathbb{R}^d$? Show that dilatations of x correspond to "boosts" of rapidity β in the plane (z_0, z_{d+1}), by giving the relation between the dilatation parameter and the rapidity.

b) Let us now consider transformations of $O(d+1,1)$ that preserve $z_0 - z_{d+1}$. Write the matrix T_a of such an infinitesimal transformation acting on coordinates $(z_0, \mathbf{z}, z_{d+1})$, and such that $\delta\mathbf{z} = \mathbf{a}(z_0 - z_{d+1})$ (to first order in $\mathbf{a}$). To which transformation of $\mathbf{x} \in \mathbb{R}^d$ does it correspond? Compute by exponentiation of T_a the matrix of a finite transformation (Hint: compute the first powers T_a^2, $T_a^3 \dots$).

c) What is finally the interpretation of the inversion in $\mathbb{R}^d$ in the Lorentz group of $\mathcal{M}_{d+1,1}$? What can be said about special conformal transformations? What is the dimension of the group $O(d+1,1)$? What can be concluded about the relation between the Lorentz group in Minkowski space $\mathcal{M}_{d+1,1}$ and the conformal group $\mathbb{R}^d$?

IV. Last question: Do you know conformal transformations in the space $\mathbb{R}^2$ that are *not* of the type discussed in **II**.1?