Simplicity in the Structure of QED and Gravity Amplitudes

Pierre Vanhove

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based on

- 0802.0868, 0805.3682 and 0806.1726 with N.E.J. Bjerrum-Bohr
- 0811.3405 + work in progress with N.E.J. Bjerrum-Bohr and S. Badger

Motivations

We have learned a lot about the structure of $\mathcal{N} = 4$ SYM amplitude, and its dual description using the AdS/CFT correspondence

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In that respect gravity is closer to QED (IR tree soft gravitons Weinberg in 1965)

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We will explain that new organisation formalism for gravity and QED is needed in order to reflect these tree level properties at loop order

Outline

1. Multiloop amplitudes in $\mathcal{N} = 8$ supergravity
2. The no triangle property of $\mathcal{N} = 8$ supergravity and QED amplitudes
3. Conclusion & Outlook

Part I

Multiloop amplitudes in $\mathcal{N} = 8$ supergravity

Constraints on $\mathcal{N} = 8$ supergravity amplitudes

An L-loop n-point gravity amplitude in D-dimensions has the dimension

$$[\mathfrak{M}_L^{(D)}] = \mathrm{mass}^{(D-2)L+2}$$

Duality argument from string theory motivates that the low-energy limit of $\mathcal{N} = 8$ amplitudes [Green,Russo,Vanhove]

$$[\mathfrak{M}_L^{(D)}] = \mathrm{mass}^{(D-2)L-6-2\beta_L} \, D^{2\beta_L} R^4 + \cdots$$

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$$D \geq D_c = 2 + \frac{6 + 2\beta_L}{L}$$

UV behaviour of $\mathcal{N} = 8$ supergravity amplitudes

- Critical dimension for UV divergence is [Green,Russo,Vanhove]

$$D \geq D_c = 2 + \frac{6 + 2\beta_L}{L}$$

- Depending on the various implementation of linearized supersymmetry

$$6 \leq 6 + 2\beta_L \leq 18$$

- Which gives a possible first divergence in $D = 4$

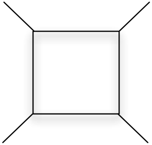
$$3 \leq L \leq 9$$

  - $L \geq 3$ [Howe, Lindstrom, Stelle]
  - $L = 8$ [Kallosh]
  - $L = 9$ [Green,Russo,Vanhove]

Loop amplitudes in $\mathcal{N} = 8$ supergravity

One loop amplitude is given by the scalar box integral with

$$\beta_1 = 0$$



$$\mathfrak{M}_1^{(D)} = R^4 \int_0^\infty \frac{dT}{T}\, T^{\frac{(8-D)}{2}} \int \prod_{i=1}^{3} du_i\, 1 \prod_{1\leq i<j\leq 4} e^{-(k^i\cdot k^j)\, G_{ij}^{(1)}}$$

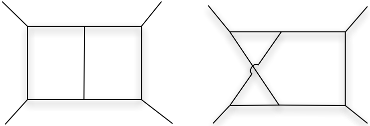
[Green,Schwarz,Brink]

At higher point the answer is as well given by boxes only (at least in four dimensions) [Bern, Bjerrum-bohr, Dunbar; Bjerrum-bohr, Dunbar, Ita, Perkins; Bjerrum-bohr, Vanhove]

Loop amplitudes in $\mathcal{N}=8$ supergravity

Two-loop amplitude is given by the sum of the planar and non-planar doubleblox scalar integral

$$\beta_2 = 2$$



[Bern, Dunbar, Dixon, Perelstein, Rozowsky],[D'Hoker, Phong; Berkovits; Berkovits, Mafra]

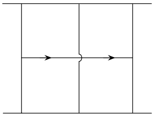
$$\mathfrak{M}_2^{(D)} = R^4 \int \prod_{i=1}^{3} dL_i \oint \prod_{i=1}^{4} du_i \, ((k_i \cdot k_j)\Delta_{ij})^2 \prod_{1\leq i<j\leq 4} e^{-(k^i \cdot k^j)\, G_{ij}^{(2)}}$$

[Green,Russo,Vanhove]

Loop amplitudes in $\mathcal{N} = 8$ supergravity

Three-loop amplitude have an integrand that satisfies the rule

$$\beta_3 = 3$$



[Bern, Dixon,Roiban; Bern, Carrasco, Dixon, Johansson, Kosower, Roiban]

[Bern, Carrasco, Dixon, Johansson, Roiban], [Berkovits]

$$\mathfrak{M}_3^{(D)} = R^4 \int \prod_{i=1}^{6} L_i \prod_{i=1}^{4} \oint du_i ((k^m \cdot k^n)(k^p \cdot \Delta_{mnp}))^2 \prod_{1\leq i<j\leq 4} e^{-(k^i \cdot k^j)\, G_{ij}^{(3)}}$$

The integrand has the structure predicted using Berkovits' pure spinor formalism

Non-renormalisation theorems and the $\beta_L = L$ rule

The $\beta_L = L$ rule

$$[\mathfrak{M}_L^{(D)}] = \mathrm{mass}^{(D-4)L-6}\, D^{2L} R^4$$

- 1-loop non-renormalisation of R^4: $\beta_L \geq 2 \qquad L \geq 2$

  First UV divergence in 4D: $L \geq 3 + \beta_L \quad \geq \quad 5$ loops

- 2-loop non-renormalisation of $D^4 R^4$: $\beta_L \geq 3 \qquad L \geq 3$

  First UV divergence in 4D: $L \geq 3 + \beta_L \quad \geq \quad 6$ loops

- 3-loop non-renormalisation of $D^6 R^4$: $\beta_L \geq 4 \qquad L \geq 4$

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- *Same critical dimension for UV divergences* as $\mathcal{N} = 4$ SYM

$$D \geq D_c = 4 + \frac{6}{L}$$

- If true to all order the theory would be perturbatively UV finite in 4D
- Up to now the linearized supersymmetry gives $\beta_L = 12$ for $L \geq 6$ which gives a first divergence at 9-loops.

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Part II

The no triangle property of one-loop amplitudes in $\mathcal{N} = 8$ supergravity and massless QED

Basis of one-loop scalar integral functions

In $D = 4 - 2\epsilon$ one expands the amplitudes on a basis of scalar integral functions with massive external legs

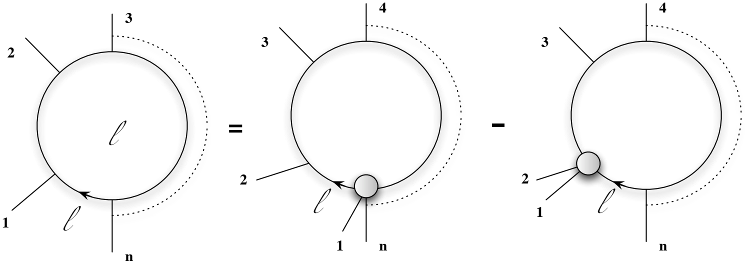
$$\mathfrak{M}_{n;1}^{(4-2\epsilon)} = \sum_i b_i \, I_{\Box}^{(i)} + \sum_i t_i \, I_{\triangleright}^{(i)} + \sum_i b_i \, I_{\circ}^{(i)} + c_{\text{rational pieces}}$$

This basis of scalar integral functions capture the IR and UV divergences of the one-loop amplitudes

The Reduction formulas

On-shell integral are reduced to the scalar integral functions using the reduction formulas stating that one can trade powers of loop momentum to massive legs ($k_i^2 = 0$)

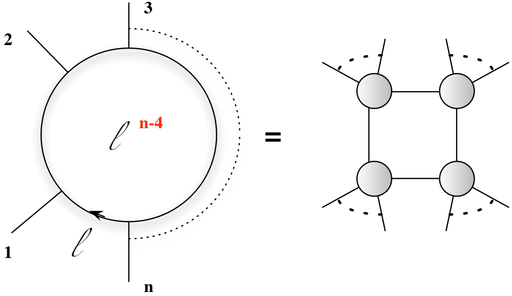
$$\int d^D\ell \, \frac{2(\ell\cdot k_1)}{\ell^2\,(\ell-k_1)^2} \times (\cdots) = \int d^D\ell \, \left(\frac{1}{(\ell-k_1)^2} - \frac{1}{\ell^2}\right) \times (\cdots)$$



These reduction formula are well adapted to the soft and collinear singularities of QCD/ $\mathcal{N} = 4$ SYM amplitudes

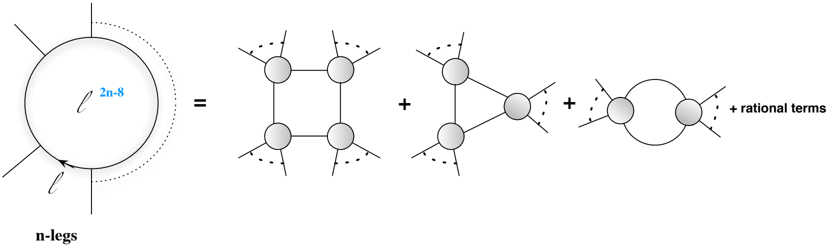
The no triangle property in $\mathcal{N} = 8$

Since $\mathcal{N} = 4$ super-Yang-Mills amplitudes have $n - 4$ powers of loop momenta they are reducible to boxes only



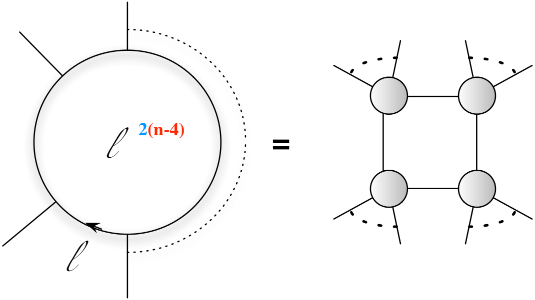
The no triangle property in $\mathcal{N} = 8$

$\mathcal{N} = 8$ amplitudes $2n - 8$ powers of loop momenta should contains boxes, triangles, bubbles and rational terms



The no triangle property in $\mathcal{N} = 8$

Explicit computations by [Bjerrum-Bohr et al., Bern et al.] showed that the amplitudes reduce to scalar box integral functions like for $\mathcal{N} = 4$ SYM



The no triangle property of $\mathcal{N} = 8$ amplitudes

This result was unexpected because the counting was based on reduction formula that did not take into account *all* the cancellations occurring in Gravity.

In particular the standard reduction formula
does not reflect the softer IR behaviour of gravity amplitudes

The no triangle property of $\mathcal{N} = 8$ amplitudes

- Gravity does not have color factor
  - summation over all the permutations at one-loop
  - Sum over all the planar and nonplanar diagrams at higher loop order
- Gauge invariance $\varepsilon_{\mu\nu} \to \varepsilon_{\mu\nu} + \partial_\mu v_\nu + \partial_\nu v_\mu$

Unordered amplitudes are more than just the sum over all ordering of color ordered amplitudes.
All the various ordering have the same tensorial structure

$$\mathfrak{M} = \sum_r t_r \int_0^\infty \frac{dT}{T} \int_0^1 \prod_{i=1}^{n-1} d\nu_i \, T^{-D/2+n} \, \mathcal{P}_n(\partial Q_n) \exp(-T \sum_{r,s} (k_r \cdot k_s) \, G^{(1)}_{r,s})$$

The no triangle property of $\mathcal{N} = 8$ amplitudes

Loop momentum is a total derivative $k_i \cdot \ell \sim \partial_{\nu_i} Q_n$ which can be freely integrated

$$\int_0^\infty \frac{dT}{T} \int_0^1 d^{n-1}\nu \, \partial_{\nu_i} Q_n(\cdots) = -\int_0^\infty \frac{dT}{T} \int_0^1 d^{n-1}\nu \, Q_n \partial_{\nu_i}(\cdots)$$

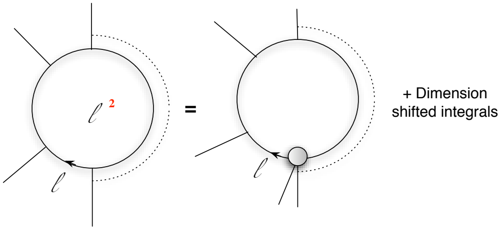
$\partial_{\nu_i}\partial_{\nu_j} Q_n \sim (k_i \cdot k_j)[\delta(\nu_i - \nu_j) - 1]$ does not contain any loop momenta

two powers of loop momentum are cancelled at each steps

The no triangle property of $\mathcal{N} = 8$ amplitudes

This implies new reduction formulas for unordered integrals

[Bjerrum-Bohr, Vanhove]

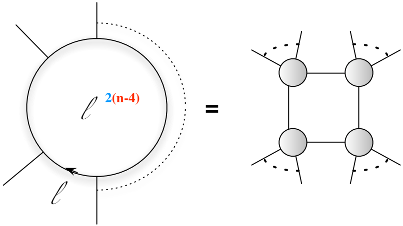


- These reduction formulas reflect that the graviton amplitudes have softer IR singularities than for QCD [Weinberg]
- The dimension shifted contributions cancel in the total amplitude

Gauge invariance implies that one can push all the ‘triangles’ into total derivative which cancel in the total amplitude (no boundary contributions) (‘cancelled propagator argument’)

The no triangle property of $\mathcal{N} = 8$ amplitudes

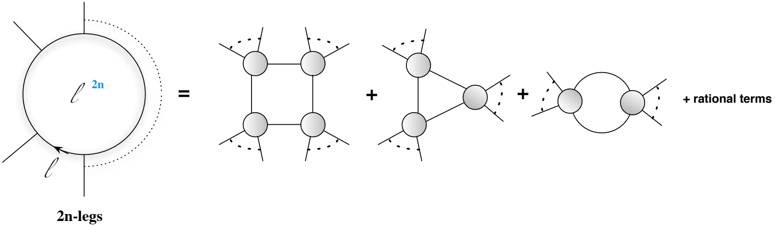
For $\mathcal{N} = 8$ sugra amplitude the no triangle property arises because the amplitude has $n - 4$ powers of ℓ^2



No triangle property in massless QED

The new reduction formula applies to QED one-loop amplitudes as well

The n-photon one-loop amplitude in massless QED has the structure



- Using the ordered reduction formula the amplitude should contain all the scalar integral functions

$$\lim_{z\to\infty} A^{tree}_{e^+e^-\to n\,\gamma}(p_a + zp_b, p_b - zp_a) \sim \frac{z^s}{z^{n-2}}$$

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The *n*-photon one-loop amplitude in massless QED has the structure

- For $n = 4$ external photons [Mahlon; Bern et al.]

$$A^{QED}_{4\,\gamma} = I_{box} + I_{Triangle} + I_{bubble}$$

- For $n = 6$ external photons [Bernicot, Guillet; Binoth, Heinrich, Gehrmann, Mastrolia]

$$A^{QED}_{6\,\gamma} = I_{box} + I_{Triangle}$$

$$\lim_{z\to\infty} A^{tree}_{e^+e^-\to n\,\gamma}(p_a + zp_b, p_b - zp_a) \sim \frac{z^s}{z^{n-2}}$$

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- For $2n \geq 8$ external photons [Badger, Bjerrum-Bohr, Vanhove]

$$A^{QED}_{2n\geq 8\,\gamma} = I_{box}$$

This is due to the very good high-energy behaviour of the tree amplitudes

$$\lim_{z\to\infty} A^{tree}_{e^+e^-\to n\,\gamma}(p_a + zp_b, p_b - zp_a) \sim \frac{z^s}{z^{n-2}}$$

Summary & Outlook

We have explained that colorless gauge theory like gravity and QED exhibit important cancellations in on-shell amplitudes. [Bjerrum-Bohr,Vanhove]

- Cancellations are already seen at tree level which have a better high-momentum limit as naively expected
- The expansion on a basis of scalar integral functions is much simplified and constrained
  - For gravity could this help extending the rule $\beta_L = L$ beyond 6 loops?
  - For QED: The one-loop answer is UV and IR finite (for $n \geq 4$ photons): Finite combination of boxes. Comparison with the dual conformal invariant basis of [Drummond, Henn, Korchemsky, Sokatchev, Smirnov], [Work in progress]

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