On string side of AdS/CFT

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Problems for string theory:

- spectrum of states (exact energies in λ)
- construction of vertex operators: closed and open (?) string ones
- computation of their correlation functions (e.g., graviton scattering, application to DIS in QCD ?)
- expectation values of various Wilson loops
- gluon scattering amplitudes (??)
- generalizations to simplest less supersymmetric cases
  – orbifolds, exactly marginal deformations, ...
- strings at finite temperature in $AdS_5 \times S^5$ ...
- solution of type 0 theory in $AdS_5 \times S^5$ ...
- non-critical superstrings: $AdS_5 \times S^1$, ...

$AdS_5 \times S^5$

Recent remarkable progress in quantitative understanding
interpolation from weak to strong 't Hooft coupling
based on/checked by perturbative gauge theory (4-loop in λ)
and perturbative string theory (2-loop in $\frac{1}{\sqrt{\lambda}}$) "data"
and assumption of exact integrability
string energies = dimensions of gauge-invariant operators

$$E(\sqrt{\lambda}, I, m, ...) = \Delta(\lambda, I, m, ...)$$

I - charges of $SO(2,4) \times SO(6)$: $S_1, S_2;\ J_1, J_2, J_3$
m - windings, folds, cusps, oscillation numbers, ...
Operators: $\mathrm{Tr}(\Phi_1^{J_1}\Phi_2^{J_2}\Phi_3^{J_3}D_+^{S_1}D_\perp^{S_2}...F_{mn}...\Psi...)$

Solve susy 4-d CFT = Solve superstring in R-R background:
compute $E = \Delta$ for any λ (and I,m)

Perturbative expansions are opposite:

$\lambda \gg 1$ in perturbative string theory

$\lambda \ll 1$ in perturbative planar gauge theory

Last 6 years: remarkable progress:

"semiclassical" string states with large quantum numbers

dual to "long" gauge operators (BMN, GKP, ...)

$E = \Delta$ – same (in some cases !) dependence on $J, m, ...$

coefficients = interpolating functions of λ

Current status:

asymptotic Bethe Ansatz (BES)

+ "phenomenological" improvement by Luscher corrections

(remarkably successful at weak coupling: Janik's talk)

Beyond restriction to long operators or

semiclassical strings with large quantum numbers ?

Need first-principles understanding of quantum

$AdS_5 \times S^5$ superstring theory:

1. Solve string theory in $AdS_5 \times S^5$ on $R^{1,1}$
relativistic 2d S-matrix including dressing phase (if any);
asymptotic BA for the spectrum
2. Generalize to finite-energy closed strings – solve on $R \times S^1$
$\rightarrow$ TBA as for standard sigma models (?)

Little progress so far ...
reformulation in terms of current variables
("Pohlmeyer reduction") seems most promising approach

In the meantime study in detail semiclassical string states
for various values of parameters including $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ corrections

Problems:

1. subleading terms in large-spin expansion?
compare to gauge theory – also partially controlled by functional relation and reciprocity?
(cf. Basso, Korchemsky; talk by Forini)
2. dependence on spin parameter is same (i.e. coefficients are interpolating functions as in cusp anomaly case) or we do need to resum also the spin dependence to compare?
3. formal small-spin limit – may shed light on dimensions of short operators at strong coupling if (!) limits commute

Principles of comparison: gauge states vs string states

1. look at states with same global $SO(2,4) \times SO(6)$ charges
e.g., (S, J) – "SL(2) sector" – $\mathrm{Tr}(D_+^S \Phi^J)$
J=twist=spin-chain length
2. assume no "level crosing" while changing λ
e.g. minimal/maximal energy states for given (S, J) should be in correspondence

Gauge theory:

$\Delta \equiv E = S + J + \gamma(S, J, m, \lambda)$, $\quad \gamma = \sum_{k=1}^{\infty} \lambda^k \gamma_k(S, J, m)$

m stands for other conserved charges labelling states

(e.g., winding in $S^1 \subset S^5$ or number of spikes in AdS_5)

fix S, J, ... and expand in λ; may then expand in large/small S, J, ...

String theory:

$E = S + J + \gamma(\mathcal{S}, \mathcal{J}, m, \sqrt{\lambda})$, $\quad \gamma = \sum_{k=-1}^{\infty} \frac{1}{(\sqrt{\lambda})^k} \widetilde{\gamma}_k(\mathcal{S}, \mathcal{J}, m)$

$\mathcal{S} = \frac{S}{\sqrt{\lambda}}$, $\quad \mathcal{J} = \frac{J}{\sqrt{\lambda}}$, $\quad m$

- semiclassical parameters fixed in the $\frac{1}{\sqrt{\lambda}}$ expansion

Various possible limits:

(i) BMN-like "fast-string" limit – "locally-BPS" long oprators

G: $J \gg 1$, $\frac{S}{J}$=fixed, m=fixed

S: $\mathcal{J} \gg 1$, $\frac{\mathcal{S}}{\mathcal{J}}$=fixed, m=fixed

direct agreement of first few orders in $\frac{1}{\mathcal{J}}$
(including 1- and 2-loop string corrections)
to 1- and 2-loop gauge theory spin chain results
including $1/J$ and $1/J^2$ finite size corrections
(Frolov, AT 03; Beisert, Minahan, Staudacher, Zarembo 03; ...)
"non-renormalization" due to susy (and structure)
no interpolation functions of λ, no need to resum J dependence
$E = S + J + \frac{\lambda}{J}\Big[h_1(\frac{S}{J}, m) + \frac{1}{J}h_2(\frac{S}{J}, m) + ...\Big] + ...$
captured by effective Landau-Lifshitz model
on both string and spin chain side
need interpolation functions at higher orders (dressing phase)

(ii) "Slow Long strings" – long non-BPS operators like $\mathrm{Tr}(\Phi D_+^S \Phi)$
G: $\ln S \gg J \gg 1$
S: $\ln \mathcal{S} \gg \mathcal{J},\ \mathcal{J} = 0$ or $\mathcal{J}$=fixed
$E = S + f(\lambda) \ln S + ...$

S dependence is same but need an interpolatig function

$f(\lambda \gg 1) = a_1\sqrt{\lambda} + ... \,, \qquad f(\lambda \ll 1) = c_1\lambda + ...$

(iii) "Fast Long strings"

G: $S \gg J \gg 1$, $j \equiv \frac{J}{\ln S}$=fixed

S: $\mathcal{S} \gg \mathcal{J} \gg 1$, $\ell \equiv \frac{\mathcal{J}}{\ln \mathcal{S}}$=fixed = $\frac{j}{\sqrt{\lambda}}$

G: $E = S + f(j,\lambda)\ln S + ...$

$f = a_1(\lambda)j + a_2(\lambda)j^3 + ...$

S: $E = S + f(\ell, \sqrt{\lambda})\ln \mathcal{S} + ...$

$f = \sqrt{\lambda}\sqrt{1+\ell^2} + (c_1 + c_2\ell^2 \ln \ell + ...) + \frac{1}{\sqrt{\lambda}}(c_3\ell^2 \ln^2 \ell + ...) + ...$

[Belitsky, Gorsky, Korchemsky 06; Frolov, Tirziu, AT 06;
Alday, Maldacena 07, Freyhult, Rej, Staudacher 07;
Roiban, AT 07; Kostov, Serban, Volin 08; Basso, Korchemsky 08;
Gromov 08, Fioravanti et al 08, ...]

need a resummation in both λ and ℓ (or j) to match

general situation – G and S limits do not commute

String Theory in $AdS_5 \times S^5$

bosonic coset $\frac{SO(2,4)}{SO(1,4)} \times \frac{SO(6)}{SO(5)}$

generalized to supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$ (Metsaev, AT 98)

$$S = T \int d^2\sigma \Big[G_{mn}(x)\partial x^m \partial x^n + \bar\theta(D + F_5)\theta\partial x + \bar\theta\theta\bar\theta\theta\partial x\partial x + ... \Big]$$

tension $T = \frac{R^2}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi}$

Conformal invariance: $\beta_{mn} = R_{mn} - (F_5)^2_{mn} = 0$

Classical integrability of coset σ-model (Luscher-Pohlmeyer 76)

also for $AdS_5 \times S^5$ superstring (Bena, Polchinski, Roiban 02)

Progress in understanding of implications of (semi)classical integrability (Kazakov, Marshakov, Minahan, Zarembo 04,...)

Computation of 1-loop quantum superstring corrections (Frolov, AT; Park, Tirziu, AT, 02-04, ...)

results were used as input for 1-loop term

in strong-coupling expansion of the phase θ in BA
(Beisert, AT05; Hernandez, Lopez 06)

Tree-level S-matrix of BMN states from $AdS_5 \times S^5$ GS string agrees with limit of elementary magnon S-matrix
(Klose, McLoughlin, Roiban, Zarembo 06)

2-loop string corrections (Roiban, Tirziu, AT; Roiban, AT 07)
2-loop check of finiteness of the GS superstring;
agreement with BA
– implicit check of integrability of quantum string theory
– non-trivial confirmation of BES exact phase in BA
(Basso, Korchemsky, Kotansky 07; Basso's talk)

Key example of weak-strong coupling interpolation:

Spinning string in AdS_5

Folded spinning string in flat space:

$X_1 = \epsilon \ \sin\sigma \ \cos\tau, \quad X_2 = \epsilon \ \sin\sigma \ \sin\tau$

$$ds^2 = -dt^2 + d\rho^2 + \rho^2 d\phi^2 = -dt^2 + dX_i dX_i$$

$$t = \epsilon\tau \ , \qquad \rho = \epsilon \sin\sigma \ , \qquad \phi = \tau$$

If tension $T = \frac{1}{2\pi\alpha'} \equiv \frac{\sqrt{\lambda}}{2\pi}$

energy $E = \epsilon\sqrt{\lambda}$ and spin $S = \frac{\epsilon^2}{2}\sqrt{\lambda}$ satisfy Regge relation:

$$\mathcal{E} = \lambda^{1/4}\sqrt{2S}$$

Absence of quantum correction from GS action:

bosonic quadratic fluctuation Lagrangian

$\widetilde{L}_B = \dot{\widetilde{t}}^2 - \widetilde{t}'^2 - \dot{\widetilde{\rho}}^2 + \widetilde{\rho}'^2 - \widetilde{\rho}^2 - \bar{\rho}^2(\dot{\widetilde{\varphi}}^2 - \widetilde{\varphi}'^2) - 4\bar{\rho}\widetilde{\rho}\dot{\widetilde{\phi}}$

after rotation

$\widetilde{\rho} = \eta_1 \cos\tau + \eta_2 \sin\tau, \ \ \widetilde{\varphi} = -\eta_1 \sin\tau + \eta_2 \cos\tau,$

becomes the Lagrangian for free massless bosons
$\widetilde{L}_B = -\partial_a \widetilde{t} \partial^a \widetilde{t} + \partial_a \eta_1 \partial^a \eta_1 + \partial_a \eta_2 \partial^a \eta_2$
GS superstring action in flat space in general coordinates

$$L_F = (\sqrt{-g} g^{ab} \delta^{IJ} - \epsilon^{ab} s^{IJ}) \bar{\theta}^I \varrho_a D_b \theta^J$$

$\varrho_a = \Gamma_A E^A_\mu \partial_a X^\mu$, $D_a = \partial_a + \frac{1}{4} \partial_a X^M \omega^{AB}_M \Gamma_{AB}$
κ-symmetry gauge $\theta_1 = \theta_2 = \theta$
After rotation of fermions
$\widetilde{\theta} = e^{-\frac{1}{2}\alpha\Gamma_0\Gamma_2}\theta, \qquad \sinh\alpha = \tan\sigma$,
$\widetilde{D}_F = i\epsilon\Big(- \Gamma_0 \cos\sigma\, \partial_0 + \Gamma_1 \cos\sigma\, \partial_1 - \frac{1}{2}\Gamma_1 \sin\sigma \Big)$
rescaling $\widetilde{\theta} = \frac{1}{\sqrt{\epsilon \cos\sigma}} \vartheta$
end up with free massless Dirac operator $D_F = i(-\Gamma_0 \partial_0 + \Gamma_1 \partial_1)$
1-loop correction cancels out

Folded spinning string in AdS_5:

(de Vega, Egusquiza 96; Gubser, Klebanov, Polyakov 02)

$$ds^2 = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\phi^2$$

$$t = \kappa\tau, \quad \phi = w\tau, \quad \rho = \rho(\sigma)$$

$$\rho'^2 = \kappa^2\cosh^2\rho - w^2\sinh^2\rho, \qquad 0 < \rho < \rho_{\rm max}$$

$$\coth\rho_{\rm max} = \frac{w}{\kappa} \equiv \sqrt{1 + \frac{1}{\epsilon^2}}$$

ϵ measures length of the string

$$\sinh\rho = \epsilon\, {\rm sn}(\kappa\epsilon^{-1}\sigma, \, -\epsilon^2)$$

periodicity in $0 \leqslant \sigma < 2\pi$

$$\kappa = \epsilon\ {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; -\epsilon^2)$$

classical energy $E_0 = \sqrt{\lambda}\mathcal{E}_0$ and spin $S = \sqrt{\lambda}\mathcal{S}$

$$\mathcal{E}_0 = \epsilon\ {}_2F_1(-\frac{1}{2}, \frac{1}{2}; 1; -\epsilon^2), \qquad \mathcal{S} = \frac{\epsilon^2\sqrt{1+\epsilon^2}}{2}\ {}_2F_1(\frac{1}{2}, \frac{3}{2}; 2; -\epsilon^2)$$

solve for ϵ as in flat space – get analog of Regge relation

$$\mathcal{E}_0 = \mathcal{E}_0(\mathcal{S}) \ , \qquad E_0 = \sqrt{\lambda}\, \mathcal{E}_0(\frac{S}{\sqrt{\lambda}})$$

Flat space – AdS interpolation:
$\mathcal{E}_0 \sim \sqrt{\mathcal{S}}$ at $\mathcal{S} \ll 1$, $\quad \mathcal{E}_0 \sim \mathcal{S}$ at $\mathcal{S} \gg 1$

Novel AdS "Long string" limit: $\epsilon \gg 1$, i.e. $\mathcal{S} \gg 1$

$$\mathcal{E}_0 = \mathcal{S} + \frac{1}{\pi} \ln \mathcal{S} + ...$$

$\mathcal{S} \to \infty$: ends of string reach the boundary ($\rho = \infty$)
solution drastically simplifies

$$t = \kappa\tau, \quad \phi \approx \kappa\tau, \quad \rho \approx \kappa\sigma \ , \quad \kappa \sim \epsilon \sim \ln \mathcal{S} \to \infty$$

string length is infinite, $R \times R$ effective world sheet
$E = S$ from massless end points at AdS boundary (null geodesic)

$E - S = \frac{\sqrt{\lambda}}{\pi} \ln S$ from tension/stretching of the string
$\rho = \kappa\sigma + ...,\ \mathcal{S} \sim e^{2\kappa}$,
$\kappa \sim \ln \mathcal{S}$=length of the string:
$\frac{1}{\mathcal{S}^n} \sim e^{n\kappa}$ – finite size corrections
For $\mathcal{S} \to \infty$ can compute quantum superstring corrections to E
remarkably, they respect the $S + \ln S$ structure:
string solution is homogeneous $\to$ const coeffs
$\kappa \sim \ln \mathcal{S} \to \infty$ is “volume factor”

Subleading terms in large S expansion

(Beccaria, Forini, Tirziu, AT 08)

string has large but finite length: does not reach boundary

$E_0 = \sqrt{\lambda}\, \mathcal{E}(\mathcal{S})$:   expand in large $\mathcal{S}$

$$E_0(\mathcal{S} \gg 1) = S + a_0 \ln \mathcal{S} + a_1 + \frac{1}{\mathcal{S}}(a_2 \ln \mathcal{S} + a_3)$$

$$+ \frac{1}{\mathcal{S}^2}(a_4 \ln^2 \mathcal{S} + a_5 \ln \mathcal{S} + a_6) + O(\frac{\ln^3 \mathcal{S}}{\mathcal{S}^3})$$

$a_0 = \frac{\sqrt{\lambda}}{\pi}, \quad a_1 = \frac{\sqrt{\lambda}}{\pi} \ln(8\pi) - 1, \ ....$

Coefficients of $\frac{\ln^k \mathcal{S}}{\mathcal{S}^k}$ terms happen to be related

to coefficient of $\ln \mathcal{S}$ as suggested by

"functional relation" (Basso, Korchemsky 06)

$$E - S = f(E + S) = a_0 \ln(S + \frac{1}{2} a_0 \ln S + ...) + ...$$

$$a_2 = \frac{1}{2} a_0^2, \qquad a_4 = -\frac{1}{8} a_0^3, ...$$

Why that happens? Simple explanation:
look at near boundary limit where for large $\mathcal{S}$
string end moves moves along nearly null line at the boundary:
pp-wave limit: cusp anomaly as “pp-wave” anomaly
(Kruczenski, AT 08)

$$ds^2 = \frac{1}{z^2}\Big[2dx_+dx_- - \mu^2(z^2 + x_i^2)dx_+^2 + dx_i dx_i + dz^2\Big]$$

locally still AdS_5 in “rotated” coordinates
boundary: Penrose limit $R \times S^3 \ \rightarrow$ 4d pp-wave
string moving in pp-wave $\sigma_1 < z < R \to \infty$
$x_+ = \tau, \qquad x_- = v\tau, \qquad z = \sigma\ , \quad \sigma_1 = \frac{\sqrt{2v}}{\mu}$
$v = 0$ when string touches the boundary: $\mathcal{S} = \infty$ limit
conserved momenta ($\mu = 1,\ T = \frac{\sqrt{\lambda}}{\pi}$):
(Ishizeki, Kruczenski, Titziu, AT 08)
$P_+ = \frac{T}{2}\Big(\ln\frac{4R^2}{\sigma_1^2} - 1\Big)\ , \quad P_- = -\frac{T}{\mu\sigma_1^2}$
$P_+ = a_0 \ln P_- + a_1\ , \qquad P_+ = E - S,\ \ P_- = E + S$

$E - S = a_0 \ln[S + \frac{1}{2}(E-S)] + ...$
pp-wave limit effectively establishes contact with collinear conformal group in the boundary theory

What about quantum string corrections?
What controls subleading coefficients?

Express in terms of $S = \sqrt{\lambda}\mathcal{S}$:

$$E = S + f \ln S + f_c + \frac{1}{S}[f_{11} \ln S + f_{10}] + \frac{1}{S^2}[f_{22} \ln^2 S + f_{21} \ln S + f_{20}] + \mathcal{O}(\frac{\ln^3 S}{S^3})$$

equivalently

$$E = S + f \ln(S/\widetilde{f}_c) + \frac{1}{S}[f_{11} \ln(S/\widetilde{f}_c) + f'_{10}] + \frac{1}{S^2}[f_{22} \ln^2(S/\widetilde{f}_c) + f'_{21} \ln(S/\widetilde{f}_c) + f'_{20}] + \mathcal{O}(\frac{\ln^3 S}{S^3})$$

string tree plus 1-loop level:

$$f = \frac{\sqrt{\lambda}}{\pi}\left[1 - \frac{3\ln 2}{\sqrt{\lambda}} + \mathcal{O}(\frac{1}{\lambda})\right]$$

$$\widetilde{f}_c = \frac{e\sqrt{\lambda}}{8\pi}\left[1 + \frac{1}{\sqrt{\lambda}}(3\ln 2 - \mathrm{c}) + \mathcal{O}(\frac{1}{\lambda})\right]$$

$$f_{11} = \frac{\lambda}{2\pi^2}\left[1 - \frac{6\ln 2}{\sqrt{\lambda}} + \mathcal{O}(\frac{1}{\lambda})\right]$$

$$f'_{10} = \frac{\lambda}{2\pi^2}\left[0 - \frac{0}{\sqrt{\lambda}} + \mathcal{O}(\frac{1}{\lambda})\right] = 0$$

thus $f_{11} = \frac{1}{2}f^2$ as at tree level equivalently

$$
\begin{aligned}
f_c = -f \ln \widetilde{f}_c = \frac{\sqrt{\lambda}}{\pi}\Big[\ln \frac{8\pi}{\sqrt{\lambda}} - 1 \\
+ \frac{1}{\sqrt{\lambda}}\big(- 3\ln 2 \ \ln \frac{8\pi}{\sqrt{\lambda}} + \mathrm{c}\big) + \mathcal{O}(\frac{1}{\lambda})\Big] \\
f_{10} = f'_{10} - f_{11} \ln \widetilde{f}_c = f'_{10} + \frac{f_c}{f} f_{11} \\
= \frac{\lambda}{2\pi^2}\Big[\ln \frac{8\pi}{\sqrt{\lambda}} - 1 \\
+ \frac{1}{\sqrt{\lambda}}\big(- 6\ln 2\big[\ln \frac{8\pi}{\sqrt{\lambda}} - \frac{1}{2}\big] + \mathrm{c}\big) + \mathcal{O}(\frac{1}{\lambda})\Big]
\end{aligned}
$$

vanishing of f'_{10} is consequence of $f_{10} - \frac{f_c}{f} f_{11} = 0$

since $f_{11} = \frac{1}{2} f^2$, equivalent to $f_{10} - \frac{1}{2} f_c f = 0$

and is actually consequences of reciprocity at strong coupling

thus expect $f'_{10} = 0$ to be true to *all orders*

in strong coupling expansion → importance of function $\widetilde{f}_c$

undetermined c is sensitive to turning point contributions
an independent way of evaluating the 1-loop correction
using algebraic curve to compute fluctuation frequencies
leads to $\mathrm{c} = 6\ln 2 + \pi$ (N. Gromov 08)

What one finds on gauge theory side?
structurally same expansion

$$E - S = \sum_{m=0}^{\infty} \frac{e_m(\lambda, \ln S)}{S^m}$$

$$e_m(\lambda, \ln S) \ = \sum_k f_{mk}(\lambda) \ln^k S$$

suggests existence of new "interpolating functions"
Clarification:
at weak coupling $k > 0$ for twist $J = 2, 3$ *minimal* dimension
for $J > 3$ one finds also $\frac{J^k}{\ln^m S}$: leading term in $E - S - J$ is
$e_0 = k_1(J) + \frac{k_2(J)}{\ln^2 S} + ...$

$k_1 = k_{11} J + k_{10}, \quad k_2 = k_{23} J^3 + k_{22} J^2 + k_{21} J + k_{20}$
visible in 1-loop approximation in $SL(2)$ sector
(Belitsky, Gorsky, Korchemsky 06)
when $J, S \gg 1$ with $j \equiv \frac{J}{\ln S}$ fixed and small

$$e_0 = (k_{11} j + k_{23} j^3 + ...) \ln S$$

but should be true for finite J too (A. Rej)
Does not contradict strong-coupling expansion:
in string semiclassical expansion one cannot distinguish
between finite values of J with $\mathcal{J} = \frac{J}{\sqrt{\lambda}} \approx 0$ and $J = 0$
for large J, S with $\ell \equiv \frac{J}{\sqrt{\lambda} \ln S} = \frac{\mathcal{J}}{\ln S}$=fixed

$$e_0 = (n_1 \ell^2 + n_2 \ell^4 + ...) \ln S = n_1 \frac{\mathcal{J}^2}{\ln \mathcal{S}} + n_1 \frac{\mathcal{J}^4}{\ln^3 \mathcal{S}} + ...$$

dependence on arbitrary J and S is different at $\lambda < 1$ and $\lambda \gg 1$
to relate the two expansions one needs a resummation

rather than just simple interpolation functions of λ:

e.g., large S and λ limits do not commute beyond $\ln S$ term

Reciprocity at strong coupling

(Basso, Korchemsky 06)

"functional relation" for $E = S + J + \gamma(S, J)$

$$\gamma(S, J) = \mathrm{f}(s; J) = \mathrm{f}\left[S + \tfrac{1}{2}J + \tfrac{1}{2}\gamma(S, J); J\right] , \qquad s = \tfrac{1}{2}(E + S)$$

if f is "simple", e.g., $\mathrm{f}(S) = f \ln S + ...$

subleading ($\frac{\ln^k S}{S^m}, \ k < m$) coefficients are

partially controlled by special properties of f

"parity invariance" or "reciprocity" (for twist 2):

large S expansion of $\mathrm{f}(S)$ should run in inverse *even* powers of

quadratic Casimir of the collinear $SL(2, R)$ (for scalar operators)

$$\mathrm{f}(S) = \sum_{n=0}^{\infty} \frac{a_n(\ln C)}{C^{2n}} , \qquad C^2 = (S + \tfrac{1}{2}J)(S + \tfrac{1}{2}J - 1)$$

then get constraints on some of coefficients of subleading terms

$$f_{10} = \tfrac{1}{2} f \, (f_c - 1 + J)$$
$$f_{32} = \tfrac{1}{16} f \, [f^3 - 2f^2 (f_c - 1 + J) - 16 f_{21}] \, , \quad ...$$

at weak coupling: relations between $f_{mk}(\lambda)$ as series in λ
at strong coupling:

$$f \equiv \sqrt{\lambda} \, \bar{f} \, , \qquad \bar{f} = a_0 + \frac{b_0}{\sqrt{\lambda}} + \frac{c_0}{(\sqrt{\lambda})^2} + ...$$
$$f_c \equiv \sqrt{\lambda} \, \bar{f}_c \, , \qquad \bar{f}_c = a_c + \frac{b_c}{\sqrt{\lambda}} + \frac{c_c}{(\sqrt{\lambda})^2} + ...$$
$$f_{mk} \equiv (\sqrt{\lambda})^{m+1} \, \bar{f}_{mk} \, , \qquad \bar{f}_{mk} = a_{mk} + \frac{b_{mk}}{\sqrt{\lambda}} + \frac{c_{mk}}{(\sqrt{\lambda})^2} + ...$$

relations between series in $\frac{1}{\sqrt{\lambda}}$
finite $J = 2, 3, ...$ not distinguished from formal case of $J = 0$

if $\mathcal{J} = \frac{J}{\sqrt{\lambda}} = 0$, $\mathcal{S} = \frac{S}{\sqrt{\lambda}}$=finite $C^2 \to S^2$ and thus

$$\bar{f}_{10} = \frac{1}{2} \bar{f} \bar{f}_c$$

$$\bar{f}_{32} = \frac{1}{16} f \left(\bar{f}^3 - 2\bar{f}^2 \; \bar{f}_c - 16\bar{f}_{21} \right) , \quad ...$$

explicitly

$$b_{11} = a_0 b_0, \qquad b_{10} = \frac{1}{2}(a_0 b_c + a_c \, b_0), \qquad b_{22} = -\frac{3}{8} a_0^2 \, b_0, ...$$

We verified that relations for b_{11} and b_{10}
are respected by tree-level and 1-loop string results!

Implications?

which coefficients in large S expansion at strong coupling
are reproduced by asymptotic BA
and which require wrapping contributions?
on string side for $\mathcal{J} = 0$ finite size corrections are controlled by
$\frac{1}{S} = e^{-\ln S} \sim e^{-\text{string length}}$
$\frac{\ln^k S}{S^k}$ coefficients are related to $\ln S$ coefficients and thus
should not receive wrapping contributions at strong coupling
but they do at weak coupling ? (Janik and Forini talks)

Another limit: short folded string
$E(\lambda \gg 1) = \lambda^{1/4}\sqrt{2S}\,\left[c_1(\lambda) + c_2(\lambda)S + c_3(\lambda)S^2 + ...\right]$
$E(\lambda \ll 1) = a_1(\lambda)S + a_2(\lambda)S^2 + ...$
Generalization to higher (non-minimal) states:
multi-spike strings with (S, J)