Open and Closed Superstring Disk Amplitudes

Stephan Stieberger, MPP München

Max-Planck-Institut für Physik
(Werner-Heisenberg-Institut)

Workshop on the interface betwen N=4 SUSY and QCD
LPTHE, Jussieu, Paris, December 12 and 13, 2008

Outline

I. Progress in higher point open superstring amplitudes (tree) (in particular multi–gluon scattering)

- SUSY Ward identities relating various amplitudes to all orders in α'
- Compact representation for amplitudes with many external massless string states

II. Open & closed vs. pure open string disk amplitudes

- Sort of generalized KLT on the disk

Review: perturbative QCD amplitudes (tree)

- Compact representation for amplitudes with many external particle legs, by use of spinor basis $(\lambda, \tilde{\lambda})$

  e.g.: MHV N–gluon amplitude

$$A(g_1^-, g_2^-, g_3^+, g_4^+, \ldots, g_N^+) = (\sqrt{2} g_{YM})^{N-2}\, \mathrm{Tr}(T^1 \ldots T^N)\, \frac{\langle 12 \rangle^4}{\prod_{k=1}^{N} \langle k\, k{+}1 \rangle}$$

Parke, Taylor 1986
Berends, Giele 1989

$\langle ij \rangle \sim \sqrt{k_i k_j}$

- SUSY Ward identities relating various amplitudes

  Grisaru, Pendleton, van Nieuwenhuizen (1977); Parke, Taylor, (1985); . . . ;
  Bianchi, Elvang, Freedman (2008)

  e.g.: $A(g_1^-, g_2^-, g_3^+, g_4^+, \ldots, g_N^+) = \frac{\langle 12 \rangle^2}{\langle 34 \rangle^2}\, A(\phi_1^-, \phi_2^-, \phi_3^+, \phi_4^+, g_5^+, \ldots, g_N^+)$

- More efficient techniques to compute QCD amplitudes; CSW, BCFW rules

Higher point amplitudes in superstring theory (tree)

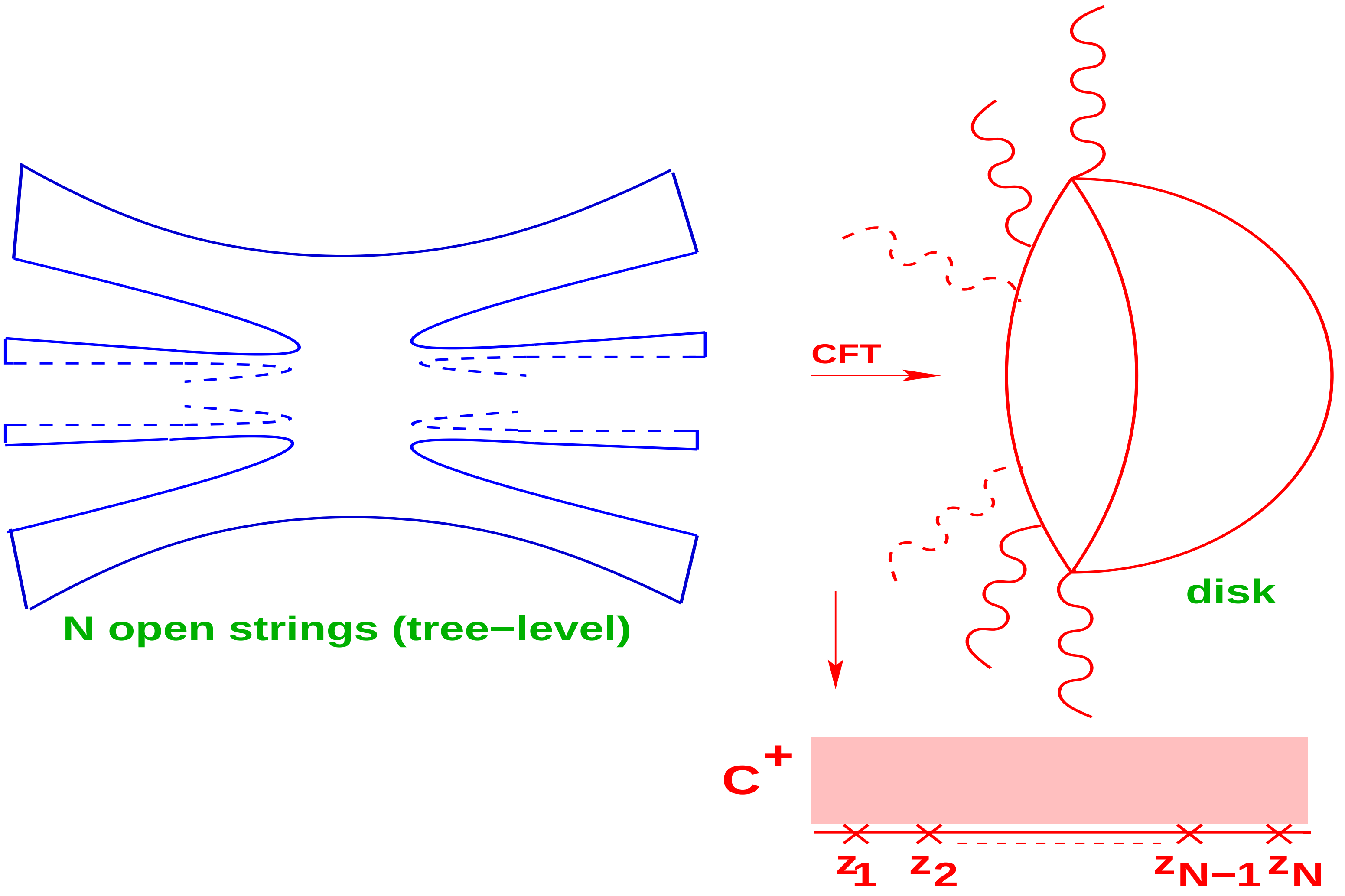
Do similar simple properties also hold for superstring (tree)–amplitudes ?

- SUSY Ward identities relating various amplitudes
  to all orders in α'
  St. St., T.R. Taylor, arXiv:0708.0574

- Compact representation for amplitudes with
  many external massless string states
  St. St., T.R. Taylor, arXiv:0711.4354

- More efficient technique to compute superstring amplitudes;
  recursion relations
  work in progress

Tree–level scattering of N open strings



Full N–point string S–matrix

$$A(1,2,\ldots,N;\alpha') = \sum_{\text{kinematics } \mathcal{K}_{\mathcal{I}}} \mathcal{K}_{\mathcal{I}}\ F\begin{bmatrix} n_a^I \\ n_{ab}^I \end{bmatrix}$$

generalized Euler integrals $\Longleftrightarrow$
multiple Gaussian hypergeometric functions

$$F\begin{bmatrix} n_a^I \\ n_{ab}^I \end{bmatrix} = \int_0^1 dx_1 \ldots \int_0^1 dx_{N-3} \prod_{a=1}^{N-3} x_a^{1+a-N+n_a} \prod_{b=a}^{N-3} x_a^{2\alpha'\, k_{b+3}\left(k_1+\sum_{j=a+3}^{b+2} k_j\right)}$$

$$\times \left(1-\prod_{j=a}^{b} x_j\right)^{2\alpha'\, k_{2+a}k_{3+b}+n_{ab}} , \qquad b \geq a = 1,2,\ldots,N-3 \ , \quad n_a, n_{ab} = 0, \pm 1$$

All F's can be expressed by $(N-3)!$ basis functions

α'–expansion $\Longleftrightarrow$ multiple Euler–Zagier sums

Oprisa, St. St., hep-th/0509042
St. St., Taylor, hep-th/0609175

N=4: Veneziano–amplitude

E.g.: $N = 4$

$$\int_0^1 dx\ x^{2\alpha' k_1 k_2 - 1}\ (1-x)^{2\alpha' k_2 k_3} = \frac{1}{s_1}\ \frac{\Gamma(1+s_1)\ \Gamma(1+s_2)}{\Gamma[1+s_1+s_2]}$$

$$= \frac{1}{s_1} - s_2\ \zeta(2) + (s_1+s_2)\ s_2\ \zeta(3) + \ldots \quad , \qquad \begin{array}{l} s_1 = 2\alpha'\ k_1 k_2 \\ s_2 = 2\alpha'\ k_2 k_3 \end{array}$$

Beta–function relevant for scattering of four open strings

Veneziano–amplitude

E.g. N=7:

$$\int_0^1 dx \int_0^1 dy \int_0^1 dz \int_0^1 dw \; \frac{x^{s_2}\,(1-x)^{s_3}\; y^{t_2}\,(1-y)^{s_4}\; z^{t_6}\,(1-z)^{s_5}\; w^{s_7}\,(1-w)^{s_6}}{(1-xy)\;(1-wz)\;(1-yz)}\;(1-wxyz)^{s_1-t_1+t_4-t_7}$$

$$\times\;(1-xy)^{-s_3-s_4+t_3}\;(1-wz)^{-s_5-s_6+t_5}\;(1-yz)^{-s_4-s_5+t_4}\;(1-wyz)^{s_5+t_1-t_4-t_5}\;(1-xyz)^{s_4-t_3-t_4+t_7}$$

$$=\mathcal{I}_0\;+\;\mathcal{I}_{1a}\;(s_1+s_2+s_3+s_4+s_5+s_6+s_7)\;+\;\mathcal{I}_{1b}\;(t_1+t_2+t_3+t_4+t_5+t_6+t_7)+\mathcal{O}(\alpha'^2)$$

with **Multiple Euler–Zagier sums** $\mathcal{I}_0,\ \mathcal{I}_{1a},\ \mathcal{I}_{1b}$:

$$\mathcal{I}_0=\int\frac{1}{(1-xy)(1-wz)(1-yz)}=\sum_{\substack{n1,n_2=0\\ n_3=1}}^{\infty}\frac{1}{n_3\;(1+n_1)\;(n_1+n_2+1)\;(n_2+n_3)}=\frac{27}{4}\zeta(4)$$

$$\mathcal{I}_{1a}=\int\frac{\ln w}{(1-xy)(1-wz)(1-yz)}=\sum_{\substack{n1,n_3=1\\ n_2=0}}^{\infty}\frac{1}{n_1\;n_3^2\;(n_1+n_2)\;(n_2+n_3)}=\frac{7}{2}\zeta(5)-4\zeta(2)\zeta(3)$$

$$\mathcal{I}_{1b}=\int\frac{\ln y}{(1-xy)(1-wz)(1-yz)}=\sum_{\substack{n1,n_2=1\\ n_3=0}}^{\infty}\frac{1}{n_1\;n_2\;(n_2+n_3)\;(n_1+n_3)^2}=-\frac{9}{2}\zeta(5)+\zeta(2)\zeta(3)$$

(Space–time) SUSY transformation on world–sheet disk

Setup: World–sheets with boundaries and $\mathcal{N}$ conserved SUSY charges $\mathcal{Q}^I_\alpha$
$I = 1, \ldots, \mathcal{N}$, with $\mathcal{Q}^I_\alpha = \oint \frac{dz}{2\pi i}\ V^I_\alpha(z)$

$$V^I_\alpha(z) = \begin{cases} \mathcal{N} \text{ supercurrents,} \\ \text{extended into double cover} \end{cases}$$

Variation of
open string vertex operator $\mathcal{O}(z)$
under (inf.) SUSY transformation

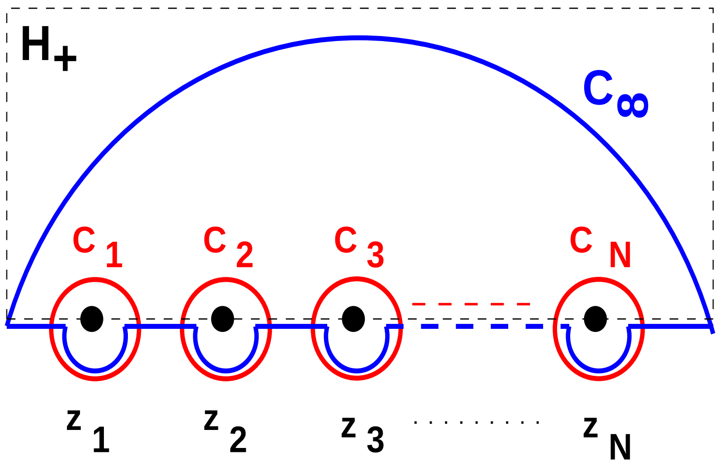
$$[\ Q^I(\eta_I)\ ,\ \mathcal{O}(z)\ ] := \oint_{C_z} \frac{dw}{2\pi i}\ \eta^\alpha_I\ V_\alpha(w)\ \mathcal{O}(z)$$

E.g.: $\mathcal{N} = 2$ gauge multiplet $(\phi, \lambda^1, \lambda^2, g)$: $Q^I(\eta_I, \bar{\eta}_I) := \eta^\alpha_I\ Q^I_\alpha + \bar{\eta}_{\dot{\alpha} I}\ \overline{Q}^{\dot{\alpha} I}$

$$\begin{aligned}
[\ Q^I(\eta_I,\bar{\eta}_I)\ ,\ \boldsymbol{\phi}(z,k)\ ] &= i\ \varepsilon^{IL}\ \langle \eta_I\ k \rangle\ \boldsymbol{\lambda}^L(z,k) \\
[\ Q^I(\eta_I,\bar{\eta}_I)\ ,\ \boldsymbol{\lambda}^L(z,k)\ ] &= -\delta^{IL}\ \langle \eta_I k \rangle\ \ \boldsymbol{g}(z,k) - i\, \varepsilon^{IL}\ [\eta_I\ k]\ \boldsymbol{\phi}(z,k) \\
[\ Q^I(\eta_I,\bar{\eta}_I)\ ,\ \boldsymbol{g}(z,k)\ ] &= -[\eta_I\ k]\ \boldsymbol{\lambda}^I(z,k)
\end{aligned}$$

SUSY transformations of open string vertices

Consider: $$\oint_{C_\infty} \frac{dw}{2\pi i}\, \eta_I^\alpha\, \langle V_\alpha^I(w)\; V_1(z_1)\; V_2(z_2)\ldots V_N(z_N)\rangle \sim \oint_{C_\infty} \frac{dw}{2\pi i}\, w^{-2} = 0$$



$$\Longrightarrow \boxed{\sum_{l=1}^{N} \langle V_1(z_1)\ldots V_{l-1}(z_{l-1})\; [Q^I(\eta_I^\alpha), V_l(z_l)]\; V_{l+1}(z_{l+1})\ldots V_N(z_N)\rangle = 0}$$

$\Longrightarrow$ SUSY Ward identity in superstring theory

N–gluon MHV amplitude in superstring theory

From supersymmetric Ward identities in string theory:

$$A(g_1^-, g_2^-, g_3^+, g_4^+, \ldots, g_N^+) \;=\; \frac{\langle 12\rangle^2}{\langle 34\rangle^2}\, A(\phi_1^-, \phi_2^-, \phi_3^+, \phi_4^+, g_5^+, \ldots, g_N^+)$$

- valid to all orders in α'
- universal to all string compactifications
- any numbers of supersymmetries

This allows for very short expressions for the N–gluon MHV amplitude

<u>*E.g.:*</u> $A(g_1^-, g_2^-, g_3^+, g_4^+, g_5^+) \;=\; \mathrm{Tr}(T^1 \ldots T^5)\; (\sqrt{2}\, g_{YM})^3\, \alpha'$ St. St., Taylor arXiv:0708.0574

$$\times\; \frac{\langle 1\,2\rangle^2}{\langle 3\,4\rangle^2 \langle 4\,5\rangle}\, \big(\, \langle 4\,1\rangle[1\,5]\ \boldsymbol{K}_1 + \langle 4\,2\rangle[2\,5]\ \boldsymbol{K}_2\, \big)$$

N–gluon MHV amplitude in superstring theory

This allows to give very short expressions for the N–gluon amplitude:

$$A(g_1^-,g_2^-,g_3^+,g_4^+,g_5^+,g_6^+) = \alpha'^2\,\frac{\langle 1\,2\rangle^2}{\langle 3\,4\rangle^2}\,[\,\tau(3,2)\ K_1+\tau(2,3)\ K_2+\tau(1,2)\ K_3$$

$$+\tau(1,3)\ K_4+\tau(2,1)\ K_5+\tau(3,1)\ K_6\,]$$

$$\tau(a,b)=\frac{\langle 4\,a\rangle\langle 4\,b\rangle[a\,5][b\,6]}{\langle 4\,5\rangle\langle 4\,6\rangle}$$

$$A(g_1^-,g_2^-,g_3^+,g_4^+,g_5^+,g_6^+,g_7^+) \ = \ \alpha'^3\,\frac{\langle 1\,2\rangle^2}{\langle 3\,4\rangle^2}\,\sum_{I=1}^{24}\tau_I\ K_I\ ,$$

$$\tau(a,b,c)=\frac{\langle 4\,a\rangle\langle 4\,b\rangle\langle 4\,c\rangle[a\,5][b\,6][c\,7]}{\langle 4\,5\rangle\langle 4\,6\rangle\langle 4\,7\rangle}$$

6–gluon NMHV amplitude in superstring theory

From SUSY Ward identities:

Full 6–gluon NMHV amplitude can be constructed from three partial subamplitudes (Mangano, Parke 1991)

$$A(g_1^-,g_2^-,g_3^+,g_4^+,g_5^-,g_6^+) = \frac{\alpha'^4}{s_{12}^2 s_{34}^2}\Big(y^2\ A_g^Y - 2\,y\,\alpha_Y\ A_\lambda^Y + \alpha_Y^2\ A_s^Y\ \Big)$$

$$A(g_1^+,g_2^+,g_3^-,g_4^-,g_5^-,g_6^+) = \frac{\alpha'^4}{s_{12}^2 s_{34}^2}\Big(x^2\ A_g^X - 2\,x\,\alpha_X\ A_\lambda^X + \alpha_X^2 A_s^X\ \Big)$$

$$A(g_1^-,g_2^+,g_3^-,g_4^+,g_5^-,g_6^+) = \frac{\alpha'^4}{s_{13}^2 s_{24}^2}\Big(z^2\ A_g^Z - 2\,z\,\alpha_Z\ A_\lambda^Z + \alpha_Z^2\ A_s^Z\ \Big)$$

$$A_g^Y = A(\phi_1^-,\phi_2^-,\phi_3^+,\phi_4^+,g_5^-,g_6^+)$$

$$A_\lambda^Y = A(\phi_1^-,\phi_2^-,\phi_3^+,\phi_4^+,\lambda_5^-,\lambda_6^+)$$

$$A_s^Y = A(\phi_1^-,\phi_2^-,\phi_3^+,\phi_4^+,\phi_5^-,\phi_6^+)$$

St. St., Taylor, arXiv:0711.4354

Final result: 6–gluon NMHV superstring amplitude

$$A^Y = A(g_1^-, g_2^-, g_3^+, g_4^+, g_5^-, g_6^+) = \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}T^{a_5}T^{a_6})\ \frac{(\sqrt{2}\,g_{YM})^4\,\alpha'^5}{s_5}$$
$$\times \left(N_1^Y \frac{\alpha_Y^2}{s_1^2 s_3^2} + N_2^Y \frac{\beta_Y^2}{s_1^2} + N_3^Y \frac{\gamma_Y^2}{s_3^2} + N_4^Y \frac{\alpha_Y \beta_Y}{s_1^2 s_3} + N_5^Y \frac{\alpha_Y \gamma_Y}{s_1 s_3^2} + N_6^Y \frac{\beta_Y \gamma_Y}{s_1 s_3} \right),$$

$$A^X = A(g_1^+, g_2^+, g_3^-, g_4^-, g_5^-, g_6^+) = \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}T^{a_5}T^{a_6})\ \frac{(\sqrt{2}\,g_{YM})^4\,\alpha'^5}{s_5}$$
$$\times \left(N_1^X \frac{\alpha_X^2}{s_1^2 s_3^2} + N_2^X \frac{\beta_X^2}{s_1^2} + N_3^X \frac{\gamma_X^2}{s_3^2} + N_4^X \frac{\alpha_X \beta_X}{s_1^2 s_3} + N_5^X \frac{\alpha_X \gamma_X}{s_1 s_3^2} + N_6^X \frac{\beta_X \gamma_X}{s_1 s_3} \right),$$

$$A^Z = A(g_1^-, g_2^+, g_3^-, g_4^+, g_5^-, g_6^+) = \mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}T^{a_5}T^{a_6})\ \frac{(\sqrt{2}\,g_{YM})^4\,\alpha'^5}{s_5}$$
$$\times \left(N_1^Z \frac{\alpha_Z^2}{s_{13}^2 s_{24}^2} + N_2^Z \frac{\beta_Z^2}{s_{13}^2} + N_3^Z \frac{\gamma_Z^2}{s_{24}^2} + N_4^Z \frac{\alpha_Z \beta_Z}{s_{13}^2 s_{24}} + N_5^Z \frac{\alpha_Z \gamma_Z}{s_{13} s_{24}^2} + N_6^Z \frac{\beta_Z \gamma_Z}{s_{13} s_{24}} \right)$$

Notation

$$Y = k_3 + k_4 + k_6 \ , \qquad y = \langle 12\rangle[34]\, Y^2 \ ,$$

$$X = k_1 + k_2 + k_6 \ , \qquad x = [12]\langle 34\rangle X^2 \ ,$$

$$Z = k_2 + k_4 + k_6 \ , \qquad z = \langle 13\rangle[24]\, Z^2 \ .$$

$$\alpha_Y = -\langle 12\rangle[34][6|Y|5\rangle \ , \qquad \beta_Y = \langle 12\rangle[46][3|Y|5\rangle \ , \qquad \gamma_Y = \langle 51\rangle[34][6|Y|2\rangle \ ,$$

$$\alpha_X = -[12]\langle 34\rangle[6|X|5\rangle \ , \qquad \beta_X = [12]\langle 45\rangle[6|X|3\rangle \ , \qquad \gamma_X = [61]\langle 34\rangle[2|X|5\rangle \ ,$$

$$\alpha_Z = -\langle 13\rangle[24][6|X|5\rangle \ , \qquad \beta_Z = \langle 13\rangle[46][2|Z|5\rangle \ , \qquad \gamma_Z = \langle 51\rangle[24][6|Z|3\rangle \ ,$$

<u>*with:*</u> $[6|Y|5\rangle = [63]\langle 35\rangle + [64]\langle 45\rangle \ ,$ etc.

$$s_{ij} = 2\alpha' k_i k_j \ ,$$

$$s_i = \alpha'(k_i + k_{i+1})^2 \ , \quad t_i = \alpha'(k_i + k_{i+1} + k_{i+2})^2 \qquad (i + 6 \equiv i)$$

α'–Expansion

E.g.:

$$N_1^X = -\zeta(2)\ s_1 s_3 + \ldots\ ,$$

$$N_2^X = \frac{s_1}{s_2 s_4 t_1} - \zeta(2)\ \left(\frac{s_1 s_6}{s_2 s_4} + \frac{s_1^2}{s_4 t_1} + \frac{s_1 s_5}{s_2 t_1}\right) + \ldots\ ,$$

$$N_3^X = \frac{s_3}{s_2 s_6 t_2} - \zeta(2)\ \left(\frac{s_3 s_4}{s_2 s_6} + \frac{s_3 s_5}{s_2 t_2} + \frac{s_3^2}{s_6 t_2}\right) + \ldots\ ,$$

$$N_4^X = \zeta(2)\ \left(\frac{s_1 t_2}{s_2} + \frac{s_1 t_3}{s_4}\right) + \ldots\ ,$$

$$N_5^X = \zeta(2)\ \left(\frac{s_3 t_1}{s_2} + \frac{s_3 t_3}{s_6}\right) + \ldots\ ,$$

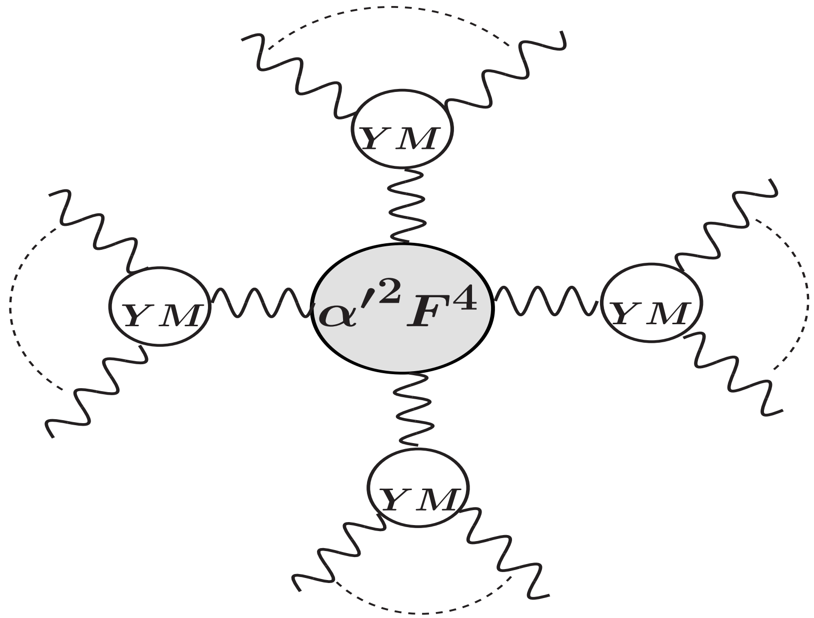
$$N_6^X = \frac{t_3}{s_2 s_4 s_6} + \zeta(2)\ \left(\frac{s_1+s_3-s_5}{s_2} - \frac{t_1 t_3}{s_2 s_4} - \frac{t_2 t_3}{s_2 s_6} - \frac{t_3^2}{s_4 s_6}\right) + \ldots$$

N–gluon MHV amplitude in superstring theory

$$A(g_1^-,g_2^-,g_3^+,g_4^+,\ldots,g_N^+;\alpha') = \left(1-\alpha'^2\frac{\zeta(2)}{2}F^{(N)}\right)A(g_1^-,g_2^-,g_3^+,g_4^+,\ldots,g_N^+) + \mathcal{O}(\alpha'^3)$$

$F^{(N)}$ polynomial in kinematic invariants:

$$\begin{aligned} F^{(4)} &= s_1s_2 \\ F^{(5)} &= s_1s_2+s_2s_3+s_3s_4+s_4s_5+s_5s_1+4i\ \epsilon_{1234} \\ F^{(6)} &= s_1s_2+s_2s_3+s_3s_4+s_4s_5+s_5s_6+s_6s_1 \\ &+ t_1t_2+t_2t_3+t_3t_1-s_1s_4-s_2s_5-s_3s_6 \\ &+ 4i\ [\epsilon_{1234}+\epsilon_{1235}+\epsilon_{1245}+\epsilon_{1345}+\epsilon_{2345}] \\ &\vdots \end{aligned}$$



St. St., Taylor, hep-th/0609175

Non–trivial duality: $\zeta(2)$ term agrees with one–loop field–theory result !

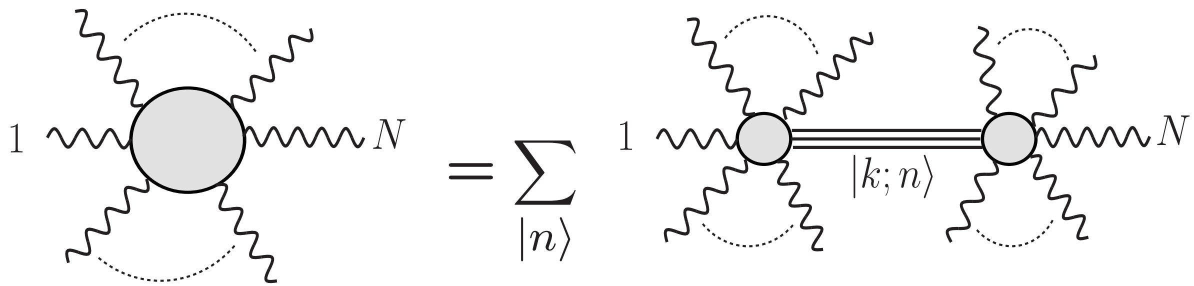
see also Dixon, Schabinger, to appear

Amplitudes from first principles

Analytic properties: Scattering amplitudes are computed *directly*

use:
- cyclic invariance
- soft–boson limit
- factorizing into collinear limits
- ...

String tree–level recursion relations:



$\Longrightarrow$ Construct amplitudes from first principles

work in progress

Effective D–brane action (α'–expansion)

Series of higher derivative terms (α'–corrections):

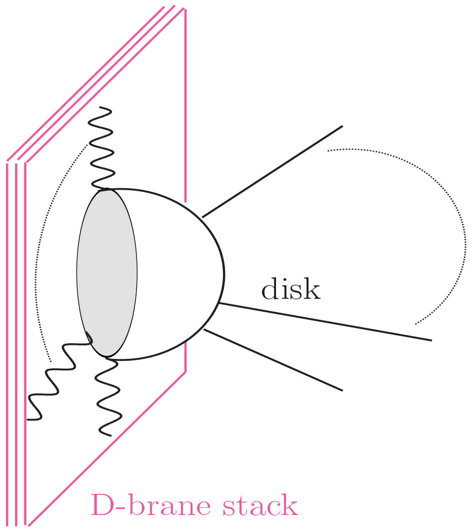
$$\mathcal{L}^{Dp}_{\text{effective}} = \text{Tr} \sum_{m\geq 4,\ n\geq 0}{}' \alpha'^{\frac{1}{2}n+m-2}\ \zeta(\frac{1}{2}n+m-2)\ D^n F^m$$

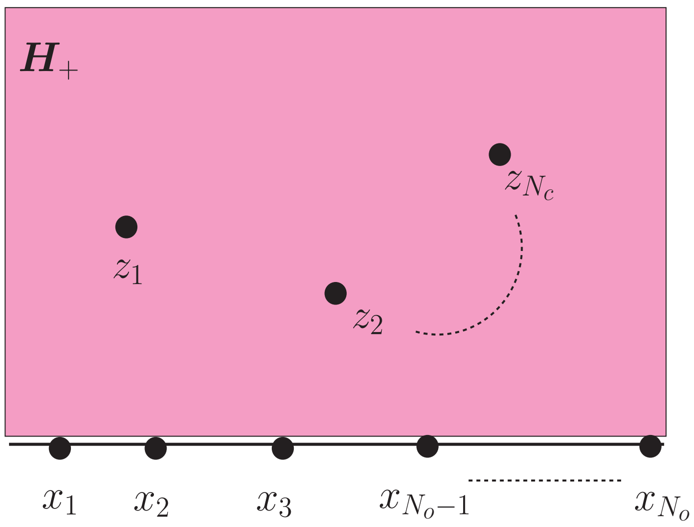
$\alpha'^0\ 1$	$\mathbf{F^2}$
$\alpha'\ 0$	$F^3 \quad D^2F^2$
$\alpha'^2\ \zeta(2)$	$\mathbf{F^4} \quad D^2F^3 \quad D^4F^2$
$\alpha'^3\ \zeta(3)$	$\mathbf{F^5} \quad \mathbf{D^2F^4} \quad D^6F^2$
$\alpha'^4\ \zeta(4)$	$\mathbf{F^6} \quad \mathbf{D^4F^4} \quad \mathbf{D^2F^5}$
$\alpha'^5\ \zeta(2)\zeta(3),\ \zeta(5)$	$\mathbf{F^7} \quad \mathbf{D^6F^4} \quad \mathbf{D^4F^5} \quad \mathbf{D^2F^6}$
$\vdots$	$\ldots \quad \ldots \quad \ldots \quad \ldots$

Degree of transcendentality $\Longleftrightarrow$ order in α'–expansion

Disk scattering of open and closed strings

$$\mathcal{A} = \sum_{\pi \in S_{N_o}/\mathbf{Z}_2} V_{\mathrm{CKG}}^{-1} \left(\prod_{j=1}^{N_o} \int_{\mathcal{I}_\pi} dx_j \prod_{i=1}^{N_c} \int_{\mathcal{H}_+} d^2 z_i \right) \langle \prod_{j=1}^{N_o} : V_o(x_j) : \prod_{i=1}^{N_c} : V_c(\overline{z}_i, z_i) : \rangle$$





$V_o(x_i)$ = open string vertex operators inserted at x_i on the boundary of the disk

$V_c(\overline{z}_i, z_i)$ = closed string vertex operators inserted at z_i inside the disk

E.g.:

$N_o = 2,\ N_c = 1 \implies$ four open strings

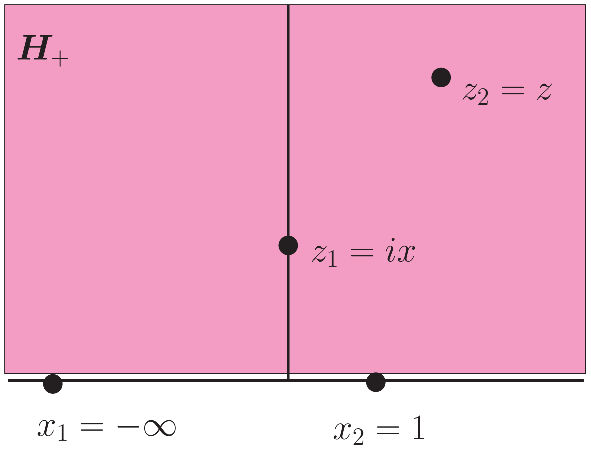
$N_o = 3,\ N_c = 1 \implies$ five open strings

$N_o = 2,\ N_c = 2 \implies$ six open strings

Two open and two closed strings on the disk

With $PSL(2,\mathbf{R})$ transformation three arbitray points $w_1, w_2 \in \mathbf{R}$ and $w_3 \in \mathbf{C}$ may be mapped to the points x_1, x_2 and z_1:

Choice: $x_1 = -\infty$, $x_2 = 1$, $\overline{z}_1 = -ix$, $z_1 = ix$, $\overline{z}_2 = \overline{z}$, $z_2 = z$



with $z \in \mathbf{H}_+$ and $x \in \mathbf{R}^+$

$$\mathcal{A}(1,2,3,4) = \int_{-\infty}^{\infty} dx \, \langle c(-\infty)c(1)c(ix)\rangle \times \int_{\mathbf{C}} d^2z \, \langle \, : V_o(-\infty) : \, : V_o(1) : \, : V_c(-ix, ix) : \, : V_c(\overline{z}, z) : \rangle$$

Two open & two closed strings versus six open strings on the disk

- generic structure of world–sheet disk amplitude of **two open** & **two closed strings**:

$$W^{(\alpha,\beta)}\begin{bmatrix}\alpha_1,\lambda_1,\gamma_1,\beta_1\\ \alpha_2,\lambda_2,\gamma_2,\beta_2\end{bmatrix} = \int\limits_{-\infty}^{\infty} dx\ x^{\beta}\ (1+x)^{\alpha_1}\ (1-x)^{\alpha_2}\int\limits_{\mathbf{C}} d^2z\ (1-z)^{\lambda_1}\ (1-\overline{z})^{\lambda_2}$$
$$\times\ \ (z-\overline{z})^{\alpha}\ (z-x)^{\gamma_1}\ (\overline{z}-x)^{\gamma_2}\ (z+x)^{\beta_1}\ (\overline{z}+x)^{\beta_2}$$

- generic structure of world–sheet disk amplitude of **six open strings**:

$$F\begin{bmatrix}n_1,n_2,n_3\\ n_4,n_5,n_6,n_7,n_8,n_9\end{bmatrix} = \int\limits_0^1 dx\int\limits_0^1 dy\int\limits_0^1 dz\ x^{p_{23}+n_1}\ y^{p_{23}+k_{24}+p_{34}+n_2}\ z^{p_{16}+n_3}$$
$$\times\ \ (1-x)^{p_{34}+n_4}\ (1-y)^{p_{45}+n_5}\ (1-z)^{p_{56}+n_6}\ (1-xy)^{p_{35}+n_7}$$
$$\times\ \ (1-yz)^{p_{46}+n_8}\ (1-xyz)^{p_{36}+n_9}\quad,\qquad n_i\in\mathbf{Z}$$

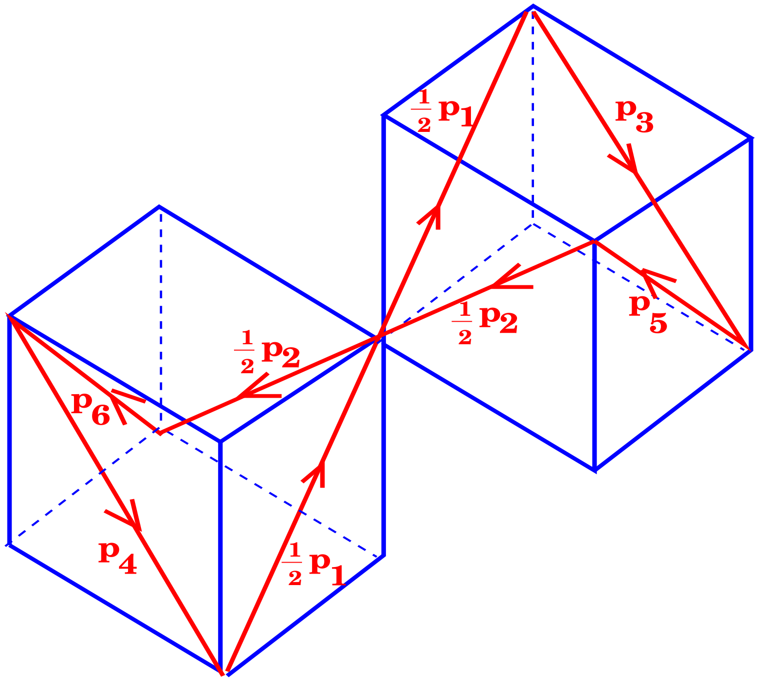
Two open & two closed strings versus six open strings on the disk

After splitting the complex integral into holomorphic and anti–holomorphic pieces

Answer: Six open strings, with:

$$z_1 = -\infty, \quad z_2 = 1, \quad z_3 = -\tau,$$
$$z_4 = \tau, \quad z_5 = \xi, \quad z_6 = \eta$$

$$p_1 = k_1, \qquad p_2 = k_2,$$
$$p_3 = p_4 = \tfrac{1}{2} k_3, \qquad p_5 = p_6 = \tfrac{1}{2} k_4$$



Four contributions, with open string ordering

$$\begin{cases} I_1: & z_1 < z_6 < z_3 < z_5 < z_4 < z_2 \\ I_2: & z_1 < z_6 < z_3 < z_5 < z_2 < z_4 \\ I_3: & z_1 < z_3 < z_4 < z_5 < z_2 < z_6 \\ I_4: & z_1 < z_3 < z_2 < z_5 < z_4 < z_6 \end{cases}$$

Two open & two closed strings versus six open strings on the disk

$$W^{(\alpha,\beta)}\begin{bmatrix}\alpha_1,\lambda_1,\gamma_1,\beta_1\\ \alpha_2,\lambda_2,\gamma_2,\beta_2\end{bmatrix}=\sigma_\gamma\ \sin(\pi\beta_2)\ (I_1+I_2)+\sin(\pi\lambda_2)\ I_3+\sigma_\lambda\sigma_\gamma\ \sin(\pi\gamma_2)\ I_4+R$$

I_1, I_2, I_3 and I_4 six open string amplitudes $F[\ldots]$

involves transformations:

$$\begin{aligned}
I_1:&\quad \tau\to-1+\frac{2}{1+yz},\quad \xi\to 1-\frac{2y}{1+yz},\quad \eta\to 1-\frac{2}{x(1+yz)}\\
I_2:&\quad \tau\to\frac{1}{1-2yz},\quad \xi\to\frac{1-2y}{1-2yz},\quad \eta\to-\frac{2-x}{x(1-2yz)}\\
I_3:&\quad \tau\to\frac{xy}{2-xy},\quad \xi\to\frac{(2-x)y}{2-xy},\quad \eta\to\frac{2-xyz}{z(2-xy)}\\
I_4:&\quad \tau\to-\frac{1}{1-2xy},\quad \xi\to\frac{1-2y}{1-2xy},\quad \eta\to-\frac{2-z}{z(1-2xy)}
\end{aligned}$$

Open & closed vs. pure open string disk amplitudes

This map reveals
important relations between
open & closed string disk amplitudes
and pure open string disk amplitudes !

E.g.:

$$\langle A_{\mu_1}(x_1)\ A_{\mu_2}(x_2)\ G_{\mu_3\mu_4}(\overline{z}_1, z_1)\ G_{\mu_5\mu_6}(\overline{z}_2, z_2)\rangle$$
$$\sim \langle A_{\mu_1}(x_1)\ A_{\mu_2}(x_2)\ A_{\mu_3}(x_3)\ A_{\mu_4}(x_4)\ A_{\mu_5}(x_5)\ A_{\mu_6}(x_6)\rangle$$

Sort of generalized KLT on the disk

St. St. `arXiv:0812.xxxx`, to appear