# Gravity, twistors and the MHV formalism

## Graviton scattering on anti-self-dual backgrounds with ideas from twistor-string theory

L.J.Mason

The Mathematical Institute, Oxford
lmason@maths.ox.ac.uk

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Based on hep-th/0808.3907, joint work with Dave Skinner (also 0706.1941 joint with Martin Wolf).

# Gravitational perturbation theory

Consider $(\mathscr{M}^4, g)$ with Einstein Hilbert action $\int_{\mathscr{M}^4} R\mathrm{d}\mu_g$.

Feynman diagrams for gravity are very complicated.

- non-polynomial action $\rightsquigarrow$ $\infty$ vertices, gauge freedom/fixing $\rightsquigarrow$ many many diagrams.
- DeWitt's 4 particle tree-level amplitude was a computational tour-de-force.

But recently, hidden structures have been found:

- Cachazo, Svrcek & Witten used Twistor-string theory $\rightsquigarrow$ MHV diagram formalism for Yang-Mills & conformal gravity.
- Boels, M. & Skinner constructed twistor actions for YM and conformal gravity (cf also Gorsky, Rosly, Selivanov, Mansfield).

Is there a similar story for Einstein Gravity?

Long term aim: reverse engineer twistor-string theory for gravity (Abou-Zeid, Hull & M hep-th/0606272, probably self-dual, Nair).

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# Twistor actions

hep-th/0507269, 0604040, 0702035

- For YM and conformal gravity can write action as

$$S_{\text{Full}} = S_{\text{ASD}} + S_{\text{MHV}}$$

  $S_{\text{ASD}}$ contains kinetic terms & interactions of ASD sector, $S_{\text{MHV}}$ contains remaining interactions of full theory.
- We reformulate on twistor space where ASD sector is of Chern-Simons type (reflecting complete integrability).

Two special gauges

1. leads directly to space-time formulation,
2. leads to MHV formalism as Feynman rules.
   - axial on twistor space, **not** accessible from space-time.
   - It linearizes the ASD sector, reducing $S_{\text{ASD}}$ to kinetic terms; realizes complete integrability of ASD sector.
   - $S_{\text{MHV}}$ = generating function for MHV amplitudes.

# Spinor helicity formalism

- Momentum eigenstates have $P^2 = 0$. In spinors $\rightsquigarrow$

$$p^{AA'} = \frac{1}{\sqrt{2}} \begin{pmatrix} P^0 + P^3 & P^1 + iP^2 \\ P^1 - iP^2 & P^0 - P^3 \end{pmatrix} \quad {\scriptstyle A=0,1,\ A'=0',1'}$$

  $P_a P^a = \det P^{AA'}$ so $P^2 = 0$ gives $P^{AA'} = p^A p^{A'}$ .
- Linear gravity represented by Weyl tensor decomposes Weyl $=$ Weyl$^+ \oplus$ Weyl$^-$ $= (+)$ helicity $\oplus (-)$ helicity.
- Polarization information $\leftrightarrow$ phase of $p^A$ i.e. $(-)$ helicity Weyl spinor:

$$\Psi_{ABCD}(x) = \mathrm{e}^{iP\cdot x} p_A p_B p_C p_D \,.$$

- Scattering amplitudes are functions of incoming/outgoing momentum spinors.

# Maximal Helicity Violating (MHV) amplitudes for gravity

- If all particles in a scattering process are $-$ helicity, or just one is $+$, the tree-level amplitudes vanish.
- The first nontrivial MHV amplitudes is $+ + - - \cdots -$:

$$M(1^+, 2^-, 3^-, \ldots, n{-}1^-, n^+) = \delta^4(\sum_i P_i) \frac{[1\,n]^8}{[1\;n-1][n-1\;n][n\,1]}$$

$$\left\{ \frac{1}{\Pi_{i=1}^{n}[i\;i+1]} \prod_{k=2}^{n-1} \frac{\langle k|P_{k+1} + \cdots + P_{n-1}|n]}{[kn]} + \Pi_{(2,\ldots,n-2)} \right\} .$$

where $\langle 1\,2\rangle := p_{1A} p_2^A\,,\ [1\,2] := p_{1A'} p_2^{A'}\,,\ \langle 1|2|3] = p_{1A} P_2^{AA'} p_{3A'}$ .

- Berends, Giele & Kuijf (1988) using Kawai Llewellyn Tye string theory relations between gravity and Yang-Mills.

Bjerrum-Bohr et. al. (2006) argue that all gravity amplitudes are generated by MHV vertices and scalar propagators.

- Uses recursion relations and asymptotic properties of amplitudes.
- Dramatic simplification; impossible to see from space-time action.
- Proof is controversial & breaks down at 12 points, Elvang & Freedman.

**Objective of this talk:** To show how this structure might arise from a twistor formulation of gravity.
(Work with D.Skinner).

# A space-time approach to MHV amplitudes

**ASD gravity:** Complexify space-time, and consider equations

$$\mathrm{Ric} = \mathrm{Weyl}^{+} = 0\,, \quad \mathrm{Weyl}^{-} \neq 0\,.$$

- Nonlinear version of $-$ helicity field, *Nonlinear graviton*.
- Newman, Penrose, Plebanski showed these equations to be completely integrable using twistor theory and H-space.
- Complete integrability of ASD eqs $\rightsquigarrow$ vanishing of $- - \ldots -$ amplitude and $+ - - \ldots -$ amplitude.
- Can we expand full theory about its ASD sector?

# Chiral action for expansion about ASD sector

Abou-Zeid, Hull hep-th/0511189

Use Plebanski-Palatini formulation with variables (on $\mathscr{M}^4$):

- $e^{AA'}$, tetrad of 1-forms s.t.

$$\mathrm{d}s^2 = e^{AA'} \odot e^{BB'} \varepsilon_{AB}\varepsilon_{A'B'}\,, \quad \varepsilon_{AB} = -\varepsilon_{BA}\,,\ \varepsilon_{01} = 1\,.$$

- $\Gamma_{A'B'} = \Gamma_{(A'B')}$ the $+$ spin connection 1-forms.
- Action

$$S = \int_{\mathscr{M}} \Sigma^{A'B'} \left( d\Gamma_{A'B'} + \kappa^2\, \Gamma_{A'}^{C'} \wedge \Gamma_{B'C'} \right)\,,$$

where $\Sigma^{A'B'} = e_A^{(A'} \wedge e^{B')A}$.

- Field equations

$$\mathrm{d}\Sigma^{A'B'} + 2\kappa^2 \Gamma_{C'}^{(A'} \wedge \Sigma^{B')C'} = 0, \ \left(\mathrm{d}\Gamma_{A'B'} + \kappa^2 \Gamma_{(A'}^{C'} \Gamma_{B')C'}\right) \wedge \mathrm{e}^{AA'} = 0.$$

$\Leftrightarrow\ \kappa^2 \Gamma_{A'B'} =$ primed spin connection 1-form, Ricci$= 0$.

# The ASD sector and perturbations around it

**ASD sector:** Set $\kappa = 0 \rightsquigarrow$ field equations

$$\mathrm{d}\Sigma^{A'B'} = 0 \Rightarrow \quad \text{metric is ASD. and}$$

$$\mathrm{d}\Gamma_{A'B'} \wedge \mathrm{e}^{AA'} = 0\,, \Rightarrow \mathrm{d}\Gamma_{A'B'} = \psi_{A'B'C'D'}\Sigma^{C'D'}$$

and $\psi_{A'B'C'D'}$ is linearized SD Weyl spinor on ASD background.

**Linearized gravitational perturbations:**
For $\delta e^{AA'} = h^{AA'}$ on the ASD background, define $\gamma_{A'B'}(h)$ by

$$\mathrm{d}(h^{A(A'} \wedge e_A^{B')}) + 2\gamma_{C'}^{(A'}\Sigma^{B')C'} = 0$$

and then $\gamma_{A'B'}$ satisfies $\mathrm{d}\gamma_{A'B'} \wedge e^{AB'} = 0$.

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# Background field formula for MHV amplitudes

- Define ASD metric perturbations to be:

$$V^- = \{h^{AA'} \; | \gamma_{A'B'}(h) = 0\}.$$

- SD perturbations are not well-defined as $\delta\psi_{ABCD} = 0$ is not diffeomorphism invariant, but, can define

$$V^+ = \{\gamma_{A'B'} \mid \mathrm{d}\gamma_{A'B'} \wedge e^{AA'} = 0\}.$$

- So the space $V$ of gravitational perturbations decomposes

$$0 \to V^- \to V \to V^+ \to 0\,.$$

- MHV scattering arises from extra term in full action

$$M(1+, 2+, e^-) = \int_{\mathscr{M}} \Sigma^{A'B'} \wedge \gamma_{1A'C'} \wedge \gamma_{2B'}^{C'}\,.$$

and obstruct the splitting $V = V^+ \oplus V^-$.

- Normal MHV amplitude arises from ASD nonlinear superposition of plane waves $\{\mathscr{M}, e^{AA'}\}$, background coupled plane waves $\gamma_1$ and $\gamma_2$ evaluated in integral.

# Twistor theory & the nonlinear graviton

Flat correspondence:

- Complex space-time $\mathbb{M} = \mathbb{C}^4$, coords $x^{AA'}$,
- Twistor space $\mathbb{T} = \mathbb{C}^4$, coords $Z^\alpha = (\omega^A, \pi_{A'})$, ${\scriptstyle \alpha=(A,A')}$.
  Projective twistors: $\mathbb{PT} = \{\mathbb{T} - 0\}/\{Z \sim \lambda Z, {\scriptstyle \lambda \in \mathbb{C}^*}\} = \mathbb{CP}^3$.
  $\mathbb{PT}' = \{Z \in \mathbb{PT} | \pi_{A'} \neq 0\}$, for finite space-time.
- Incidence relation

$$\omega^A = i x^{AA'} \pi_{A'}.$$

$$\{\text{Point } x \in \mathbb{M}\} \longleftrightarrow \{L_x = \mathbb{CP}^1 \subset \mathbb{PT}\}, \text{ hgs coords } \pi_{A'}.$$

**Theorem (Penrose, 1976)**

$$\left\{\begin{array}{l}\textit{Deformations of complex}\\ \textit{structure, } \mathbb{PT}' \rightsquigarrow \mathscr{PT}\end{array}\right\} \overset{1-1}{\longleftrightarrow} \left\{\begin{array}{l}\textit{anti-self-dual deforma-}\\ \textit{tions of conformal struc-}\\ \textit{ture } (\mathbb{M}, \eta) \rightsquigarrow (M, [g]).\end{array}\right\}$$

**For Ricci-flat** $g \in [g]$, *$\mathscr{PT}$ must have fibration $p : \mathscr{PT} \to \mathbb{CP}^1$ and holomorphic symplectic form on fibres valued in $p^*\mathcal{O}(2)$.*

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## Main ideas

- **Ricci-flat deformation:**
  choose $h \in \Omega^{0,1}_{\mathbb{PT}}(2)$ & define almost $\mathbb{C}$-structure

$$\Omega^{1,0}_{\mathscr{PT}} = \{\mathrm{d}\omega^A + \varepsilon^{AB}\frac{\partial h}{\partial \omega^B}, \mathrm{d}\pi_{A'}\}\,.$$

- Almost complex structure is integrable iff

$$\bar{\partial} h + \varepsilon^{AB}\frac{\partial h}{\partial \omega^A} \wedge \frac{\partial h}{\partial \omega^B} = 0\,.$$

- Satisfied when $h = \sigma(\omega, \pi, \bar{\pi})\bar{\pi}^A \mathrm{d}\bar{\pi}_A$ (Newman's H-space).
- Space-time $M$ = holomorphic sections of $p : \mathscr{PT} \to \mathbb{CP}^1$:
  a section $\omega^A = F^A(x, \pi)$ is holomorphic if

$$\bar{\partial} F^A = \varepsilon^{AB}\frac{\partial h}{\partial \omega^B}(F, \pi, \bar{\pi})\,.$$

- Full 4-parameter family survives small deformations.
- $x, y \in M$ on light ray $\Leftrightarrow$ incidence $\mathbb{CP}^1_x \cap \mathbb{CP}^1_y \neq \emptyset$.
- $\rightsquigarrow$ conformal structure with Weyl$^+ = 0$ on $M$
  (and metric with Ricci$= 0$ with more work).

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Linear fields $\{\gamma_{B'C'} = \gamma_{AA'B'C'} e^{AA'} \mid \mathrm{d}\gamma_{B'C'} \wedge e^{CC'} = 0\}$ on $\mathscr{M}$ are obtained from $B_A \in H^1(\mathscr{PT}, \Omega^1_p(-5))$ by

$$\gamma_{AA'B'C'}(x) = \int_{\omega^A = F^A(x,\pi)} \pi_{A'}\pi_{B'}\pi_{C'}\, \Lambda_A^B\, B_B(F, \pi, \bar{\pi})\, \pi^{E'} \mathrm{d}\pi_{E'} \,.$$

where $\mathrm{d}F^A = \pi_{B'} \Lambda^A_B e^{BB'}$.

- Eigenstates of momentum $P^{AA'} = p^A p^{A'}$ are obtained as

$$B_A = \beta_A \mathrm{e}^{\langle\omega|P|\alpha]/[\alpha\,\pi]} \bar{\delta}_{-5}([\pi\, p])\,, \quad \bar{\delta}_n([\pi\, p]) = \frac{[\alpha\,\pi]^{n+1}}{[\alpha\, p]^{n+1}} \bar{\partial} \frac{1}{[\pi\, p]}$$

- Since ‘asymptotic twistors’ are attached to $\mathscr{I}^\pm$, these classes give asymptotic momentum eigenstates.

Background coupled fields inserted into MHV formula give

$$M(1+,2+,e-) = \int_{\mathscr{M}\times\mathbb{CP}^1_x\times\mathbb{CP}^1_x} \mathrm{d}^4x\,[\pi_1\,\mathrm{d}\pi_1][\pi_2\,\mathrm{d}\pi_2][\pi_1\,\pi_2]^3\langle B_1|\Lambda_1|\Lambda_2|B_2\rangle$$

where on $\mathbb{CP}^1_x$, $\omega_1^A = F^A(x,\pi_1,\bar{\pi}_1)$ etc.. Rewrite as

$$M(1+,2+,e-) = \int_{\mathscr{M}\times\mathbb{CP}^1_x\times\mathbb{CP}^1_x} \mathrm{d}^4x[\pi_1\,\mathrm{d}\pi_1]B_{1C}\Lambda_1^{AC}\bar{\partial}^{-1}\left([\pi_1\,\pi_2]^4\Lambda^B_{2A}B_{2B}\right)$$

with $\bar{\partial}^{-1}f = \int_{\mathbb{CP}^1}\frac{f}{[\pi_1\,\pi_2]}[\pi_2\,\mathrm{d}\pi_2]$.

- To eliminate $F^A(x,\pi)$ use $\pi$ dependent diffeomorphism

$$y^{AA'} = \frac{F^A(x,\pi)\alpha^{A'} - F^A(x,\alpha)\pi^{A'}}{\langle\pi\,\alpha\rangle} \quad \rightsquigarrow \quad F^A = y^{AA'}\pi_{A'}\,.$$

- Under this diffeo $\bar{\partial}\to\bar{\partial}-V$ where

$$V = \bar{\partial}y^{AA'}\frac{\partial}{\partial y^{AA'}} = \frac{\partial h}{\partial\omega_A}\alpha^{A'}\frac{\partial}{\partial y^{AA'}}\,.$$

- Lippman-Schwinger expansion of $(\bar\partial - V)^{-1}$ gives

$$
\begin{aligned}
&M(1+,2+,e-)\\
&\quad = \int_{\mathscr{M}\times\mathbb{CP}^1_y\times\mathbb{CP}^1_y} \mathrm{d}^4 y[\pi_1\, \mathrm{d}\pi_1] B_{1C}\Lambda_1^{AC}(\bar\partial - V)^{-1}\left([\pi_1\,\pi_2]^4\Lambda^B_{2A}B_{2B}\right)\\
&\quad = \sum_i \int_{\mathscr{M}\times\mathbb{CP}^1_y\times\mathbb{CP}^1_y} \mathrm{d}^4 y[\pi_1\, \mathrm{d}\pi_1] B_{1C}\Lambda_1^{AC}\bar\partial^{-1}(V\bar\partial^{-1})^i\left([\pi_1\,\pi_2]^4\Lambda^B_{2A}B_{2B}\right)
\end{aligned}
$$

- Choose momentum eigenstates for $B_{iA}$, $i=1,2$ and

$$
h = \sum_{i=3}^{n} \epsilon_i e^{\langle\omega|P_i|\alpha]/[\alpha\,\pi]}\bar\delta_2([\pi\, i])\,.
$$

- Pick out term in $\Pi^n_{i=3}\epsilon_i$ and integrate out delta-functions $\rightsquigarrow$ BGK MHV formula.
- Gives self-contained derivation from first principles.

# Twistor action for ASD gravity

M. & Wolf hep-th/0706.1941

For a twistor gravity action, we first need one for ASD sector:

- Action

$$S^-[h, B_A] = \int_{\mathbb{PT}} B^A \frac{\partial}{\partial \omega^A} \left(\bar\partial h + \{h, h\}\right) \wedge \mathrm{d}^3 Z$$

- with field equations

$$\frac{\partial}{\partial \omega^A} \left(\bar\partial h + \{h, h\}\right) = 0\,, \quad \bar\partial_h \left(\frac{\partial}{\partial \omega^A} B^A\right) = 0\,.$$

- $\Rightarrow$ integrability of almost $\mathbb{C}$ structure $\bar\partial_h = \bar\partial + \varepsilon^{AB} \frac{\partial h}{\partial \omega^A} \frac{\partial}{\partial \omega^B}$ and $B_A \in H^1(\mathscr{PT}, \Omega^{1,0}_V(-5))$.
- **Complete integrability:** $\exists$ coordinates ('axial gauge') such that $B^A \wedge \partial\{h, h\}/\partial\omega^A = 0$ so equations become linear.

- This formulation of anti-self-dual gravity works 'off-shell' and can include interactions of full gravity etc..
- The MHV vertices are all generated as above by

$$S_{MHV}[B_A, h] = \int_{\mathscr{M}\times\mathbb{CP}^1_x\times\mathbb{CP}^1_x} \mathrm{d}^4 x\, [\pi_1\, \mathrm{d}\pi_1][\pi_2\, \mathrm{d}\pi_2][\pi_1\, \pi_2]^3 \langle B_1|\Lambda_1|\Lambda_2|B_2\rangle$$

- Set

$$S_{Full}[B_A, h] = S^-[B_A, h] + S_{MHV}[B_A, h]$$

$\rightsquigarrow$ an action that gives the MHV formalism perturbatively.

- $N = 4$ sugra extension is easy. $N = 8$ possible, but more gauge dependent.

Summary

- Self-contained proof of the BGK MHV gravity amplitudes.
- Twistor gravity action that yields Gravity MHV formalism.
- Off-shell, but gauge fixed, so does not express full symmetries.
- Does **not** yet give self-contained proof of MHV formalism; if MHV formalism for gravity is incomplete, then so is this action.

Outlook

- Twistor actions for Yang-Mills and conformal gravity are **not** gauge fixed and **do** give self-contained proof of MHV formalism for Yang-Mills and conformal gravity;
- suggests full gauge invariant twistor action should exist; if MHV formalism is incomplete, it would give completion.