MHV amplitudes in N=4 SUSY Yang-Mills theory and quantum geometry of the momentum space

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ITEP

Paris, December 13, to appear

MHV amplitudes ($++$——) are the simplest objects to discuss within the gauge/string duality

Simplification at large N - MHV amplitudes are described by the single function of the kinematical variables

Properties of the tree amplitudes

- Holomorphy - it depends only on the "'half'' of the momentum variables $p_{\alpha,\dot{\alpha}} = \lambda_\alpha \bar{\lambda}_{\dot{\alpha}}$
- Fermionic representation (Nair,88) - tree amplitudes are the correlators of the chiral fermions of the sphere

- Tree amplitudes admit the twistor representation(Witten,04). Tree MHV amplitudes are localized on the curves in the twistor space. Twistor space - $CP(3\|4)$
- Twistor space emerges if we make a Fourier transform with respect to the "half" of the momentum variables $\int d\lambda e^{i\mu\lambda} f(\lambda\bar{\lambda})$. Point in the Minkowski space corresponds to the plane in the twistor space
- Localization follows from the holomorphic property of the tree MHV amplitude. Possible link to integrability via fermionic representation
- Stringy interpretation - auxiliary fermions are the degrees of freedom on the D1-D5 open strings ended on the Euclidean D1 instanton.

- The tree MHV amplitude has very simple form

$$A(1^-, 2^-, 3^+ \ldots, n^+) = g^{n-2} \frac{< 12 >^4}{< 12 >< 23 > \cdots < n1 >}$$

- The on-shell momenta of massless particle in the standard spinor notations read as $p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$, λ_a and $\tilde{\lambda}_{\dot{a}}$ are positive and negative helicity spinors.
- Inner products in spinor notations $< \lambda_1, \lambda_2 >= \epsilon_{ab} \lambda_1^a \lambda_2^b$ and $[\tilde{\lambda}_1 \tilde{\lambda}_2] = \epsilon_{\dot{a}\dot{b}} \tilde{\lambda}_1^{\dot{a}} \tilde{\lambda}_2^{\dot{b}}$.

- The generating function for the tree MHV amplitudes - solution to the self-duality equation with the particular boundary conditions (Bardeen 96, Rosly-Selivanov 97). It substitutes the sum of the plane waves in the nonlinear theory

$$A_{\dot\alpha} = g^{-1}\partial_{\dot\alpha} g$$

$$g_{ptb}(\rho) = 1+\sum_{J} g_J(\rho)E_J+\dots+\sum_{J_1\dots J_L} g_{J_1\dots J_L}(\rho)E_{J_1}\dots E_{J_L}+\dots$$

- The E_{J_1} is the solution to the free equation of motion
- The coefficients are derived from self-duality condition

$$g_{J_1\dots J_L}(\rho) = \frac{<\rho, q^{j_1}><j_1, q^{j_2}><j_2, q^{j_3}>\cdots<j_{L-1}, q^{j_L}>}{<\rho, j_1><j_1, j_2><j_2, j_3>\cdots<j_{L-1}, j_L>}$$

- The resummation of the tree amplitudes can be done resulting into the so-called MHV Lagrangian (Cachazo-Scwrcek-Witten). The tree MHV amplitude corresponds to the vertex in this formulation
- The same solution to the self-duality equation provides the canonical transformation from the tree light-cone YM Lagrangian to the MHV Lagrangian (Rosly-A.G., Mansfield, Boels-Mason,Skinner).
- The proper analogy: instanton solution to the selfduality equation generates t'Hooft vertex in QCD. Here the different solution to the selfduality equation (perturbiner) generates the infinite set of MHV vertexes

Properties of the loop MHV amplitudes

- Exponentiation of the ratio $\frac{M_{all-loop}}{M_{tree}}$ which contains the IR divergent and finite parts.
- BDS conjecture for the all loop answer

$$\log \frac{M_{all-loop}}{M_{tree}} = (IR_{div} + \Gamma_{cusp}(\lambda) M_{one-loop})$$

- It involves only two main ingredients - one-loop amplitude and all-loop $\Gamma_{cusp}(\lambda)$
- $\Gamma_{cusp}(\lambda)$ obeys the integral equation (Beisert-Eden-Staudacher) and can be derived recursively
- The conjecture fails starting from six external legs at two loops (Bern -Dixon-Kosower, Drummond-Henn-Korchemsky-Sokachev) and at large number of legs at strong coupling(Alday-Maldacena)

- One more all-loop conjecture - $\frac{M_{all-loop}}{M_{tree}}$ coincides with the Wilson polygon built from the external light-like momenta p_i.
- The conjecture was formulated at strong coupling (Alday-Maldacena, 06) upon the T-duality at the worldsheet of the string in the AdS_5 geometry
- Checked at weak coupling (one and two loops) as well (Drummond- Henn- Korchemsky- Sokachev, Bern-Dixon-Kosover, Brandhuber-Heslop-Travagnini 07).
- Important role of Ward identities with respect to the special conformal transformation in determination of the Wilson polygon (Drummond-Henn-Korchemsky-Sokachev). It fixes the form of the amplitudes at small number of legs

There is no satisfactory stringy explanation of the loop MHV amplitudes and Wilson polygon-amplitude duality. Suspicion - closed string modes contribute (Cachazo-Swrchek-Witten)that is perturbative diagrams in YM theory are sensitive to the gravity degrees of freedom.

The T-duality in the radial AdS direction supplemented by the fermionic T-duality is the symmetry of the sigma model (Berkovits-Maldacena, Beisert et.al) hence it restricts the amplitudes

Main Questions

- Is there fermionic representation of the loop MHV amplitudes similar to the tree case?
- Is there link with integrability at generic kinematics ? The integrability behind the amplitudes is known at low-loop Regge limit (Lipatov 93, Faddeev-Korchemsky 94) only
- Is there trace of the weak-strong coupling S-suality of N=4 SYM in the amplitudes?
- What is the stringy geometrical origin of the BDS conjecture, if any?
- What is the physical origin of MHV amplitude-Wilson polygon duality?

c=1 example

- Consider c=1 string (1d-target space + Liouville direction). The only degrees of freedom - massless tachyons with the discrete momenta
- Exact answer for the tachyonic amplitudes (Dijkgraaf,Plesser,Moore 94)
- Generating function for the amplitude - solution (τ function) for the Toda integrable systems. "'Times"'- generating parameters for the tachyon operators with the different momenta

- Generating function admits representation via chiral fermions or bosons on the Riemann surface=Fermi surface for the auxiliary fermions
$$x^2 - y^2 = 1$$
in the background of the particular abelian gauge field A(z) which provides the "S-matrix"

- This Riemann surface parameterizes the particular moduli space.

- The "fermions" represent the intersection of noncompact Lagrangian branes (they cover the half of the whole dimension) so-called FZZT branes. They are not literally fermions - better to think of as Wigner functions on the phase space. Two types of branes ZZ branes - localized in the Lioville direction but extended on the Riemann surface. FZZT branes- extended(semiinfinitely) in the Lioville direction and localized on the Riemann surface

- The generating function for the amplitude

$$\tau(t_k) =< 0|exp(\sum t_k V_k) exp \int (\bar{\psi} A \psi) exp(\sum t_{-k} V_{-k})|0 >$$

- That is all scattering amplitudes can be described in terms of the fermionic currents on the Fermi surface

- The amplitude can be represented in terms of the "‘Wilson polygon"’ for the auxiliary abelian gauge field! This gauge field has nothing to do with the initial tachyonic scalar degrees of freedom. The auxiliary abelian gauge field A(z) yields the choice of the vacuum state in the theory of fermions.
- Riemann Fermi surface reflects the hidden moduli space of the theory (chiral ring) and it gets quantized. Equation of the Riemann surface becomes the operator acting on the wave function(the analogue of the secondary quantization). The following commutation relation is implied

$$[x, y] = i\hbar$$

- This procedure of the quantization of the Riemann surface is familiar in the theory of integrable systems. Quantum Riemann surface =so-called Baxter eqution
- Degrees of freedom on the Riemann surface - Kodaira-Spencer gravity reduced to two dimensions (Dijkgraaf-Vafa,07)
- Solution to the Baxter equation - wave function of the single separated variable - Lagrangian brane or Lagrangian branes intersection (Nekrasov-Rubtsov-A.G. 2000)
- Polynomial solution to the Baxter equation - Bethe equations for the roots

- Why moduli space? Naively we have set of external momenta which yield a set of points in the momentum space. These set of points provides the moduli space of the complex structures. More carefully - the marked points in the rapidity space yield the desired moduli space.
- From the Feynman diagrams - integration over the loop Schwinger parameters in the first quantized language amounts to the integration over $M_{0,n}$ (Gopakumar. Aharony et.al)
- At strong coupling. To have the proper interpretation of the Wilson loop as the wave function the integration over the diffeomorphisms $F(s)$ of the contour is necessary (Polyakov). In the amplitude case infinite dimensional integral over F is reduced to finite dimensional integration at the vertexes.

- The moduli space and more precisely Teichmuller space is closely related to the Liouville theory. Classically the universal Teichmueller space is the coadjoint Virasoro orbit. On the other hand Liouville Lagrangian is nothing but free PdQ system on this manifold.

- The discrete Liouville system is related to the Teichmueller space of the disc like surface with n-points at the boundary. The mapping class group generator is identified with the Hamiltonian of the discrete Liouville system (Faddeev-Kashaev).

- Hence we expect that the transition from the tree to loop amplitude involves the proper dressing by the discrete Liouville modes

- Consider the moduli space of the complex structures for genus zero surface with n marked points, $M_{0,n}$. Inequivalent triangulations of the surface can be mapped into set of geodesics on the upper half-plane

- This manifold has the Poisson structure and can be quantized in the different coordinates (Kashaev-Fock-Chekhov, 97-01). The generating function of the special canonical transformations (flip) on this symplectic manifold is provided by $Li_2(z)$ where z- is so-called shear coordinate related to the conformal cross-ratio of four points on the real axe

$$exp(z) = \frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_3)(x_2 - x_4)}$$

- The natural objects geodesics can be determined in terms of shear variables z_a
- The symplectic structure in terms of these variables is simple $\sum_a dz_a \wedge dz_b$ where a corresponds to oriented edge and b is edge next to the right
- Upon quantization

$$[Z_a, Z_b] = 2\pi\hbar\{z_a, z_b\}$$

- Quantum mechanically there is operator of the "‘duality"’ K acting on this phase space with the property $\hat{K}^5 = 1$. It is the analogue of the Q-operator in the theory of the integrable systems since it is build from the eigenfunction of the "‘quantum spectral curve operator"’ Classically this curve looks as

$$e^u + e^v + 1 = 0$$

and gets transformed quantum mechanically into the Baxter equation

$$(e^{i\hbar\partial_v} + e^v + 1)Q(v) = 0$$

- The pair of Baxter equations for the discrete Liouville reads as (Kashaev)

$$Q(x + ib^{\pm}/2) + (1 - e^{4\pi x b^{\pm}})^N Q(x - ib^{\pm}/2) = t(x)Q(x)$$

- Let us use the representation for the finite part of the one-loop amplitude as the sum of the following dilogariphms. The whole amplitude is expressed in terms of the sums of the so-called two easy-mass box functions

$$\sum_i \sum_r Li_2\left(1 - \frac{x^2_{i,i+r} x^2_{i+1,i+r+1}}{x^2_{i,i+r+1} x^2_{i-1,i+r}}\right)$$

$$x_{i,k} = p_i - p_k$$

where p_i are the external on-shell momenta of gluons

- One-loop amplitude with n-gluons is described in terms of the "fermions" living on the spectral curve=Fermi surface which is embedded into the four dimensional complex space! MHV loop amplitude - fermionic current correlator on the spectral curve. Fermi surface lives in the space $T^*M_{0,4}$
- BDS conjecture for all-loop answer=quasiclassics of the fermionic correlator with the identification

$$\hbar^{-1} = \Gamma_{cusp}(\lambda)$$

- Is any ground behind this identification?
- In the limit describing the operators with large Lorentz spin the ground state energy of the corresponding string O(6)(Alday-Maldacena) behaves as

$$E \propto \Gamma_{cusp} logS \propto TL$$

  that is Γ_{cusp} plays the role of the effective tension when the boundary of the string worldsheet is light-like
- For the Wilson loop with cusps and without self-intersections the loop equations reads as

$$\Delta W(c) = \sum_i \Gamma_{cusp}(\lambda, \theta_i) W(c)$$

  that is Γ_{cusp} plays the same role

- "Fermions" on Fermi surface represent the noncompact branes (IR regulator) in the B model . In the mirror dual A model geometry fermions represent Lagrangian branes. Arguments of the brane wave-functions are the points on the moduli space of the complex structures. Fermions are transformed nontrivially on the Fermi surface because of its quantum nature
- Geometry: The spectral curve is embedded as the holomorphic surface in the internal 4-dimensional complex space

$$xy = e^u + e^v + 1$$

- Two branes in C^4 have the geometry

$$x = 0 \quad e^u + e^v + 1 = 0$$

and

$$y = 0 \quad e^u + e^v + 1 = 0$$

- Classically we have degrees of freedom on the intersection of the Lagrangian branes. There are also open strings, representing gluons with the disk geometry ending on the noncompact branes. These strings correspond to the external gluons.
- The tree amplitudes are localized at the points in the Minkowski space. Is there similar "localization" of the loop amplitudes? The suggestive relation - $Gr(2,4)//T = \bar{M}_{0,4}$ where T-maximal torus. The complexified Minkowski space M_c is $Gr(2,4)$ that is localization at points in $M_{0,4}$ can be considered as a kind of localization at the submanifold in M_c.
- The space where the string propagates is essentially noncommutative because of the conventional Planck constant. This is essential when the loop effects in the gauge theory are calculated.

- We expect weak-strong coupling duality valid in N=4 theory. D3 Lagrangian branes are self-dual but F1 strings gets substituted dy D1 strings that is the two-dimensional YM theory with the marked points on their worldvolume is relevant
- The origin of the Riemann surface. It corresponds to the summation of all anomalous relations in the gauge theory. Nontrivial effect of regulator degrees of freedom.
- Similar emergence of the Riemann surfaces. N=2 SYM theory-surface follows from the summation of the infinite number of the instantons. N=1 SYM- the surface is the result of the account of all generalized Konishi anomalies under the transformations $\Phi \to F(\Phi)$.

- Quantization of the Fermi surface involves the YM coupling constant

$$\frac{1}{g_{YM}^2} = \frac{\int B_{NS-NS}}{g_s}$$

  Usually it is assumed that g_s yields the "Planck constant" for the quantization of the moduli space of the complex structures in the Kodaira-Spenser gravity. However equally some function of Yang-Mills coupling can be considered as the quantization parameter.

- The YM coupling constant yields the quantization of the gravity degrees of freedom in the box diagram (light-on-light scattering)

- Quasiclassics for the solution to the equation of the quantized Fermi surface

$$\Psi(z,\hbar) = \int \frac{e^{ipz}}{p \times sinh(\pi p) sinh(\pi \hbar p)} dp$$

reduces to

$$\Psi(x) \to exp(\hbar^{-1} Li_2(x) + ...)$$

- Arguments of the Li_2 in the expression for the amplitudes correspond to the shear coordinates on the moduli space.
- The quantum dilogariphm has the dual-symmetric form

$$\Psi(z,\hbar) = \frac{e_q(\omega)}{e_{\tilde{q}}(\tilde{\omega})}$$

$q\tilde{q} = 1$ and $e_q(z) = \prod(1 - zq^n)$

- The one-loop MHV amplitude can be presented in the following form

$$M_{one-loop} \propto < 0|J(z_1)...J(z_n)exp(\psi_k A_{nk} \psi_n)|0 >$$

- The variables ψ_k are the modes of the fermion on the spectral curve and $J(z)$ is the fermionic current. The matrix $A_{n,k}$ for the corresponding spectral curve is known (Aganagic-Vafa-Klemm-Marino 03)

- Towards the Regge limit
- One can recover the fundamental R-matrix for the Regge limit from the R-matrix for the modular double
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- In the Regge limit the Baxter operator plays the important role; it provides the Lipatov,s duality transformation between the coordinates and momenta. Surprise- strong coupling limit in Liouville
- From the worldsheet viewpoint one considers the discretization of the Liuville mode and the Faddeev-Volkov model yields the good candidate for the correct S-matrix. In the target space the natural integrable system is described by the model with the universal R-matrix based on the modular double

- Candidates for reggeons - open strings between the IR regulator branes. These states have momenta depending masses
- Possible link with the Reggeon field theory

$$L_{int} = -\frac{1}{g}\partial_+ Pexp(-\frac{g}{2}\int_{-\infty}^{x^+} A_+ dx_-)\partial^2 V_-$$

$$-\frac{1}{g}\partial_- Pexp(-\frac{g}{2}\int_{-\infty}^{x^-} A_- dx_+)\partial^2 V_+$$

where x_+, x_- are the light-cone coordinates and A is the conventional gluon field. Reggeons are the sources for the Wilson lines in accordance with holographic approach.

- The universal R-matrix for the modular double acting in $D \otimes D$ involves the product of four quantum dilogs desired to describe the "basic block"-box diagram
- Natural link with the spin chains for the Regge limit when the Wilson polygon becomes very thin in one direction

Conclusion

- The representation of the loop MHV amplitude as the chiral fermionic correlator on the quantized Fermi surface of the effective degrees of freedom is suggested. Nontrivial effect of closed string degrees of freedom(Kodaira-Spencer gravity) in the box diagram!
- Link to the integrability behind generic MHV amplitudes via fermionic= IR brane representation. Particular solutions to 3-KP integrable system which correspond to the Faddeev-Volkov model of the discrete conformal mappings (discrete Lioville) with the good S-duality properties. The corresponding statistical model with the positive weights is Bazhanov-Mangazeev-Segreev one (2007)

- BDS conjecture can be reformulated in terms of the quantum geometry of the momentum space with $\Gamma_{cusp}(\lambda)$ as the quantization parameter. Way to improve-take into account the cubic vertex (screening operator) on the world-sheet in the Kodaira-Spencer gravity and loops in the 2d theory. Hopefully this improves the matching with the Regge limit of the amplitudes lost in BDS anzatz

- The degrees of freedom responsible for the dual description of the gluon amplitudes - IR branes=hypersurfaces in the "momentum" space. They are analogue of the D1-instantons (Witten) or IR branes of (Alday-Maldacena)
- Positions of the branes are fixed by the Bethe anzatz equations. Example of the same nature. Extremization of the superpotential on the brane worldvolume yields its position in the embedding space
- There are some candidates for the "reggeon" degrees of freedom - open strings between two regulator branes. They are analogue of "W-bosons" with masses depending on the momenta. This could explain the same universality class of the N=2 SQCD at $N_f = 2N_c$ and Reggeon Hamiltonian. The brane geometry is similar.