Reciprocity in AdS/CFT

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References:
V.F., M. Beccaria arXiv: **0710.0217, 0803.3768 (0810.0101)**
M. Beccaria, V.F., A. Tirziu, A. A. Tseytlin arXiv: **0809.5234**

N=4 SUSY and QCD (FRIF workshop)
LPTHE Jussieu, Paris 12-13 December 2008

Outline

- **Introduction**

  the central role of N=4 SYM, closed formulas for anomalous dims

- **Reciprocity at weak coupling**

- **Reciprocity at strong coupling** **+ Tseytlin talk!**

  classical string solutions dual to twist operators

- **Conclusions & Outlook**

see also Belitsky, Basso and Janik talks!

The central role of N=4 SYM

from string theory to strong interactions

energies ⟷ anomalous dimensions ⟶ splitting functions

$$AdS_5 \times S^5 \longleftrightarrow \begin{matrix}\mathcal{N}=4\\ \text{SYM}\end{matrix} \longrightarrow \begin{matrix}\mathcal{N}=1,\,2\\ \text{SYM}\end{matrix} \longrightarrow \text{QCD}$$

I (from $AdS_5 \times S^5$ to $\mathcal{N}=4$ SYM)

II (from $\mathcal{N}=4$ SYM to QCD)

I. AdS/CFT duality conjecture **[Maldacena, 97]**

type IIB strings on $AdS_5 x S^5$ ↔ $\mathcal{N}$=4 Super Yang Mills in d=3+1

- Agreement of the underlying symmetry supergroup PSU(2,2|4)
- Weak/strong coupling duality $\frac{4\pi\lambda}{N} = g_s, \qquad \sqrt{\lambda} = \frac{R^2}{\alpha'}, \qquad (\lambda = N\, g^2_{\text{YM}})$
- Prediction $E_{\text{string}} = \Delta_{\text{CFT}}$
- Planar limit $N \to \infty \Rightarrow g_s = 0$ free string. Integrability!

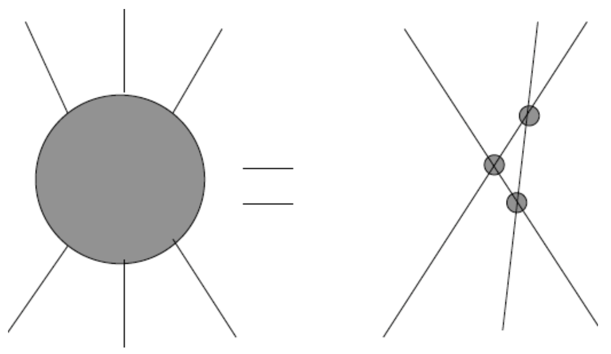
II. Superconformal (β=0) vs confined (β<0), SU(N) vs SU(3), adj. vs fundam.
And presence/lack of a dual stringy description!

Integrability in N=4 SYM

Quantitative duality: remarkable boost, ***data*** coming from perturbative gauge theory (4-loop in λ) and perturbative string theory (2-loop in $1/\sqrt{\lambda}$).

Framework: integrable structures on both sides of AdS/CFT (planar limit!)

Integrable CFT: ***not in the sense of factorised space-time scattering!***



Observables: correlation functions of *gauge invariant local composite operators*

$$\mathcal{O} = \mathrm{Tr}(\mathcal{X}\mathcal{Y}\mathcal{Z}\mathcal{F}_{\mu\nu}\Psi(\mathcal{D}_\mu\mathcal{Z})\ldots)$$

It is integrable the evolution of the composite operators with the RG scale.

[Minahan, Zarembo 02]

The planar dilatation operator $\mathfrak{D}$, measuring the scaling dimensions, maps to a spin chain Hamiltonian, ***integrable - thus solvable*** by means of a Bethe Ansatz.

$$\mathfrak{D}(\lambda) = \mathfrak{D}_0 + \sum_{\ell\geq 1} \lambda^\ell \mathcal{H}^{(\ell)}_{integrable}$$

Features not exclusive of N=4 SYM!

[Lipatov 93, Fadeev, Korchemsky 94, Korchemsky 95]
[Belitsky, Braun, Derkachov, Korchesky, Manashov, 98-99]
[Belitsky, Gorsky, Korchemsky, 03]

Solvability

Easier is working with S-matrix: constrained by the global symmetry [Beisert, 05]
crossing symmetry [Janik, 06] *plus* string data [Beisert, Tseytlin, 05, Beisert, Hernandez, Lopez, 06, Beisert Eden Staudacher 06]

- All-loop formulation of the full PSU(2,2 | 4) asymptotic Bethe equations

[Beisert, Staudacher 05]

$$1=\prod_{j=1}^{K_4}\frac{x_{4,j}^+}{x_{4,j}^-},$$

$$1=\prod_{j=1}^{K_2}\frac{u_{1,k}-u_{2,j}+\frac{i}{2}\eta_1}{u_{1,k}-u_{2,j}-\frac{i}{2}\eta_1}\prod_{j=1}^{K_4}\frac{1-g^2/2x_{1,k}x_{4,j}^{+\eta_1}}{1-g^2/2x_{1,k}x_{4,j}^{-\eta_1}},$$

$$1=\prod_{\substack{j=1\\ j\neq k}}^{K_2}\frac{u_{2,k}-u_{2,j}-i\eta_1}{u_{2,k}-u_{2,j}+i\eta_1}\prod_{j=1}^{K_3}\frac{u_{2,k}-u_{3,j}+\frac{i}{2}\eta_1}{u_{2,k}-u_{3,j}-\frac{i}{2}\eta_1}\prod_{j=1}^{K_1}\frac{u_{2,k}-u_{1,j}+\frac{i}{2}\eta_1}{u_{2,k}-u_{1,j}-\frac{i}{2}\eta_1},$$

$$1=\prod_{j=1}^{K_2}\frac{u_{3,k}-u_{2,j}+\frac{i}{2}\eta_1}{u_{3,k}-u_{2,j}-\frac{i}{2}\eta_1}\prod_{j=1}^{K_4}\frac{x_{3,k}-x_{4,j}^{+\eta_1}}{x_{3,k}-x_{4,j}^{-\eta_1}},$$

$$1=\left(\frac{x_{4,k}^-}{x_{4,k}^+}\right)^L\prod_{\substack{j=1\\ j\neq k}}^{K_4}\left(\frac{x_{4,k}^{+\eta_1}-x_{4,j}^{-\eta_1}}{x_{4,k}^{-\eta_2}-x_{4,j}^{+\eta_2}}\,\frac{1-g^2/2x_{4,k}^+x_{4,j}^-}{1-g^2/2x_{4,k}^-x_{4,j}^+}\,\sigma^2(x_{4,k},x_{4,j})\right)$$
$$\times\prod_{j=1}^{K_1}\frac{1-g^2/2x_{4,k}^{-\eta_1}x_{1,j}}{1-g^2/2x_{4,k}^{+\eta_1}x_{1,j}}\prod_{j=1}^{K_3}\frac{x_{4,k}^{-\eta_1}-x_{3,j}}{x_{4,k}^{+\eta_1}-x_{3,j}}\prod_{j=1}^{K_5}\frac{x_{4,k}^{-\eta_2}-x_{5,j}}{x_{4,k}^{+\eta_2}-x_{5,j}}\prod_{j=1}^{K_7}\frac{1-g^2/2x_{4,k}^{-\eta_2}x_{7,j}}{1-g^2/2x_{4,k}^{+\eta_2}x_{7,j}}$$

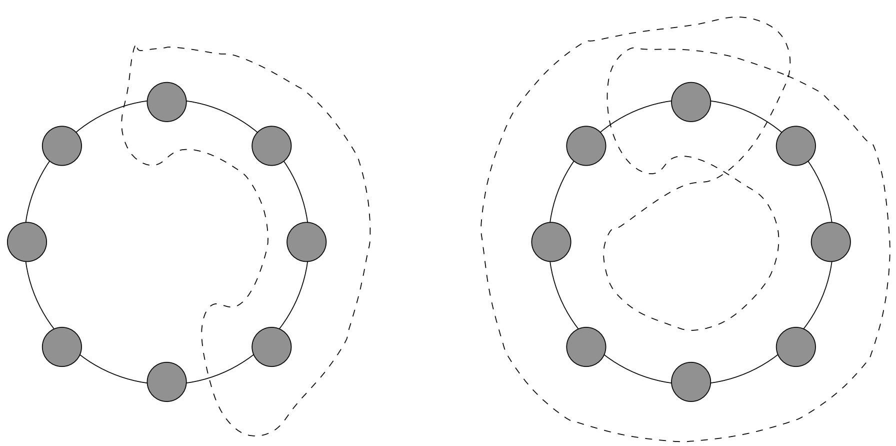
$$1=\prod_{j=1}^{K_6}\frac{u_{5,k}-u_{6,j}+\frac{i}{2}\eta_2}{u_{5,k}-u_{6,j}-\frac{i}{2}\eta_2}\prod_{j=1}^{K_4}\frac{x_{5,k}-x_{4,j}^{+\eta_2}}{x_{5,k}-x_{4,j}^{-\eta_2}},$$

$$1=\prod_{\substack{j=1\\ j\neq k}}^{K_6}\frac{u_{6,k}-u_{6,j}-i\eta_2}{u_{6,k}-u_{6,j}+i\eta_2}\prod_{j=1}^{K_5}\frac{u_{6,k}-u_{5,j}+\frac{i}{2}\eta_2}{u_{6,k}-u_{5,j}-\frac{i}{2}\eta_2}\prod_{j=1}^{K_7}\frac{u_{6,k}-u_{7,j}+\frac{i}{2}\eta_2}{u_{6,k}-u_{7,j}-\frac{i}{2}\eta_2},$$

$$1=\prod_{j=1}^{K_6}\frac{u_{7,k}-u_{6,j}+\frac{i}{2}\eta_2}{u_{7,k}-u_{6,j}-\frac{i}{2}\eta_2}\prod_{j=1}^{K_4}\frac{1-g^2/2x_{7,k}x_{4,j}^{+\eta_2}}{1-g^2/2x_{7,k}x_{4,j}^{-\eta_2}},$$

$$Q_r=\frac{1}{r-1}\sum_{j=1}^{K_4}\left(\frac{i}{(x_{4,j}^+)^{r-1}}-\frac{i}{(x_{4,j}^-)^{r-1}}\right),\qquad \delta D=g^2Q_2=g^2\sum_{j=1}^{K_4}\left(\frac{i}{x_{4,j}^+}-\frac{i}{x_{4,j}^-}\right).$$

CAVEAT: wrapping!



Bethe eqs correct up to $O(\lambda^L)$
(L: length of operators)

[Kotikov, Lipatov, Rej, Staudacher, Velizhanin, 07]

Remarkable outcomes

The large spin limit

Spectacular interpolation weak/strong coupling: the scaling function/cusp anomaly

- **QCD**: ***logarithmic scaling*** in leading ***twist operators*** at large spin S (x→1)

$$\gamma(S) = 2\,\Gamma_{\rm cusp}(\alpha)\ \log S + \mathcal{O}(S^0) \qquad\qquad \mathcal{O}_S = \bar{q}(\gamma_+ \mathcal{D}^+)^S q$$

$\Gamma_{\rm cusp}$ governs the renormalization of a Wilson loops evaluated over a closed contour with a cusp.

- **SYM**: Twist two operators in sl(2) ⊂ psu(2,2|4) via Bethe Ansatz

$$\gamma(S) = f(\lambda)\ \log S + O(S^0) \qquad\qquad \mathcal{O}_S = \mathrm{Tr}(\varphi\,\mathcal{D}_+^S\,\varphi)$$

- At weak coupling $f(\lambda) = \dfrac{\lambda}{2\pi^2} - \dfrac{\lambda^2}{96\pi^2} + \cdots$

✓ Agreement with MHV 4-point gluon amplitudes of N=4 SYM at four loops !

[Bern, Czakon, Dixon, Kosower, Smirnov, 06]

- At strong coupling $f(\lambda) = \dfrac{\sqrt{\lambda}}{2\pi} - \dfrac{3\log 2}{2\pi} + \cdots$

numerically **[Benvenuti et al 07, Alday et al 07, Kostov et al 07, Beccaria VF 0703]**

analytically! **[Basso, Korchemsky 07]**

- **STRINGS**: Agreement with two loops string calculations ! **[Roiban, Tseytlin 07]**

Remarkable “byproducts”

▸ Wilson loops vs gluon scattering amplitudes at weak and strong coupling

[Alday, Maldacena 07]

[Korchemsky et al. 07, Bern et al. 08, ...]

▸ Generalised scaling function f(λ, J)

[Freyult Rej Staudacher 07, Roiban Tseytlin 07,]

[Basso, Korchemsky 08]

[Gromov 08, Fioravanti et al. 1& 2 08, Beccaria 08]

and natural question:

How this “*triality*” survives in the large spin expansion
***beyond the leading order* ?**

▸ *CFT*: at higher twist? Sectors other than sl(2)?

▸ ***Strings?***

Closed formulas
Digression (but interesting!)

Field strength operators: $\mathcal{O}_L = \text{Tr}\,\mathcal{F}^L$ **[Beisert, Ferretti, Heise, Zarembo 04]**

▶ Solving (nested) Bethe equations in the thermodynamical limit **[Rej, Staudacher, Zieme, 07]**

$$\frac{\gamma(g)}{L} \sim 2 + 3\,g^2 - \frac{51}{4}\,g^4 + \frac{393}{4}\,g^6 + \ldots$$

$$g^2 = \frac{\lambda}{16\,\pi^2}$$

▶ Solving numerically, one might find a ***formula closed in L*** **[Beccaria Forini, 07]**

$$\gamma_L(g) = L\left[2 + 3\,g^2 + \sum_{n\geq 2} c_n(L)\,g^{2n}\right]$$

$$c_2(L) = -\frac{51}{8} - \frac{9}{8}\frac{1}{(-1)^L\,2^{L-1}+1}$$

and *then* expand it at large L

$$\frac{\gamma(g)}{L} \sim f_0(g) + f_1(g,L)\,e^{-L\,\ln 2} + f_2(g,L)\,e^{-2\,L\,\ln 2} + \ldots$$

This reveals

- *length-changing* effects!
- which *string dual* description?

N=4 SYM and QCD interplay: I

QCD-inspired closed formulas

Maximum transcendentality principle (MTP) **[Kotikov, Lipatov, 01, 02]**

The N=4 universal twist two anomalous dimension at n loops is a linear combination of harmonic sums of transcendentality $2n-1$.

3-loop $\gamma(S)_{\text{uni}}$ extracted from the "most complicated terms" of the QCD result of **[Moch, Vermaresen, Vogt, 04]** with $C_F=C_A=N_C$ **[Kotikov, Lipatov, Onishchenko, Velizhanin, 04]**

✓ Applied to the numbers coming from the Bethe eqs it works!

$$\gamma^{(1)}(S) = \sum_{|\tau|=1} c_\tau S_\tau(S) = c\, S_1(S)$$

$$\gamma^{(2)}(S) = \sum_{|\tau|=3} c_\tau S_\tau(S)$$

$$\gamma^{(3)}(S) = \sum_{|\tau|=5} c_\tau S_\tau(S)$$

(Nested) harmonic sums

$$S_a(S) = \sum_{n=1}^{S} \frac{(\text{sign}(a))^n}{n^{|a|}}$$

$$S_{a,\mathbf{b}}(S) = \sum_{n=1}^{S} \frac{(\text{sign}(a))^n}{n^{|a|}} S_{\mathbf{b}}(n)$$

Recent development (and confirmation of MTP): *analytic* solutions to the Baxter equation for operator of twist two and three! **[Kotikov, Rej, Zieme, 08]**

Generalising MTP:

- Twist-3 **sl(2)**, three scalars $\mathcal{O}_S = \mathrm{Tr}\mathcal{D}_+^S(\varphi)^3$ [Beccaria 07] [Kotikov et al 07]

confirmed by [Kotikov, Rej, Zieme, 08]

- Twist-3 **sl(2|1)**, three gauginos $\mathcal{O}_S = \mathrm{Tr}\mathcal{D}_+^S(\lambda)^3$ [Beccaria 07]

- Twist three, full **psu(2,2|4)!** $\mathcal{O}_S = \mathrm{Tr}\mathcal{D}_+^S(A)^3$ [Belitsky 99] [Beccaria 07] [Beccaria, Forini 08]

  - Generalization of the MTP principle

$$\gamma_n = \sum_{\tau=0}^{2n-1} \gamma_n^{(\tau)} \qquad \gamma_n^{(\tau)}(S) = \sum_{k+\ell=\tau} \frac{\mathcal{H}_{\tau,\ell}(s)}{(s+1)^k}$$

$$\gamma_1 = 4\,S_1 + \frac{2}{s+1} + 4$$

$$\gamma_2 = -2\,S_3 - 4S_1\,S_2 - \frac{2S_2}{s+1} - \frac{2S_1}{(s+1)^2} - \frac{2}{(s+1)^3} - 4\,S_2 + \frac{2}{(s+1)^2} - 8$$

Harmonic sums evaluated in s=S/2+1.

N=4 SYM and QCD interplay: II

x $\rightarrow$ 0, *negative* spin S (BFKL equation)

The (analitically continued) anomalous dimension for twist two sl(2) operators *maximally violates, at four loops,* the BFKL prediction for the (Regge) poles

- ➡ Breakdown of the Bethe eqs. at four loops **[Kotikov, Lipatov, Rej, Staudacher, Velizhanin, 07]**
- ➡ Correctness of the twist two anomalous dimension! **[Bajnok, Janik Lukowsky, 08]**

N=4 SYM and QCD interplay: III

x $\rightarrow$ 1, *large* spin S (quasi elastic limit)

▶ ***Leading* logarithmic behavior**: ***universal in twist and flavors for all gauge theories***!

$$\gamma(S) = 2\,\Gamma_{\text{cusp}}(\alpha)\,\log S + \mathcal{O}(S^0)$$

[Belitsky, Gorsky, Korchemsky, 06]

▶ ***Subleading* logarithmic behavior**: much less clear!

- MVV relations for twist two operators **[Moch, Vermaresen, Vogt, 04]**

$$\gamma(S) = A\,\log S + B + C\,\frac{\log S}{S}$$

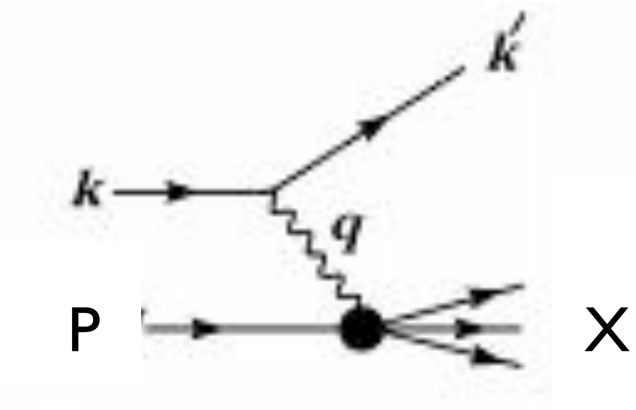
$$C = \frac{A^2}{2}$$

⟶ Non trivial! A, B, C depend on the coupling!

Structural explanation: **revisiting the parton evolution** in the x $\rightarrow$ 1 regime

Anomalous dimensions vs. splitting functions I

The deep inelastic scattering (DIS) process



$x = -\frac{q^2}{2\,p\cdot q}$ Bjorken variable

$-q^2 = Q^2$ momentum transfer.

The ***hadronic tensor***

$$W(p,q) = \frac{1}{4\pi}\int d^4y\, e^{i\,q\,y} < p\,|\,j(y)\,j(0)\,|\,p >$$

can expanded in the DIS regime (Q large, x fixed)

$$j(y)\,j(0) = \sum_{a,S} C_a^S(y^2)\, y^{\mu_1}\ldots y^{\mu_S}\, \boxed{\mathcal{O}^S_{\mu_1\ldots\mu_S,a}(0)}$$

Twist operators!

$$\sim \frac{1}{y^{2(2\dim j - t)}}$$

Leading = minimal twist = twist two

The ***evolution*** of the C_a^S is governed by the ***anomalous dimensions*** of the twist ops.

The hadronic tensor is parametrized by the structure function ↔ ***distribution functions***

Anomalous dimensions vs. splitting functions II

- The ***DGLAP evolution equations*** for the parton distribution function f_S (DIS)

$$\frac{df_\sigma(x,Q^2)}{d\log Q^2} = \int_x^1 \frac{dz}{z} P_\sigma(z,\alpha_s) f_\sigma\left(\frac{x}{z},Q^2\right) \qquad \sigma = S,T$$

space-like (DIS)
time-like (e^+e^- ann)

also describe evolution of fragmentation functions f_T (e^+e^- annihilation into hadrons, the cross channel of DIS).

$$\frac{df_\sigma(x,Q^2)}{d\log Q^2} = \gamma_\sigma(S)\, f_\sigma(S,Q^2) \qquad \gamma_\sigma(S) = \int_0^1 dx\, x^{S-1} P_\sigma(x)$$

The Mellin transforms of the space or time-like splitting function P_σ are anomalous dimensions → in the space-like case, those of twist-2 operators.

- Proposal by Gribov-Lipatov: the two splitting functions ***coincide*** and

$$P(x) = -x\, P\left(\frac{1}{x}\right)$$

GL reciprocity **[Gribov, Lipatov, 72]**

Broken in QCD beyond 1-loop **[Curci, Furmansky, Petronzio, 80]**

▶ Proposal (Dokshitzer-Marchesini-Salam): ***Reciprocity Respecting Evolution Equations***

$$\frac{df_\sigma(x,Q^2)}{d\log Q^2} = \int_x^1 \frac{dz}{z}\mathcal{P}(z,\alpha_s) f_\sigma\left(\frac{x}{z}, z^\sigma Q^2\right)$$

RREE

with a universal kernel $\mathcal{P}$.

[Dokshitzer, Marchesini, Salam, 05]

[Dokshitzer, Khoze, Troian 95]

Evolution is now solved by the ***non-linear relation***

$$\gamma_\sigma = \mathcal{P}\left(S - \frac{1}{2}\sigma\gamma_\sigma(S)\right)$$

- If the kernel satisfies the GL reciprocity

$$\mathcal{P}(x) = -x\,\mathcal{P}\left(\frac{1}{x}\right)$$

✓ MVV relations automatically satisfied !

✓ Verified at three loops for non singlet twist two **[Mitov, Moch, Vogt 06]**

The reciprocity property can be ***formulated in Mellin space as***

$$\mathcal{P}(S) \equiv \mathcal{P}\left(C^2\right), \qquad C^2 = S(S+1)$$

Conformal symmetry approach

[Basso Korchemsky 06]

If conformal symmetry was exact, operators as $\mathcal{O} = \mathrm{Tr}\{D_+^{\kappa_1} X ... D_+^{\kappa_J} X\}$ can be classified according to representations of the ***collinear SL(2; R) subgroup*** of the full SO(2, 4) conformal group.

[Ohrndorf 82]

Conformal invariance: different SL(2; R) multiplets cannot mix under renormalization and their anomalous dimension depends on the ***conformal SL(2; R) spin s***

$$s = \frac{(S+\Delta)}{2}$$

S = Lorentz spin Δ = scaling dimension

The scaling dimension receives anomalous contribution due to interaction

$$\Delta(\lambda) = S + J + \gamma_\lambda(S)$$

⇒ the *conformal spin gets modified in higher loops* (e.g. scalars) [Belitsky Mueller 98]

$$s(\lambda = 0) = S + \frac{J}{2} \longrightarrow s(\lambda) = S + \frac{J}{2} + \frac{1}{2}\gamma(S)$$

The anomalous dimension is actually a (twist-dep) function of the conformal spin

$$\gamma(S) = \mathrm{f}\left(S + \frac{1}{2}\gamma(S)\right)$$

Generalized Reciprocity in (QCD and) N=4 SYM

generic twist!

[Basso, Korchemsky 06]

The function f defined by the *functional relation*

$$\gamma = \mathrm{f}\left(S + \tfrac{1}{2}\gamma\right) \star$$

is a *reciprocity respecting* (RR) function if its ***large spin*** expansion takes the form

$$\mathrm{f}\left(S\right) = \sum_n \frac{a_n(\ln C)}{C^{2n}}$$

with C^2 the suitable ***Casimir of the collinear conformal subgroup*** SL(2,R) ⊂ SO(4,2)

$$C^2 = (S + J\,\ell)(S + J\,\ell - 1)$$

$$\mathcal{O} = \mathrm{Tr}\{D^{k_1}X ... D^{k_J}X\}$$

$$k_1 + ... + k_J = S$$

J : twist ℓ:

φ	λ	A
$\frac{1}{2}$	1	$\frac{3}{2}$

★ In QCD $\gamma_\sigma(S) = f\left(S - \frac{\sigma}{2}\gamma_\sigma(S) - \frac{J}{4}\beta(\alpha_S)\right)$

Checks of Reciprocity

- Step 1: solve for

$$f(S) = \sum_{k=1}^{\infty} \left(-\frac{1}{2}\partial_S\right)^{k-1} [\gamma(S)]^k = \gamma - \frac{1}{4}(\gamma^2)' + \frac{1}{24}(\gamma^3)'' + \dots$$

As the anomalous dimension, f will be a perturbative series (weak, or strong!, coupling)

- Step 2: Expand in large S, rewrite in terms of C, check parity invariance

Ex. twist two

$$f^{(1)} = 8\,\ln C + \frac{4}{3}\frac{1}{C^2} - \frac{4}{15}\frac{1}{C^4} + \dots$$

$$f^{(2)} = -\frac{8}{3}\pi^2\,\ln C - 24\,\zeta_3 + \left(8 - \frac{4\pi^2}{9}\right)\frac{1}{C^2} + \left(4 + \frac{4\pi^2}{45}\right)\frac{}{C^4} + \dots$$

✓ All twist-2 anomalous dimensions in QCD and $\mathcal{N}$=4 SYM [Basso, Korchemsky 06]

✓ Twist-3 scalars, gauginos, gluons (in ***closed*** form!) in $\mathcal{N}$=4 SYM

[Beccaria, Marchesini, Dokshitzer 07] [Beccaria 07] [Beccaria Forini 08]

Example: twist 3 gluons in closed form!

[Beccaria, Forini 08]

- RR wrt Casimir $C^2 = (S + J\ell)(S + J\ell - 1) \equiv s(s+2),$ $\quad s = \frac{S}{2} + 1$

1. The linear map

$$\Phi_a(S_{b,c}) := S_{a,b,c} - \frac{1}{2} S_{a+b,c}$$

defines combinations

$$I_a(S_a) = S_a \qquad I_{a,b,c,\ldots} = \Phi_a(\Phi_b(\ldots))$$

that are *RR wrt s(s+1)* iff all positive a_i are odd

2. If $I(s)$ is RR wtr C^2=*s(s+1)* **then** $\tilde{I}(s) = I(s) + I(s+1)$ is RR wtr C^2=*s(s+2)*

e.g. at 2 loops $\quad \mathrm{f}^{(2)} = -\widetilde{I}_3 - \frac{1}{3}\pi^2\,\widetilde{I}_1 - 8 - \frac{2\pi^2}{3}$

- The function f is ***simpler*** than γ : reduction of log. singularities in large S expansion !

[Dokshitzer, Marchesini,Salam, 05]

The functional relation $\gamma(S) = \mathrm{f}\left(S + \frac{1}{2}\gamma(S)\right)$ *is not predictive as it is.*

(Physical) assumption: f is "simpler"

In particular, it does not contain *(ln S/S)ᵖ* terms

Then considering that *f starts as the anomalous dimension*, the leading ~ $(\ln S/S)^p$ power corrections can be resummed into

$$\gamma(S) = f_0 \,\ln(S + \frac{1}{2} f_0 \ln S) + \ldots = f_0 \,\ln S + \frac{f_0^2}{2}\frac{\ln S}{S} - \frac{f_0^3}{8}\frac{\ln^2 S}{S^2} + \frac{f_0^4}{24}\frac{\ln^3 S}{S^3} + \ldots$$

The *leading logs* $(\ln S/S)^p$ are governed ***at all loops*** by the cusp anomaly (f_0)!

⇒ they should be ***universal*** in twist (and flavor)

✓ Verified up to three loops for all the known anomalous dimensions!
(twist two and three, all flavors)

Twist 2 scalar

$$
\begin{aligned}
\gamma^{\varphi}_{L=2}(\hat\lambda) \quad = \quad & \left[\mathbf{8}\hat\lambda - \frac{\mathbf{8}\pi^{\mathbf{2}}}{\mathbf{3}}\hat\lambda^{\mathbf{2}} + \frac{\mathbf{88}\pi^{\mathbf{4}}}{\mathbf{45}}\hat\lambda^{\mathbf{3}} + \left(-\frac{\mathbf{584}\pi^{\mathbf{6}}}{\mathbf{315}} - \mathbf{64}\zeta_{\mathbf{3}}^{\mathbf{2}}\right)\hat\lambda^{\mathbf{4}}\right] \ln \bar{\mathbf{S}} \\
& -24\zeta_3\hat\lambda^2 + \left(\frac{16}{3}\pi^2\zeta_3 + 160\zeta_5\right)\hat\lambda^3 + \left(-\frac{56}{15}\pi^4\zeta_3 - \frac{80}{3}\pi^2\zeta_5 - 1400\zeta_7\right)\hat\lambda^4 \\
& + \left[\mathbf{32}\hat\lambda^{\mathbf{2}} - \frac{\mathbf{64}\,\pi^{\mathbf{2}}}{\mathbf{3}}\hat\lambda^{\mathbf{3}} + \frac{\mathbf{96}\,\pi^{\mathbf{4}}}{\mathbf{5}}\hat\lambda^{\mathbf{4}}\right] \frac{\ln \bar{\mathbf{S}}}{\mathbf{S}} \\
& + \left[4\hat\lambda - \frac{4\pi^2}{3}\hat\lambda^2 + \left(\frac{44\pi^4}{45} - 96\zeta_3\right)\hat\lambda^3 + \left(-\frac{292\pi^6}{315} + \frac{160}{3}\pi^2\zeta_3 - 32\zeta_3^2 + 640\zeta_5\right)\hat\lambda^4\right] \frac{1}{S} \\
& + \left[-\mathbf{64}\hat\lambda^{\mathbf{3}} + (\mathbf{64}\pi^{\mathbf{2}} - \mathbf{128}\zeta_{\mathbf{3}})\hat\lambda^{\mathbf{4}}\right] \frac{\ln^{\mathbf{2}} \bar{\mathbf{S}}}{\mathbf{S}^{\mathbf{2}}} \\
& + \left[-16\hat\lambda^2 + \left(128 + \frac{16\,\pi^2}{3}\right)\hat\lambda^3 + \left(-128\pi^2 - \frac{32\,\pi^4}{15} + 448\zeta_3\right)\hat\lambda^4\right] \frac{\ln \bar S}{S^2} \\
& + \left[-\frac{2}{3}\hat\lambda + \left(24 + \frac{2\pi^2}{9}\right)\hat\lambda^2 + \left(-\frac{32\pi^2}{3} - \frac{22\pi^4}{135} + 48\zeta_3\right)\hat\lambda^3\right. \\
& \left. + \left(\frac{136\pi^4}{15} + \frac{146\pi^6}{945} - 384\zeta_3 - \frac{32}{3}\pi^2\zeta_3 + \frac{16\zeta_3^2}{3} - 320\zeta_5\right)\hat\lambda^4\right] \frac{1}{S^2} \\
& + \left[\frac{\mathbf{512}}{\mathbf{3}}\hat\lambda^{\mathbf{4}}\right] \frac{\ln^{\mathbf{3}} \bar{\mathbf{S}}}{\mathbf{S}^{\mathbf{3}}} \\
& + \left[64\,\hat\lambda^3 + \left(-768 - \frac{64\,\pi^2}{3} + 128\zeta_3\right)\hat\lambda^4\right] \frac{\ln^2 \bar S}{S^3} \\
& + \left[\frac{16}{3}\hat\lambda^2 + \left(-256 + \frac{16\pi^2}{9}\right)\hat\lambda^3 + \left(512 + \frac{512\pi^2}{3} - \frac{64\pi^4}{15} - 576\zeta_3\right)\hat\lambda^4\right] \frac{\ln \bar S}{S^3} \\
& + \left[-\frac{56}{3}\hat\lambda^2 + \left(96 + \frac{40\pi^2}{9} - 16\zeta_3\right)\hat\lambda^3 + \left(-\frac{224\pi^2}{3} - \frac{32\pi^4}{15} + 800\zeta_3 - \frac{64}{9}\pi^2\zeta_3 + \frac{320\zeta_5}{3}\right)\hat\lambda^4\right] \frac{1}{S^3}
\end{aligned}
$$

Where $\bar S = e^{\gamma_E}\, S$ and $\hat\lambda = \dfrac{\lambda}{16\pi^2}$

Twist 3 scalar

$$
\begin{aligned}
\gamma^{\varphi}_{L=3} \;=\;& \left[\mathbf{8}\hat\lambda - \frac{\mathbf{8}\pi^{\mathbf{2}}}{\mathbf{3}}\hat\lambda^{\mathbf{2}} + \frac{\mathbf{88}\pi^{\mathbf{4}}}{\mathbf{45}}\hat\lambda^{\mathbf{3}} + \left(-\frac{\mathbf{584}\pi^{\mathbf{6}}}{\mathbf{315}} - \mathbf{64}\zeta_{\mathbf{3}}^{\mathbf{2}}\right)\hat\lambda^{\mathbf{4}}\right]\ln\bar{\mathbf{S}} - 8\ln 2\hat\lambda + \left(\frac{8}{3}\pi^2\ln 2 - 8\zeta_3\right)\hat\lambda^2 \\
&+ \left(-\frac{88}{45}\pi^4\ln 2 + \frac{8}{3}\pi^2\zeta_3 - 8\zeta_5\right)\hat\lambda^3 + \frac{8}{315}\left(73\pi^6\ln 2 - 84\pi^4\zeta_3 + 2520\ln 2\zeta_3^2 + 105\pi^2\zeta_5 + 17325\zeta_7\right)\hat\lambda^4 \\
&+ \left[\mathbf{32}\hat\lambda^{\mathbf{2}} - \frac{\mathbf{64}\,\pi^{\mathbf{2}}}{\mathbf{3}}\hat\lambda^{\mathbf{3}} + \frac{\mathbf{96}\,\pi^{\mathbf{4}}}{\mathbf{5}}\hat\lambda^{\mathbf{4}}\right]\frac{\ln\bar{\mathbf{S}}}{\mathbf{S}} + \left[8\hat\lambda + \left(-\frac{8\pi^2}{3} - 32\ln 2\right)\hat\lambda^2 + \left(\frac{88\pi^4}{45} + \frac{64}{3}\pi^2\ln 2 - 32\zeta_3\right)\hat\lambda \right. \\
&\quad \left. - \frac{8}{315}\left(73\pi^6 + 756\pi^4\ln 2 - 840\pi^2\zeta_3 + 2520\zeta_3^2 + 1260\zeta_5\right)\hat\lambda^4\right]\frac{1}{S} \\
&+ \left[-\mathbf{64}\hat\lambda^{\mathbf{3}} + \mathbf{64}\pi^{\mathbf{2}}\hat\lambda^{\mathbf{4}}\right]\frac{\ln^{\mathbf{2}}\bar{\mathbf{S}}}{\mathbf{S^2}} \\
&+ \left[-32\hat\lambda^2 + \left(128 + \frac{64\pi^2}{3} + 128\ln 2\right)\hat\lambda^3 + \left(-256 - 128\pi^2 - \frac{96\pi^4}{5} - 128\pi^2\ln 2\right.\right. \\
&\quad \left.\left. + 256\zeta_3\right)\hat\lambda^4\right]\frac{\ln\bar S}{S^2} + \left[-\frac{8\lambda}{3} + \left(48 + \frac{8\pi^2}{9} + 32\ln 2\right)\hat\lambda^2 + \left(32 - \frac{80\pi^2}{3} - \frac{88\pi^4}{135} - 128\ln 2 - \frac{64}{3}\pi^2 \mathrm{l}\right.\right. \\
&\quad \left. - 64\ln^2 2 + 32\zeta_3\right)\hat\lambda^3 + \left(-512 - \frac{32\pi^2}{3} + \frac{352\pi^4}{15} + \frac{584\pi^6}{945} + 256\ln 2 + 128\pi^2\ln 2\right. \\
&\quad \left.\left. + \frac{96}{5}\pi^4\ln 2 + 64\pi^2\ln^2 2 - 128\zeta_3 - \frac{64}{3}\pi^2\zeta_3 - 256\ln 2\zeta_3 + \frac{64\zeta_3^2}{3} + 32\zeta_5\right)\hat\lambda^4\right]\frac{1}{S^2} \\
&+ \left[\frac{\mathbf{512}}{\mathbf{3}}\hat\lambda^{\mathbf{4}}\right]\frac{\ln^{\mathbf{3}}\bar{\mathbf{S}}}{\mathbf{S^3}} \\
&+ \left[128\hat\lambda^3 + \left(-768 - 128\pi^2 - 512\ln 2\right)\hat\lambda^4\right]\frac{\ln^2\bar S}{S^3} \\
&+ \left[\frac{64}{3}\hat\lambda^2 + \left(-512 - \frac{128\pi^2}{9} - 256\ln 2\right)\hat\lambda^3 + \left(768 + \frac{1408\pi^2}{3} + \frac{64\pi^4}{5} + 1536\ln 2\right.\right. \\
&\quad \left.\left. + 256\pi^2\ln 2 + 512\ln^2 2 - 512\zeta_3\right)\hat\lambda^4\right]\frac{\ln\bar S}{S^3} \\
&+ \left[\left(-\frac{224}{3} - \frac{64\ln 2}{3}\right)\hat\lambda^2 + \left(128 + \frac{352\pi^2}{9} + 512\ln 2 + \frac{128}{9}\pi^2\ln 2 + 128\ln^2 2 - \frac{64\zeta_3}{3}\right)\hat\lambda^3\right. \\
&\quad + \left(896 - \frac{448\pi^2}{3} - \frac{512\pi^4}{15} - 768\ln 2 - \frac{1408}{3}\pi^2\ln 2 - \frac{64}{5}\pi^4\ln 2 - 768\ln^2 2 - 128\pi^2\ln^2 2\right. \\
&\quad \left.\left. - \frac{512\ln^3 2}{3} + 640\zeta_3 + \frac{128}{9}\pi^2\zeta_3 + 512\ln 2\zeta_3 - \frac{64\zeta_5}{3}\right)\hat\lambda^4\right]\frac{1}{S^3}
\end{aligned}
$$

Twist 3 "gauge sector"

$$\begin{aligned}\gamma^A(S) \quad &= \quad \left[\mathbf{8}\hat\lambda - \frac{\mathbf{8}\pi^{\mathbf{2}}}{\mathbf{3}}\hat\lambda^{\mathbf{2}} + \frac{\mathbf{88}\pi^{\mathbf{4}}}{\mathbf{45}}\hat\lambda^{\mathbf{3}} + \left(-\frac{\mathbf{584}\pi^{\mathbf{6}}}{\mathbf{315}} - \mathbf{64}\zeta_{\mathbf{3}}^{\mathbf{2}}\right)\hat\lambda^{\mathbf{4}}\right]\ln\bar{\mathbf{S}} + c^A \\ &+ \left[\mathbf{32}\hat\lambda^{\mathbf{2}} - \frac{\mathbf{64}\,\pi^{\mathbf{2}}}{\mathbf{3}}\hat\lambda^{\mathbf{3}} + \frac{\mathbf{96}\,\pi^{\mathbf{4}}}{\mathbf{5}}\hat\lambda^{\mathbf{4}}\right]\frac{\ln\bar{\mathbf{S}}}{\mathbf{S}} + L^A_{10}\frac{1}{S} \\ &+ \left[-\mathbf{64}\hat\lambda^{\mathbf{3}} + \mathbf{64}\pi^{\mathbf{2}}\hat\lambda^{\mathbf{4}}\right]\frac{\ln^{\mathbf{2}}\bar{\mathbf{S}}}{\mathbf{S}^{\mathbf{2}}} + L^A_{21}\frac{\ln\bar S}{S^2} + L^A_{20}\frac{1}{S^2} \\ &+ \left[\frac{\mathbf{512}}{\mathbf{3}}\hat\lambda^{\mathbf{4}}\right]\frac{\ln^{\mathbf{3}}\bar{\mathbf{S}}}{\mathbf{S}^{\mathbf{3}}} + L^A_{32}\frac{\ln^2\bar S}{S^3} + L^A_{31}\frac{\ln\bar S}{S^3} + L^A_{30}\frac{1}{S^3}\end{aligned}$$

Also twist 2 gauginos and gauge, and twist 3 gauginos!
Their closed formulas are obtained by the twist two scalar case by just *shifting* the argument of the harmonic sums ⇒ and *shifts* do not affect *(ln S/S)^p* coefficients.

What we learn at weak coupling
about the large Lorentz spin expansion of anomalous dimensions

- ***Structure*** (for twist $J \leq 3$)

$$\gamma(S)_{S\gg 1} \quad = \quad f\ \ln S + f_c + \frac{f_{11}\ \ln\ S + f_{10}}{S} + \frac{f_{22}\ \ln^2\ S + f_{21}\ \ln\ S + f_{20}}{S^2} + \mathcal{O}\Big(\frac{\ln^3\ S}{S^3}\Big)$$

- ***Functional/inheritance relation***: ***all-orders*** description for the 'leading logs'!

The function f derived by the anomalous dimension, $\gamma = \mathrm{f}\big(S + \tfrac{1}{2}\gamma\big)$
assumed to be *simpler*, implies all leading logs in terms of the cusp anomaly

- ***Reciprocity***, or parity invariance (for ***minimal*** anomalous dimensions!)

$$\mathrm{f}\big(S\big) = \sum_n \frac{a_n(\ln C)}{C^{2n}} \qquad\qquad C^2 = (S + J\,\ell)(S + J\,\ell - 1)$$

For twist-J higher than 2, anomalous dimensions, as functions of S, occupy a *band*!
Our discussion was restricted to the ***minimal energy***, lower edge of the band.

Such (empirical) evidence *at strong coupling*?

Reciprocity and AdS/CFT?

Anomalous dimensions of operators in N=4 SYM ⟷ Energies of semiclassical strings in $AdS_5 \times S^5$

Bosonic $AdS_5 \times S^5$ σ-model $$I = \frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \, G_{\mu\nu}(X) \, \partial_a X^\mu \partial^a X^\nu + \dots$$

states belonging to reprs of $SO(2,4)$ x $SO(6)$ ➡ Classified by $(E, S_1, S_2; J_1, J_2, J_3)$.

- Classical energy $E = \sqrt{\lambda}\, \mathcal{E}(\omega)$ and classical spins $S = \sqrt{\lambda}\, \mathcal{S}(\omega)$, $J = \sqrt{\lambda}\, \mathcal{J}(\omega)$

- The energy organizes in the semiclassical expansion

$$E = \sqrt{\lambda} \left[E_0 + \frac{E_1}{\sqrt{\lambda}} + \frac{E_2}{(\sqrt{\lambda})^2} + \dots \right]$$

- Operators with large Lorentz spin and minimal energy correspond to *folded* strings rotating in AdS_3: the *classical* result for the energy reproduces the logarithmic behavior!

$$E = S + \frac{\sqrt{\lambda}}{\pi} \log \frac{S}{\sqrt{\lambda}} + \dots \qquad S \gg \sqrt{\lambda}$$

[Gubser, Klebanov, Polyakov 02]

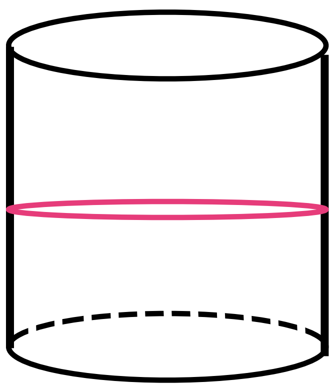
i.e. the behavior of the (classical + anomalous) dimension $(\,\gamma(S) = E - S\,)$ of twist operators with minimal energy. Confirmed *at one loop*!

[Frolov, Tseytlin 02]

Folded string in AdS_3: I

The S^5 momentum J of the string state can be ignored ⟷ twist of the gauge theory operator small compared to Lorentz spin.

Folded string: closed string folded on itself to form a segment line rotating in AdS



Large spin limit: the cusps approach the boundary of AdS, responsible for logs

- Ansatz for a stationary solution rotated and boosted

$$\rho = \rho(\sigma) \qquad t = \kappa\,\tau \qquad \phi = \omega\,\tau$$

where $\rho(\sigma)$ varies from *0* to ρ_0 defined via $\coth^2 \rho_0 = 1 + \eta$

- Exact solution

$$\sinh \rho = \frac{1}{\sqrt{\eta}}\, \mathrm{sn}\big[\kappa\sqrt{\eta}\,\sigma, -\frac{1}{\eta}\big]\ , \qquad 0 \leq \sigma \leq \frac{\pi}{2}$$

To construct the fully periodic solution: 4 such functions glue together (1-fold).

Folded string in AdS_3: II

- Integrals of motion: energy and spin

$$E = P_t = \sqrt{\lambda}\,\kappa \int_0^{2\pi} \frac{d\sigma}{2\pi}\, \cosh^2\rho \equiv \sqrt{\lambda}\,\mathcal{E} \qquad\qquad S \;=\; P_\phi = \sqrt{\lambda}\,\omega \int_0^{2\pi} \frac{d\sigma}{2\pi}\, \sinh^2\rho \equiv \sqrt{\lambda}\,\mathcal{S}$$

In parametric form

$$\mathcal{E} = \frac{2}{\pi\,\sqrt{\eta}}\,\mathbb{E}\left(-\frac{1}{\eta}\right) \qquad\qquad \mathcal{S} = \frac{2}{\pi\,\sqrt{\eta}}\left[\mathbb{E}\left(-\frac{1}{\eta}\right) - \mathbb{K}\left(-\frac{1}{\eta}\right)\right]$$

- In the "long string" limit $\eta \to 0$

✔ ***Same structure!*** $\mathcal{E} = \mathcal{S} + \dfrac{\ln\bar{\mathcal{S}} - 1}{\pi} + \dfrac{\ln\bar{\mathcal{S}} - 1}{2\,\pi^2\,\mathcal{S}} - \dfrac{2\ln^2\bar{\mathcal{S}} - 9\ln\bar{\mathcal{S}} + 5}{16\,\pi^3\mathcal{S}^2} + \ldots \qquad \bar{\mathcal{S}} \equiv 8\,\pi\,\mathcal{S}$

✔ ***Leading logs!*** $E - S = \dfrac{\sqrt{\lambda}}{\pi}\ln\left[S + \dfrac{1}{2}\dfrac{\sqrt{\lambda}}{\pi}\ln S + \ldots\right] + \ldots$

✔ ***Reciprocity!*** the function f runs in even negative powers of the Casimir $\mathcal{C} \equiv \mathcal{S}$

$$\tilde{\mathrm{f}}(\mathcal{S}) \equiv \frac{\mathrm{f}}{\sqrt{\lambda}} = \frac{1}{\pi}\left[\ln\bar{\mathcal{S}} - 1 + \frac{\ln\bar{\mathcal{S}} + 1}{16\pi^2\mathcal{S}^2} + \mathcal{O}\left(\frac{1}{\mathcal{S}^4}\right)\right] + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)$$

Folded string in $AdS_3 \times S^1$

- Integrals of motion: $E = \sqrt{\lambda}\,\mathcal{E}$ and two angular momenta $S = \sqrt{\lambda}\,\mathcal{S}$ and $J = \sqrt{\lambda}\,\mathcal{J}$
  We are interested in the limit $\mathcal{S} \gg \mathcal{J}$

- "Slow" long strings $\mathcal{J} \ll \ln \mathcal{S}$

✓ ***Same structure!***

$$\mathcal{E} - \mathcal{S} - \mathcal{J} \approx \frac{1}{\pi}(\ln \bar{\mathcal{S}} - 1) + \frac{\pi\,\mathcal{J}^2}{2\,\ln \bar{\mathcal{S}}} - \frac{\pi^3\,\mathcal{J}^4}{8\,\ln^3 \bar{\mathcal{S}}}\left(1 - \frac{1}{\ln \bar{\mathcal{S}}}\right) + \ldots$$
$$+ \frac{4}{\bar{\mathcal{S}}}\left[\frac{1}{\pi}(\ln \bar{\mathcal{S}} - 1) + \frac{\pi\,\mathcal{J}^2}{2\ln^2 \bar{\mathcal{S}}} - \frac{3\pi^3\,\mathcal{J}^4}{4\ln^4 \bar{\mathcal{S}}}\left(1 - \frac{2}{3\,\ln \bar{\mathcal{S}}}\right) + \ldots\right] + \ldots$$

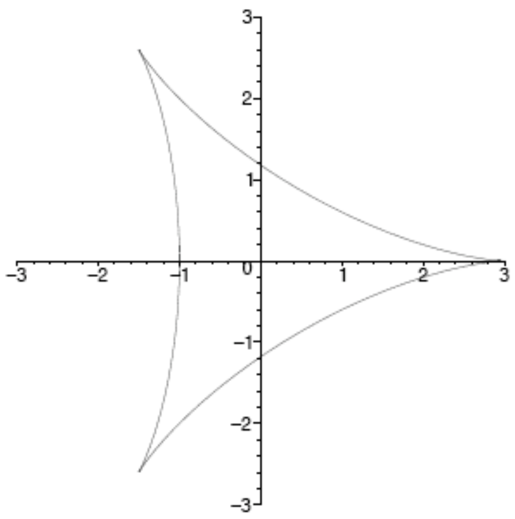
✓ ***Leading logs!***

✓ ***Reciprocity!*** f runs in even negative powers of the Casimir $\mathcal{C} \equiv \mathcal{S} + \frac{1}{2}\mathcal{J}$

- "Fast" long strings $\ln \mathcal{S} \ll \mathcal{J} \ll \mathcal{S}$: again reciprocity respecting .

[Basso, Korchemsky 06]

Spiky strings
vs. higher twist on excited trajectories



Rigidly rotating , n cusps or spikes.

Same logarithmic asymptotics of anomalous dimensions BUT proportional to the number of spikes, $n > 2 \Rightarrow$ higher energy for given spin!

$$\mathcal{E} - \mathcal{S} = \frac{n}{2\,\pi} \ln \frac{16\,\pi\,\mathcal{S}}{n} + \ldots$$

[Belitsky, Gorsky, Korchemsky, 03]

[Kruczenski. 04]

Beyond the leading large spin limit: the ends of the spikes do not approach the bndry

✓ *Leading logs: YES!* $$\mathcal{E} - \mathcal{S} = \frac{n}{2\pi} \ln \mathcal{S} + \frac{n^2}{8\,\pi^2 \mathcal{S}} \ln \mathcal{S} - \frac{n^3}{64\,\pi^3 \mathcal{S}^2} \ln^2 \mathcal{S} + \ldots$$

✓ ~~*Reciprocity*~~*: NO!* $$\tilde{\mathrm{f}}(\mathcal{S}) \quad = \quad \frac{n}{2\pi}\Big[\ln \bar{\mathcal{S}} + q_1 + \frac{q_2}{\bar{\mathcal{S}}} + \ldots\Big]$$

$\Rightarrow$ Proper correspondent to twist *J*=*n* operators with *NON MINIMAL* anomalous dim!

✓ Exactly as it happens at weak coupling! [Belitsky, Korchemsky, Pasechnik 08]

String perturbation theory: I

- With the 1-loop corrections included:
  the structure of the large spin expansion remains the same?
  are the "MVV" constraints still satisfied? procedure for leading (1 loop)

Gauge and string perturbative expansions are different!

Gauge theory	$\lambda \ll 1$	$S =$	fixed and then	$S \gg 1$
String theory	$\sqrt{\lambda} \gg 1$	$\frac{S}{\sqrt{\lambda}} =$	fixed and then	$\frac{S}{\sqrt{\lambda}} \gg 1$

Quantization of folded string: doable via semiclassical considerations and in the **special** limits of short and ***long*** string ! [Frolov, Tseytlin 02]

- ***Standard procedure*** for leading (1 loop) quantum corrections:

Fluctuation lagrangean $\tilde{L}$ ⟶ 2-d Effective action Γ_1

Euclidean ⟶ $$E_1 = \frac{\Gamma_1}{\kappa \mathcal{T}}\,, \qquad \mathcal{T} \equiv \int d\tau \to \infty$$ ⟶ **Tseytlin**

Conclusions...

CFT/

- Long range Bethe equations → multiloop anomalous dimensions.
- Closed formulas *QCD-inspired*
- Reciprocity in the full PSU(2,2 | 4). Why?

/AdS

- Reciprocity holds (and perturbatively! see Tseytlin)
- Reciprocity in AdS. Why?

QCD

- Complete solution of N =4 SYM (and string theory) should provide a "one-line-all-orders description" of the *major* part of QCD parton dynamics.

[Dokshitzer, DIS 2007]

The N=4 SYM twist two universal anomalous dimension at 4 loops..

The anomalous dimension

$$\gamma_4(S) = \gamma_4^{\text{Bethe}}(S) + \gamma_4^{\text{wrapping}}(S)$$

[Kotikov, Lipatov, Rej, Staudacher, Velizhanin, 07]

$$\begin{aligned}
&\boldsymbol{4\,S_{-7} + 6\,S_7} + 2\,(S_{-3,1,3} + S_{-3,2,2} + S_{-3,3,1} + S_{-2,4,1}) + 3\,(-S_{-2,5}\\
&+ S_{-2,3,-2}) + 4\,(S_{-2,1,4} - S_{-2,-2,-2,1} - S_{-2,1,2,-2} - S_{-2,2,1,-2} - S_{1,-2,1,3}\\
&- S_{1,-2,2,2} - S_{1,-2,3,1}) + 5\,(-S_{-3,4} + S_{-2,-2,-3}) + 6\,(-S_{5,-2}\\
&+ S_{1,-2,4} - S_{-2,-2,1,-2} - S_{1,-2,-2,-2}) + 7\,(-S_{-2,-5} + S_{-3,-2,-2}\\
&+ S_{-2,-3,-2} + S_{-2,-2,3}) + 8\,(S_{-4,1,2} + S_{-4,2,1} - S_{-5,-2} - S_{-4,3}\\
&- S_{-2,1,-2,-2} + S_{1,-2,1,1,-2}) + 9\,S_{3,-2,-2} - 10\,S_{1,-2,2,-2} + 11\,S_{-3,2,-2}\\
&+ 12\,(-S_{-6,1} + S_{-2,2,-3} + S_{1,4,-2} + S_{4,-2,1} + S_{4,1,-2} - S_{-3,1,1,-2} - S_{-2,2,-2,1}\\
&- S_{1,1,2,3} - S_{1,1,3,-2} - S_{1,1,3,2} - S_{1,2,1,3} - S_{1,2,2,-2} - S_{1,2,2,2} - S_{1,2,3,1} - S_{1,3,1,-2}\\
&- S_{1,3,1,2} - S_{1,3,2,1} - S_{2,-2,1,2} - S_{2,-2,2,1} - S_{2,1,1,3} - S_{2,1,2,-2} - S_{2,1,2,2}\\
&- S_{2,1,3,1} - S_{2,2,1,-2} - S_{2,2,1,2} - S_{2,2,2,1} - S_{2,3,1,1} - S_{3,1,1,-2} - S_{3,1,1,2} - S_{3,1,2,1}\\
&- S_{3,2,1,1}) + 13\,S_{2,-2,3} - 14\,S_{2,-2,1,-2} + 15\,(S_{2,3,-2} + S_{3,2,-2})\\
&+ 16\,(S_{-4,1,-2} + S_{-2,1,-4} - S_{-2,-2,1,2} - S_{-2,-2,2,1} - S_{-2,1,-2,2} - S_{-2,1,1,-3}\\
&- S_{1,-3,1,2} - S_{1,-3,2,1} - S_{1,-2,-2,2} - S_{2,-2,-2,1} + S_{-2,1,1,-2,1} + S_{1,1,-2,1,-2}\\
&+ S_{1,1,-2,1,2} + S_{1,1,-2,2,1}) - 17\,S_{-5,2} + 18\,(-S_{4,-3} - S_{6,1} + S_{1,-3,3})\\
&+ 20\,(-S_{1,-6} - S_{1,6} - S_{4,3} + S_{-5,1,1} + S_{-4,-2,1} + S_{-3,-2,2} + S_{-2,-4,1}\\
&+ S_{-2,-3,2} + S_{1,3,3} + S_{3,1,3} + S_{3,3,1} - S_{1,1,-2,3} - S_{1,2,-2,-2} - S_{2,1,-2,-2})\\
&- 21\,S_{3,4} + 22\,(S_{1,-2,-4} + S_{2,2,3} + S_{2,3,2} + S_{3,-2,2} + S_{3,2,2}) + 23\,(-S_{-3,-4}\\
&- S_{5,2} + S_{2,-2,-3}) + 24\,(-S_{-4,-3} + S_{1,-4,-2} - S_{1,-3,1,-2} - S_{1,1,1,4} - S_{1,1,4,1}\\
&- S_{1,3,-2,1} - S_{1,4,1,1} - S_{3,-2,1,1} - S_{3,1,-2,1} - S_{4,1,1,1} + S_{-2,-2,1,1,1} + S_{-2,1,-2,1,1}\\
&+ S_{1,-2,-2,1,1} + S_{1,-2,1,-2,1} + S_{1,1,-2,-2,1} + S_{1,1,1,-2,-2} + S_{1,1,2,-2,1} + S_{1,2,1,-2,1}\\
&+ S_{2,1,1,-2,1}) + 25\,S_{2,-3,-2} + 26\,(-S_{2,5} + S_{1,4,2} + S_{2,4,1} + S_{4,1,2} + S_{4,2,1})\\
&+ 28\,(S_{1,2,4} + S_{2,1,4} - S_{-3,1,-2,1} - S_{-2,1,-3,1} - S_{1,-2,1,-3}) + 30\,S_{-3,1,-3}\\
&+ 32\,(S_{1,5,1} + S_{5,1,1} - S_{-3,-2,1,1} - S_{-2,-3,1,1} - S_{1,-3,-2,1} - S_{1,-2,-3,1}\\
&- S_{2,2,-2,1} + S_{1,2,-2,1,1} + S_{2,1,-2,1,1} - S_{1,1,1,-2,1,1}) + 36\,(S_{1,1,5} + S_{1,3,-3}\\
&+ S_{3,1,-3} - S_{1,1,-3,-2} - S_{1,1,-2,-3} - S_{1,1,2,-3} - S_{1,2,-2,2} - S_{1,2,1,-3} - S_{2,1,-2,2}\\
&- S_{2,1,1,-3}) + 38\,S_{-3,-3,1} + 40\,(-S_{1,-4,1,1} - S_{2,-3,1,1} + S_{1,1,1,-2,2})\\
&- 41\,S_{3,-4} + 42\,(-S_{2,-5} + S_{1,-4,2} + S_{1,-3,-3}) + 44\,(S_{1,-5,1} + S_{2,-3,2} + S_{3,-3,1})\\
&+ 46\,S_{2,2,-3} + 48\,S_{1,1,-3,1,1} + 60\,(S_{1,1,-5} - S_{1,1,-3,2}) + 62\,S_{2,-4,1} + 64\,S_{1,1,1,-3,1}\\
&+ 68\,(S_{1,2,-4} + S_{2,1,-4} - S_{1,2,-3,1} - S_{2,1,-3,1}) - 72\,S_{1,1,1,-4} - 80\,S_{1,1,-4,1}\\
&\boldsymbol{-\zeta(3)\,S_1(S_3 - S_{-3} + 2\,S_{-2,1})}.
\end{aligned}$$

Luscher-type correction

[Bajnok, Janik, Lukowsky 08]

$$\begin{aligned}
\gamma_4^{\text{wrapping}} &= 256\,S_1^2\,(-S_5 + S_{-5} + 2S_{4,1} - 2S_{3,-2} + 2S_{-2,-3} - 4S_{-2,-2,1}) \\
&\quad -512\,\zeta_3\,S_1^2\,S_{-2} - 640\,\zeta_5\,S_1^2
\end{aligned}$$

- At large S $\quad \gamma_4^{\text{wrapping}} \sim \dfrac{\ln^2 S}{S^2}$

  which *should be* in terms of the cusp anomaly, if..

- The f function ***is not simpler*** at four loops

... and Reciprocity

- The ***wrapping contribution*** is reciprocity respecting!

- The ***asymptotic*** anomalous dimension at 4 loops is RR! **[Forini, Beccaria 08]**

- The asymptotic and the wrapping part are ***separately*** RR!

  $\longrightarrow$ Reciprocity seems a property ***built in*** the Bethe equations.

Outlook 1, N=4 SYM

Some numerology

How much <u>MTP + Reciprocity</u> constrain on their own the anomalous dimension?

2 loops: of 6 unknown coeffs fix 3 (+1 scaling function)
3 loops: of 40 unknown coeffs fix 24 (+ 3 from large S)
4 loops: $\gamma_4 = T_7 + \zeta_3 T_4 + \zeta_5 T_2$, of 258 unknown coefficients Reciprocity + large M + Konishi values fix about 120 (up to $1/S^{50}$)

- **MTP + Reciprocity** + **Bethe equations** + **Luescher** to write down the anomalous dimension of ***sl(2) twist-3 operators at 5 loops***?

Outlook 2, Strings

Further orders in $1/S$, generalization to (S,J)

Outlook 3: Universalities at the level of the amplitudes in N=4 SYM

The first two coefficients are connected with singularities in the amplitudes.

What about subleading?

[Dixon, Magnea, Sterman 08]