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ADs in AdS

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Outline

- Classical symmetries of gauge theories
- Quantum symmetries
- Rank-one sector of $\mathcal{N}$=4 dilatation operator: one loop and all loops
- Strong/weak interplay
- Full $\mathcal{N}$=4 SYM
- Conclusions

Classical symmetries of gauge theories

QCD = (3+1)D Yang-Mills theory with matter in fundamental representation of SU(3)

$$L_{\mathrm{QCD}} = -\tfrac{1}{2}\,\mathrm{tr} F_{\mu\nu}^2 + \overline{\psi}\,(i\not{D} - m_q)\,\psi$$

- Symmetries of the classical theory:
  - gauge symmetry
  - chiral symmetry (for m_q=0)
  - SO(4,2) symmetry (for m_q=0): 4D rotations, translations, dilatations, special conformal boosts
- Supersymmetric theory: fermions in adjoint reps, scalars, adjust couplings:

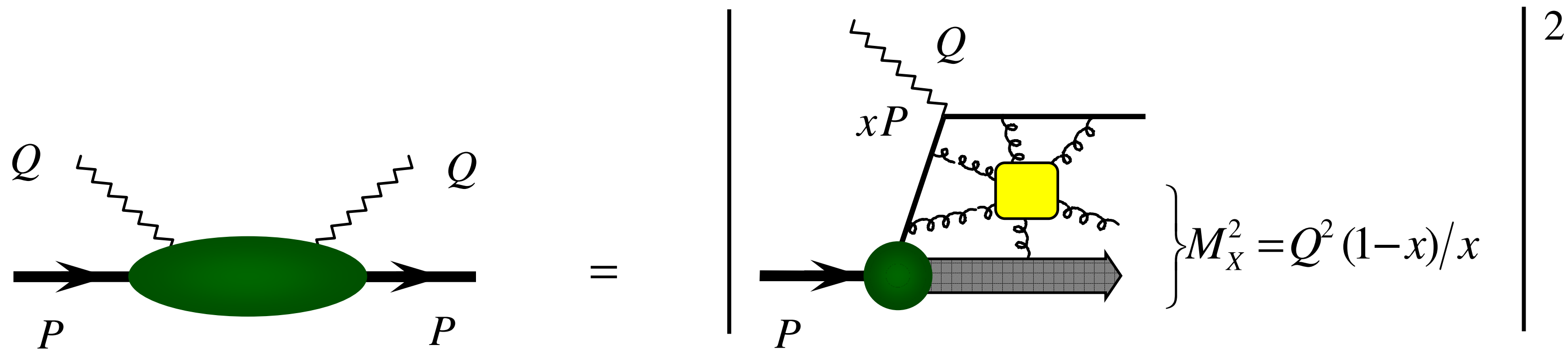
QCD → adj. quarks → $\mathcal{N}$=1SYM → + 2 scalars → $\mathcal{N}$=2 SYM → + 4 scalars → $\mathcal{N}$=4 SYM

- Many of classical symmetries are broken on quantum level due to anomalies:
  - chiral anomaly: $$\partial_\mu J_5^\mu = \frac{g^2}{8\pi^2}\,\mathrm{tr} F_{\mu\nu}\tilde{F}^{\mu\nu}$$
  - conformal (trace) anomaly: $$\Theta_\mu{}^\mu = \frac{\beta(g)}{g}\,\mathrm{tr}\; F_{\mu\nu}F^{\mu\nu}$$

Q: Are there any hidden symmetries which would help to solve the theory?

Wilson operators and anomalous dimensions

Classic example: deeply inelastic scattering of hard probes off hadrons



- Operator product expansion:

$$\int_0^1 dx\, x^N F(x,Q^2) = \sum_{L\geq 2} \frac{c_{N,L}(g)}{Q^L} \langle p | O_{N,L}(0) | p \rangle_{\mu^2=Q^2}$$

Wilson operators of high spin $N >> 1 \Leftrightarrow x \to 1$ asymptotics of structure functions

- For $x \to 1$, the final state has a small invariant mass and is dominated by soft gluons: quark fields $\to$ Wilson lines

- The anomalous dimensions at large spin N scale at most logarithmically

$$\langle p | O_{N,L}(0) | p \rangle_{\mu^2} \sim \exp\left(-\gamma_{N,L} \ln \mu^2\right), \qquad \gamma_{N,L} \sim \Gamma_{\text{cusp}} \ln N$$

- Sudakov scaling is a universal feature of all gauge theories, from QCD to $\mathcal{N}$=4 SYM

Callan-Symanzik equation

Wilson operators

$$O_{N_1 N_2 \ldots N_L} = \mathrm{tr}\left\{\partial_+^{N_1} X(0)\,\partial_+^{N_2} X(0)\ldots\partial_+^{N_L} X(0)\right\}$$

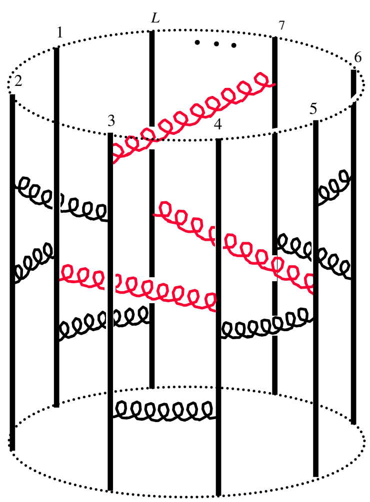
mix under the action of the dilatation operator

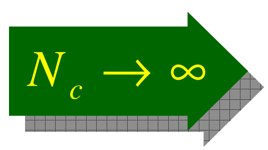
$$\left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g}\right) O_{N_1 N_2 \ldots N_L} = H \cdot O_{N_1 N_2 \ldots N_L} = \sum_{K_j} V(N_i \mid K_j)\, O_{K_1 K_2 \ldots K_L}$$

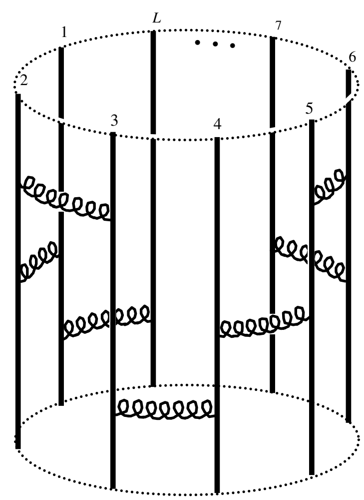
The eigenvalues of V (H) determine the anomalous dimensions of multiplicatively renormalizable Wilson operators:

$$H \cdot \Psi = \gamma(g)\Psi$$

$$g = g_{\mathrm{YM}}\sqrt{N_c}\big/(2\pi)$$







Only planar interactions survive!

Perturbative structure of dilop

The dilop admits perturbative expansion in ’t Hooft coupling constant:

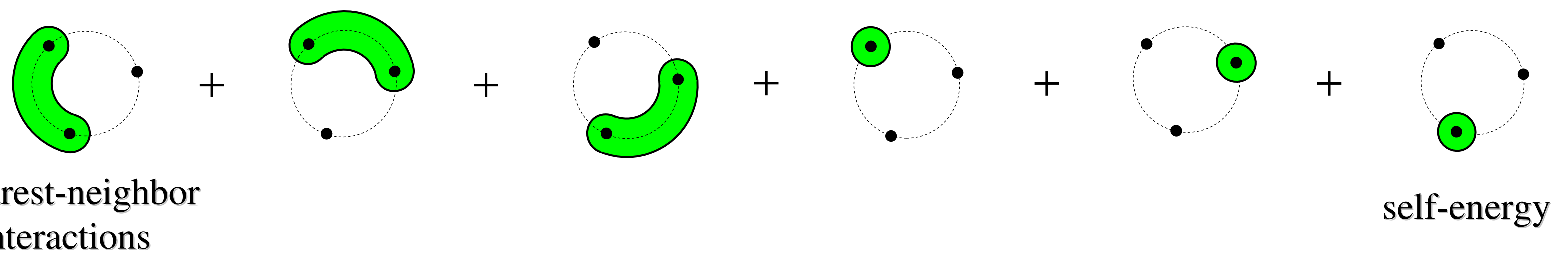
$$H_L = g^2 H_L^{(0)} + g^4 H_L^{(1)} + \dots$$

The range of interaction increases with order the coupling (e.g., *L*=3):

Bukhvostov, Frolov, Kuraev, Lipatov ’85

- $O(g^2)$:

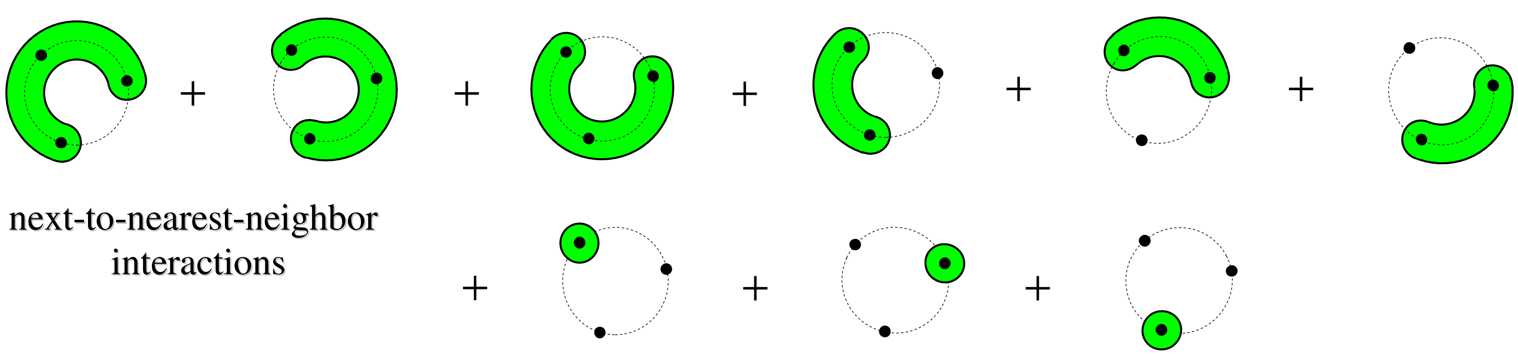
$H_{L=3}^{(0)} =$



$$H_L^{(0)} = H_{12}^{(0)} + H_{23}^{(0)} + H_{34}^{(0)} + \dots + H_{L1}^{(0)}$$

- $O(g^4)$:

$H_{L=3}^{(1)} =$



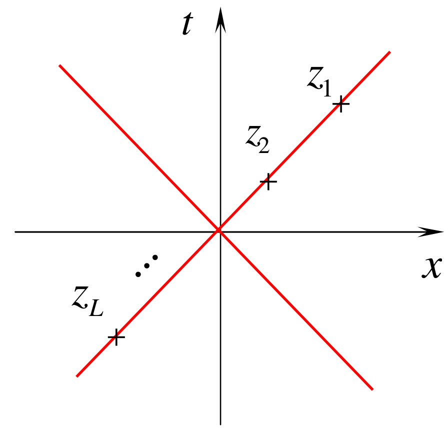
- ...

Generating functions

Transition from local Wilson operators to generating functions:

$$O_{N_1 N_2 \dots} = \mathrm{tr}\,\left(i\vec{\partial}_+\right)^{N_1} X(0)\left(i\vec{\partial}_+\right)^{N_2} X(0)\dots\left(i\vec{\partial}_+\right)^{N_L} X(0)$$

$$O(z_1, z_2, \dots) = \mathrm{tr}\, X(z_1) X(z_2) \dots X(z_L)$$



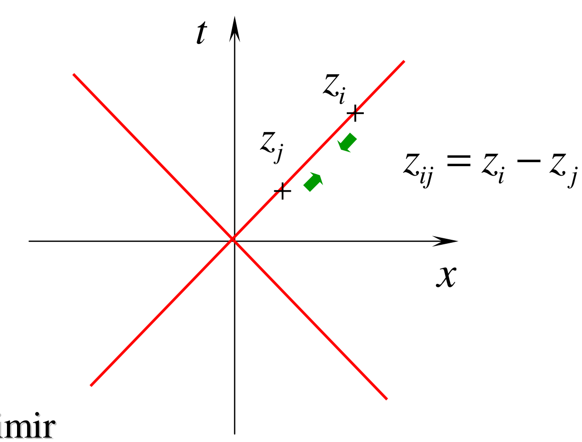
The RG equation:

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}\right) O(z_1, z_2, \dots, z_L) = H \cdot O(z_1, z_2, \dots, z_L)$$

Geyer, Robaschik ’80
Balitsky, Braun ’87

The pairwise dilop generates the shift of light-cone positions of the fields in the direction of each other (e.g., for the aligned-helicity spin-*s* sl(2) sector of any gauge theory):

$$H_{ij}^{(0)} O(\dots, z_i, z_j, \dots) = \int_0^1 \frac{d\alpha}{\alpha}\Big\{2O(\dots, z_i, z_j, \dots) - (1-\alpha)^{2s-1} O(\dots, z_i - \alpha z_{ij}, z_j, \dots) - (1-\alpha)^{2s-1} O(\dots, z_i, z_j + \alpha z_{ij}, \dots)\Big\}$$



Diagonalization:

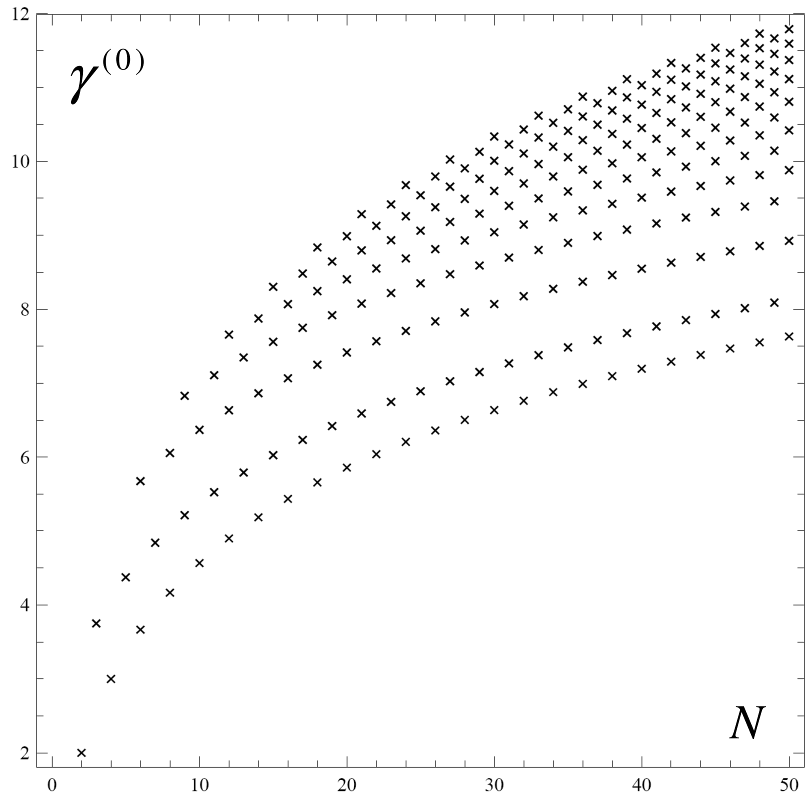
$$H_{ij}^{(0)} (z_i - z_j)^N = 2[\psi(N + 2s) - \psi(2s)](z_i - z_j)^N$$

eigenvalue of sl(2) conformal Casimir

Degeneracies and integrability

Braun, Derkachov, Manashov, Korchemsky, AB '98

Diagonalization of the mixing matrix for L=3 yields the spectrum:



- Anomalous dimensions occupy a band
- Anomalous dimensions seem to lie on trajectories
- Each eigenvalue is double-degenerate (except for the lowest trajectory)

integrability

There exists a new quantum number q:

$$\gamma^{(0)}(N,-q) = \gamma^{(0)}(N,q)$$

Implies the existence of a conserved charge.

- Nontrivial charge: $\mathbf{q}_3 = \varepsilon^{ijk} S_1^i S_2^j S_3^k$
- Conformal Casimir: $\mathbf{q}_2 = (S_1^i + S_2^i + S_3^i)^2$

$$[H^{(0)}, \mathbf{q}_2] = [H^{(0)}, \mathbf{q}_3] = 0$$

complete set of charges!

Non-compact SL(2,R) magnet

The spin operators admit representation in terms of differential operators of variable *z*:

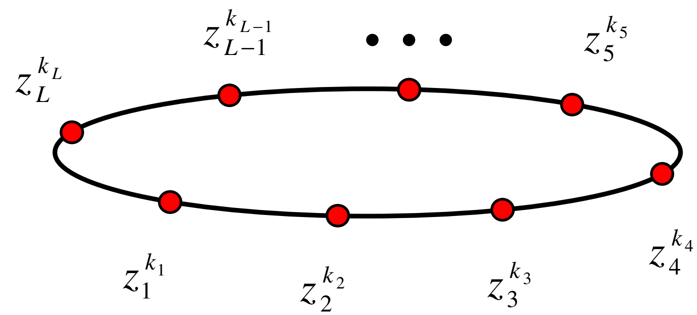
$$S^{+} = z^{2}\partial_{z} + 2sz, \quad S^{-} = -\partial_{z}, \quad S^{0} = z\partial_{z} + s$$

(Recall that for the compact magnet *s*=-1/2.)

On each site, the spin can take an infinite number of values:

$$|\text{state}\rangle \rightarrow \{1, z, z^{2}, z^{3}, \ldots, z^{\infty}\}$$

$$z^{k} \quad \longleftrightarrow \quad \vec{\partial}_{+}^{k} X(0)$$



There is the lowest weight, but not the highest weight.

The pair-wise Casimir operator:

$$(\mathbf{S}_{k} + \mathbf{S}_{k+1})^{2} = J_{k,k+1}(J_{k,k+1} - 1)$$

The integrable pair-wise Hamiltonian:

$$h_{L} = \sum_{k=1}^{L} \{\psi(J_{k,k+1}) - \psi(2s)\}$$

Kulish, Reshetikhin, Sklyanin '81
Tarasov, Takhtajan, Faddeev '83

One-loop dilop is the Hamiltonian of the noncompact XXX magnet!

One-loop Baxter equation

There exists a family of commutative operators:

Baxter '82

$$[\hat{\mathbf{Q}}_{(0)}(u), \hat{\mathbf{Q}}_{(0)}(v)] = [\hat{\mathbf{Q}}_{(0)}(u), \mathbf{q}_k] = 0$$

The sl(2) Q-operator satisfies the Baxter equation (for L-particles):

$$(u+is)^L \hat{\mathbf{Q}}_{(0)}(u+i) + (u-is)^L \hat{\mathbf{Q}}_{(0)}(u-i) = t(u)\hat{\mathbf{Q}}_{(0)}(u)$$

The (auxiliary) transfer matrix is a polynomial in u:

$$t(u) = 2u^L + \mathbf{q}_2 u^{L-2} + \ldots + \mathbf{q}_{\mathrm{L}}$$

The Hamiltonian of the magnet:

$$H^{(0)} = i\hat{\mathbf{Q}}'_{(0)}(is)\big/\hat{\mathbf{Q}}_{(0)}(is) - i\hat{\mathbf{Q}}'_{(0)}(-is)\big/\hat{\mathbf{Q}}_{(0)}(-is)$$

The eigenvalues of the Baxter operator:

$$Q_{(0)}(u) = \prod_{k=1}^{N}(u - u_{(0)k})$$

Bethe roots
(obey Bethe Ansatz equations)

Fine structure of spectrum (L=3)

Asymptotic solution to Baxter equation

Systematic expansion in the inverse conformal spin

$$Q_{(0)}(u/\eta) \sim \exp\left(\frac{1}{\eta} S(u)\right)$$

$$\eta = (N + sL)^{-1} << 1$$

Eigenvalues lie on trajectories:

$$\gamma_{(0)}(N,n) = \ln \frac{q^{(0)} N^3}{\sqrt{3}} + \frac{1}{N}\frac{q^{(1)}}{q^{(0)}} + \frac{1}{N^2}\left(\frac{q^{(2)}}{q^{(0)}} - \frac{4\left(q^{(1)}\right)^2 - 11}{8\left(q^{(0)}\right)^2}\right) - 3\psi(1) + O(N^{-3})$$

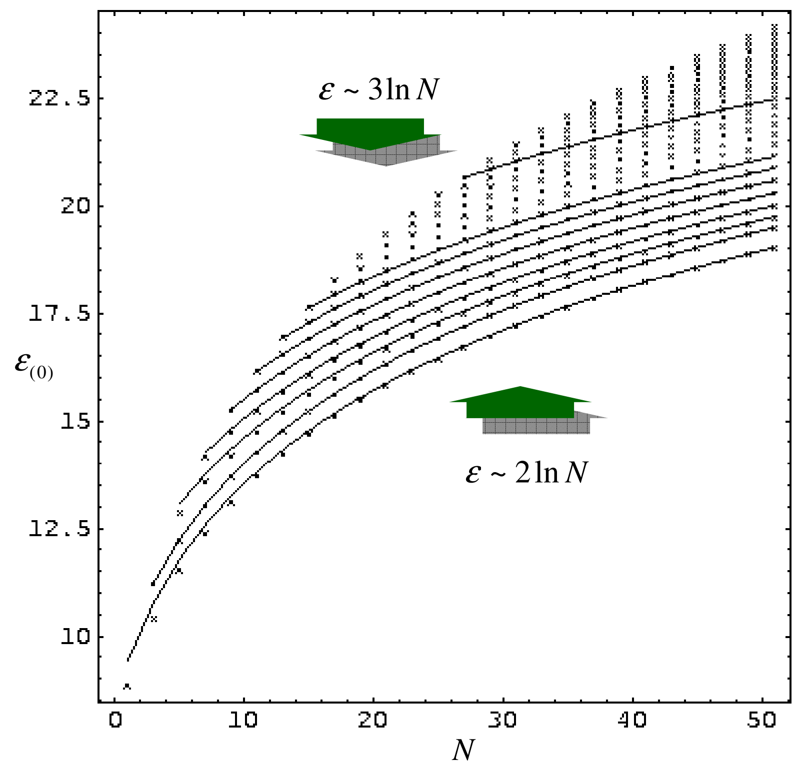
The nontrivial charge is quantized:

$$q = \frac{N^3}{\sqrt{3}}\left(q^{(0)} + q^{(1)}/N + \ldots\right)$$

$$q^{(0)} = \frac{1}{3},$$

$$q^{(1)} = \frac{7}{2} - n,$$

$$q^{(3)} = \frac{85}{9} - \frac{22}{3}n - \frac{2}{3}n^2$$



Braun, Derkachov, Korchemsky, Manashov, AB'98

From the light cone to superspace

Brink, Lindgren, Nilsson'83
Mandelstam '83

Complex scalar $\mathcal{N}$=4 chiral light-cone superfield:

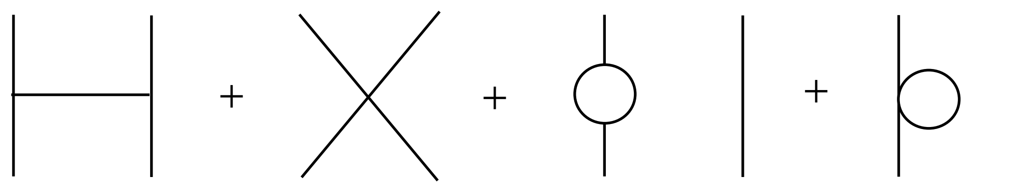
$$\Phi = e^{\frac{1}{2}\bar{\theta}\cdot\theta\partial_+}\left\{\partial_+^{-1}A + \theta^A\partial_+^{-1}\bar{\lambda}_A + \tfrac{i}{2!}\theta^A\theta^B\bar{\phi}_{AB} - \tfrac{1}{3!}\varepsilon_{ABCD}\theta^A\theta^B\theta^C\lambda^D - \tfrac{1}{3!}\varepsilon_{ABCD}\theta^A\theta^B\theta^C\theta^D\partial_+\bar{A}\right\}$$

All single-trace operators of $\mathcal{N}$=4 SYM:

$$O(Z_1, Z_2, \ldots, Z_L) = \mathrm{tr}\left[\Phi(Z_1)\Phi(Z_2)\ldots\Phi(Z_L)\right]$$

One-loop dilatation operator:

$$H_{ij}^{(0)} =$$

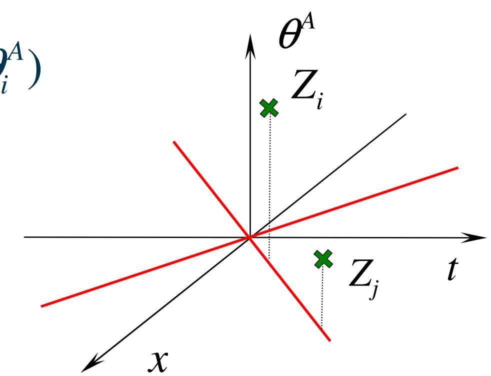


Light-cone kernel of the dilatation operator acting in superspace $Z_i = (z_i, \theta_i^A)$

$$H_{ij}^{(0)}O(\ldots,Z_i,Z_j,\ldots) = \int_0^1 \frac{d\alpha}{\alpha}\Big\{2\,O(\ldots,Z_i,Z_j,\ldots) - (1-\alpha)^{-2}O(\ldots,Z_i-\alpha Z_{ij},Z_j,\ldots)$$
$$-(1-\alpha)^{-2}O(\ldots,Z_i,Z_j+\alpha Z_{ij},\ldots)\Big\}$$

$2s_\Phi - 1$ with $s_\Phi = -\frac{1}{2}$

disaplacement
in superspace



AB, Derkachov, Korchemsky, Manashov '04

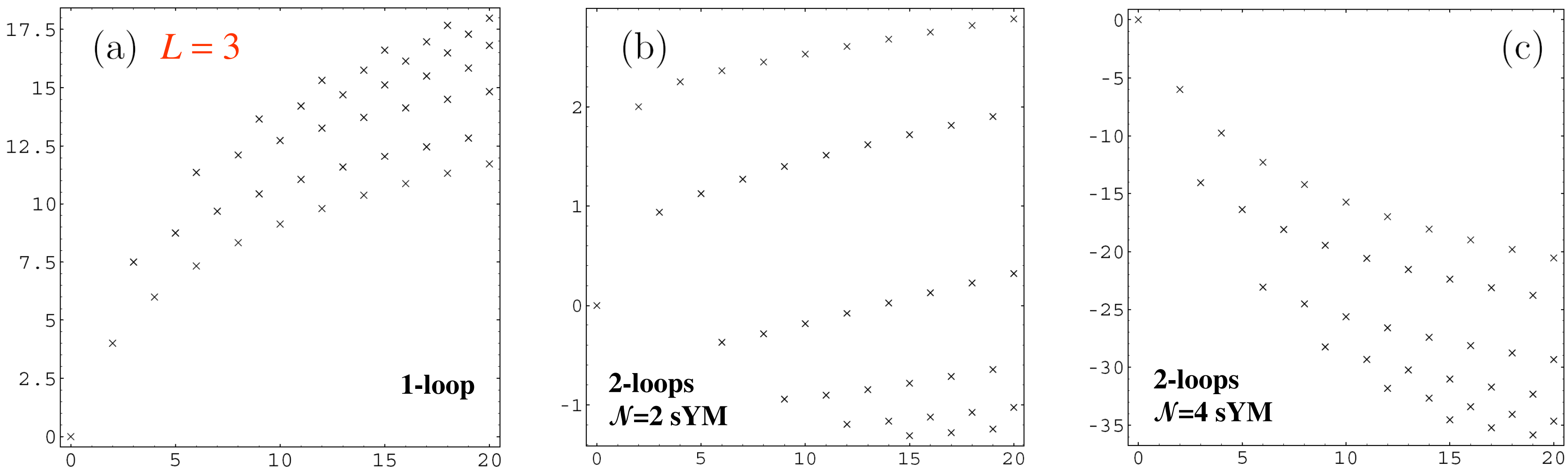
One-loop dilop in $\mathcal{N}$=4 SYM is psu(2,2|4) graded magnet!

Lipatov '97
Minahan, Zarembo '03
Beisert, Staudacher '04

Two-loop scalar operators in $\mathcal{N}$=2,4 SYM

$$O(z_1, z_2, \ldots, z_L) = \mathrm{tr}\{X(z_1)X(z_2)\ldots X(z_L)\}$$

$$X = \begin{cases} \phi, & \mathcal{N}=2 \\ \phi_1 + i\phi_2, & \mathcal{N}=4 \end{cases}$$



- $\mathcal{N}$=2 SYM

$$\gamma^{\pm}_{N=2}(g^2) = g^2\, 2 + g^4\, \tfrac{1}{2}$$

$$\gamma^{\pm}_{N=3}(g^2) = g^2\, \tfrac{15}{4} + g^4\, \tfrac{15}{64}$$

- $\mathcal{N}$=4 SYM

$$\gamma^{\pm}_{N=2}(g^2) = g^2\, 2 - g^4\, \tfrac{3}{2}$$

$$\gamma^{\pm}_{N=3}(g^2) = g^2\, \tfrac{15}{4} - g^4\, \tfrac{225}{64}$$

The spectra are degenerate in both $\mathcal{N}$=2 and $\mathcal{N}$=4!

Relation between anomalous dimensions:

$$\gamma_{\mathcal{N}=2}(g^2) = (1+g^2)\gamma_{\mathcal{N}=4}(g^2)$$

AB, Korchemsky, Mueller'06

Multiloop sl(2) Baxter equation

Beyond one-loop, the Bethe roots admit perturbative corrections:

$$u_k(g) = u_k^{(0)} + g^2\, u_k^{(1)} + g^4\, u_k^{(2)} + \dots$$

The Baxter function remains a polynomial in the spectral parameter *u*:

$$Q(u) = \prod_{k=1}^{N} (u - u_k(g)) = Q_{(0)}(u) + g^2\, Q_{(1)}(u) + g^4 Q_{(2)}(u) + \dots$$

The Baxter function obeys a Baxter-like equation:

$$d_+(u) Q(u+i) + d_-(u) Q(u-i) = t\;(u) Q(u)$$

- The dressing factors are functions of the renormalized spectral parameter (string-inspired):

$$x[u] = \tfrac{1}{2} u + \tfrac{1}{2}\sqrt{u^2 - g^2}, \qquad x_\pm = x[u_\pm], \qquad u_\pm = u \pm \tfrac{i}{2}$$

Beisert, Dippel, Staudacher ’04

- ... depend on Q at previous order (!)

$$d_+(u) = x_+^L e^{\sigma_+[x_+]}, \qquad \sigma_+[x] = -\frac{g^2}{2x}\left[\ln Q\left(\tfrac{i}{2}\right)\right]' + O(g^4)$$

AB, Korchemsky, Mueller’06

is a consequence of renormalization of the conformal spin of operators beyond one loop:

$$J_{(0)} = N + \tfrac{1}{2} L \quad \Rightarrow \quad J = N + \tfrac{1}{2} L + \tfrac{1}{2}\gamma(g^2)$$

AB, Mueller ’98

All-loop Baxter equation

Asymptotic all-order Baxter equation: AB '06

$$x_+^L \mathbf{e}^{\Delta_+[x_+]} Q(u+i) + x_-^L \mathbf{e}^{\Delta_-[x_-]} Q(u-i) = t\ (u) Q(u)$$

The all-order dressing factor

$$\Delta_\pm[x] = \tfrac{1}{\pi}\int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}} \ln Q(\pm\tfrac{i}{2} - gt)\left(1 - \tfrac{\sqrt{u^2-g^2}}{u-gt}\right)$$
$$- g\int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}} \ln \frac{Q(-\frac{i}{2}-gt)}{Q(+\frac{i}{2}-gt)} \int_{-1}^{1} \frac{ds}{s-t} \int_{[i,i\infty]} \frac{d\kappa}{2\pi i} \frac{1}{\sinh^2 \pi\kappa} \ln\left(1 + \tfrac{g^2}{4x\cdot x[\kappa+gs]}\right)\left(1 - \tfrac{g^2}{4x\cdot x[\kappa-gs]}\right)$$

The second lines provides smooth strong-weak interpolation.

Beisert, Eden, Staudacher '06
Beisert, Hernandez, Lopez '06

Anomalous dimension:

$$\gamma(g) = ig^2 \int_{-1}^{1} \frac{dt}{\pi} \sqrt{1-t^2} \left(\ln \frac{Q(+\frac{i}{2} - gt)}{Q(-\frac{i}{2} - gt)} \right)'$$

Asymptotic Bethe Ansatz equations:

$$t(u_k) = \text{pole - free} \quad \Rightarrow \quad \left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{j=1,\, j\neq k}^{N} \frac{x_k^- - x_j^+}{x_k^+ - x_j^-} \frac{1 - \frac{g^2}{4x_k^+ x_j^-}}{1 - \frac{g^2}{4x_k^- x_j^+}} \mathbf{e}^{i\theta(x_k, x_j)}$$

Beisert, Staudacher '06

WKB of Baxter equation

$$(u+\tfrac{i}{2})^L Q_{(0)}(u+i)+(u-\tfrac{i}{2})^L Q_{(0)}(u-i)=t(u)Q_{(0)}(u)$$

Quasiclassical limit

$\eta=(N+jL)^{-1}<<1$ "Planck" constant

$Q(u/\eta)\sim\exp\left(\frac{1}{\eta}S(u)\right)$ Hamilton-Jacobi function

Flaschka, McLaughlin' 76
Pasquier, Gaudin' 92
Korchemsky'95

$$\beta=\frac{L}{2N+L}$$

The Baxter equation reduces to a complex curve:

$$\Gamma_L:\quad y^2=4-t^2(u),\quad y=2\cos p(u),\quad p(u)=\left(S'(u)+\frac{\beta}{u}\right)$$

The problem of determining the spectrum is reduced to finding the quasimomentum $p(u)$. It is a double-valued function on the complex plane with square-root branch points at $|t(u)|=2$.

$$\gamma^{(0)}\sim p\left(+i\tfrac{\beta}{L}\right)-p\left(-i\tfrac{\beta}{L}\right)$$

Spectral curve

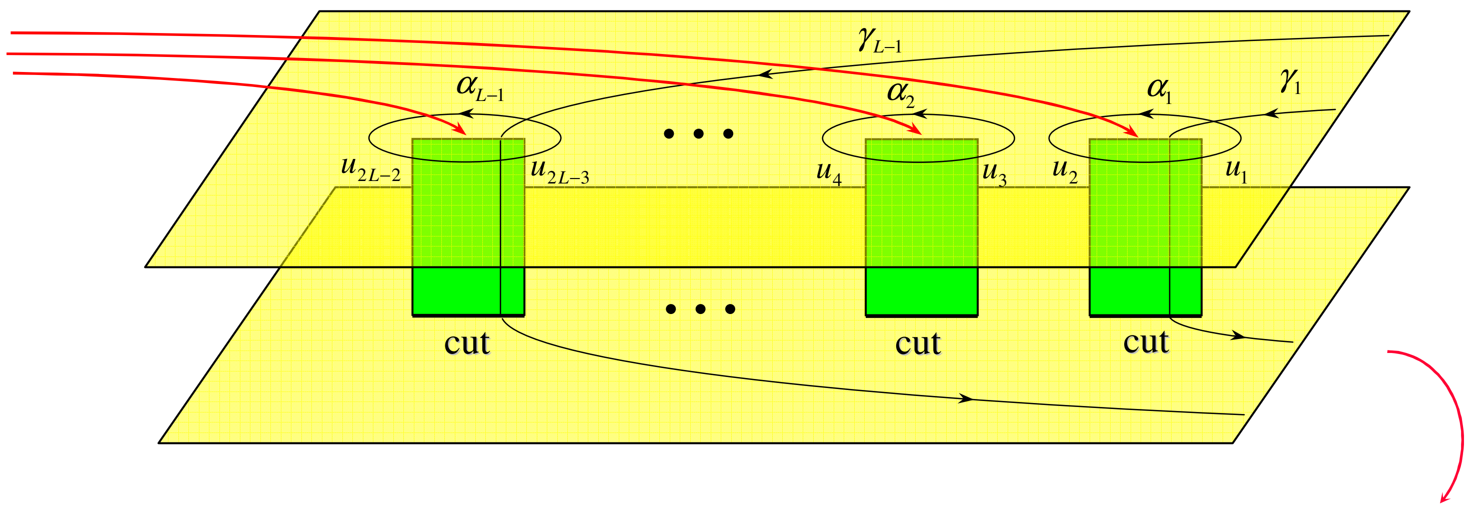
Novikov, Manakov, Pitaevsky, Zakharov '84
Kazakov, Minahan, Marshakov, Zarembo '03

The quasimomentum $p(x)$ becomes a single valued function on the hyperelliptic Riemann surface of genus L-2.

accumulation of Bethe roots on cuts:

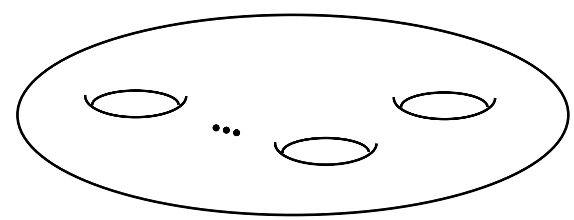
$$t^2(u) \geq 4$$

(zones of allowed classical motion in separated variables)



The curve is endowed with a meromorphic differential

$$dp = -\frac{t'(x)\,dx}{\sqrt{4 - t^2(x)}}$$



and a complete set of cycles α_j and γ_j:

$$\oint_{\alpha_j} dp = 0, \qquad \oint_{\gamma_j} dp = 2\pi j$$

Lowest trajectory in thermodynamic limit

Thermodynamic limit: $L \to \infty$ — AB, Gorsky, Korchemsky'05

In thermodynamic limit, the minimal energy arises from two-cut configuration:

$$dp = \frac{-1 + \frac{\beta ab}{x^2}}{\sqrt{(x^2 - a^2)(x^2 - b^2)}}, \quad a = \frac{1}{2\,\mathrm{E}(\tau)}, \quad b = \frac{\beta}{2\,\mathrm{K}(\tau)}, \quad \beta = \sqrt{1-\tau}\,\frac{\mathrm{K}(\tau)}{\mathrm{E}(\tau)}$$

- $L \gg N$: $\gamma^{(0)} = \frac{\pi^2}{2s}\frac{N}{L^2} + \dots$

  Berenstein, Maldacena, Nastase '02

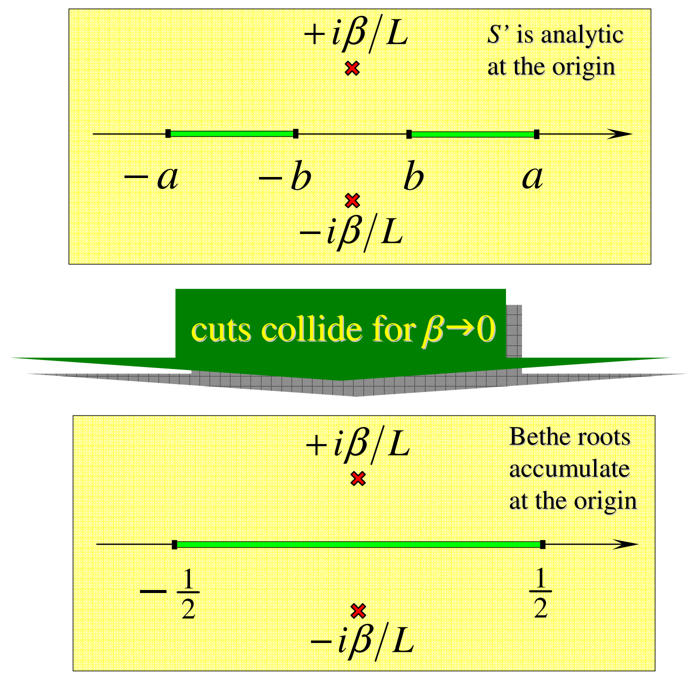
  depends on single-particle conformal spin

- $L \ll N$, $Le^L > N$: $\gamma^{(0)} = \frac{2}{L}\ln^2\frac{N}{L} + \dots$

  Beisert, Frolov, Staudacher, Tseytlin'03

- $Le^L \ll N$: $\gamma^{(0)} = 2\ln N + \dots$

  does not depend on twist



Does not depend on twist, i.e., same for any L!
Its coefficient is (one-loop) cusp anomalous dimension.

Cusp equation

Cusp anomalous dimension determines Sudakov asymptotics of Wilson anomalous dimensions ($Le^L << N$):

$$\gamma(g) = 2\Gamma_{\text{cusp}}(g)\ln N + O(N^0)$$

Korchemsky '88

The equation for cusp anomalous dimension in all orders of perturbation theory:

$$e^{\sigma_+[x_+]}\frac{Q(u+i)Q_{(0)}(u)}{Q(u)Q_{(0)}(u+i)} = 1$$

AB'06
Kostov, Serban, Volin '07

$$\Gamma_{\text{cusp}}(g)\ln N = i\lim_{u\to\infty} u\left(\ln Q(u)/Q_{(0)}(u)\right)'$$

Fourier transform yields the "momentm-space" cusp equation.

Beisert, Eden, Staudacher '06

Perturbative solution post-/pre-dicts terms in the expansion:

$$\Gamma_{\text{cusp}}(g) = g^2 -$$

Polyakov '80

$$-\tfrac{\pi^2}{12}g^4$$

AB, Gorsky, Korchemsky '03
Kotikov, Lipatov '03

$$-\tfrac{11\pi^4}{720}g^6 -$$

Kotikov, Lipatov, Onischenko, Velizhanin'05

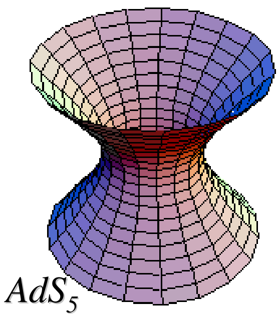
$$-\left(\tfrac{73\pi^6}{20160} + \tfrac{1}{8}\zeta^2(3)\right)g^8 + \ldots$$

Bern, Czakon, Dixon, Kosower, Smirnov'06

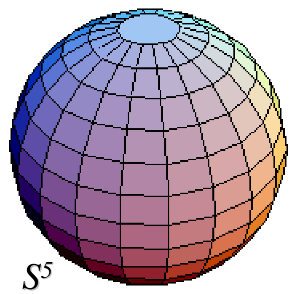
Gauge/string correspondence

Maldacena'97

$\mathcal{N}$=4 sYM is dual to type IIB string theory on curved superspace:



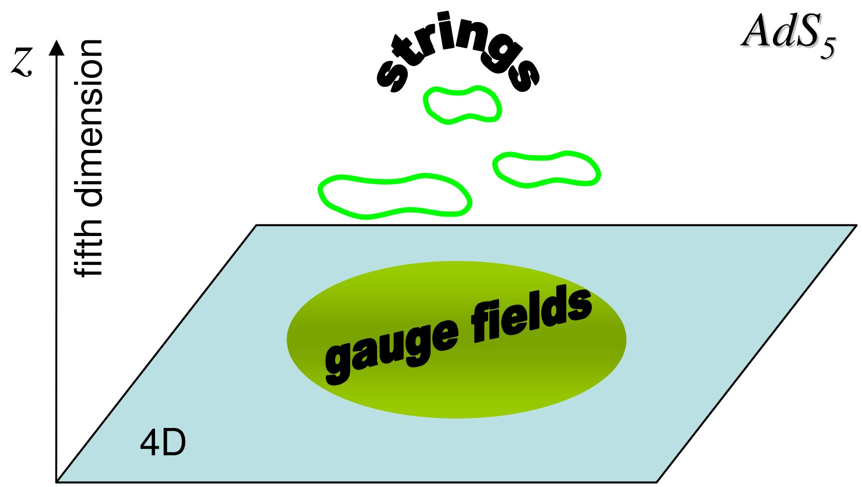
AdS_5 $\times$



S^5 $\times$ fermi $\sim PSU(2,2|4)$

Strings in 4d behave as if they are living in the 5D space, the fifth (Liouville) dimension being a result of quantum fluctuations.

Polyakov'89



AdS_5

$$ds^2 = \frac{dx_\mu dx^\mu + dz^2}{z^2}$$

Dictionary:

$$\sqrt{\lambda} = R/\alpha', \quad g_s = \lambda/(4\pi N_c)$$

Wilson operators in gauge/string duality

string states (energies) ⟷ local operators (anomalous dimensions)

Gubser, Klebanov, Polyakov'98
Witten'98

- Spectra should coincide
- Integrability of gauge theory should manifest itself in string theory (and vice versa)

Strong/weak duality:

perturbative gauge theory ⟷ quantum strings

strong coupling gauge theory ⟷ classical strings

Resolution of the problem: consider operators with large number of fields
= string states with large quantum numbers

Dual string states: folded rotating string

Energy/anomalous dimension relation:

$$\gamma = E - L - N$$

- Short strings $L \gg N$:

  Berenstein, Maldacena, Nastase ’02

$$\gamma = \pi^2 \frac{N}{L^2} + \ldots$$

- Long strings
  - $L \ll N$, $L >$ ln(N/L):

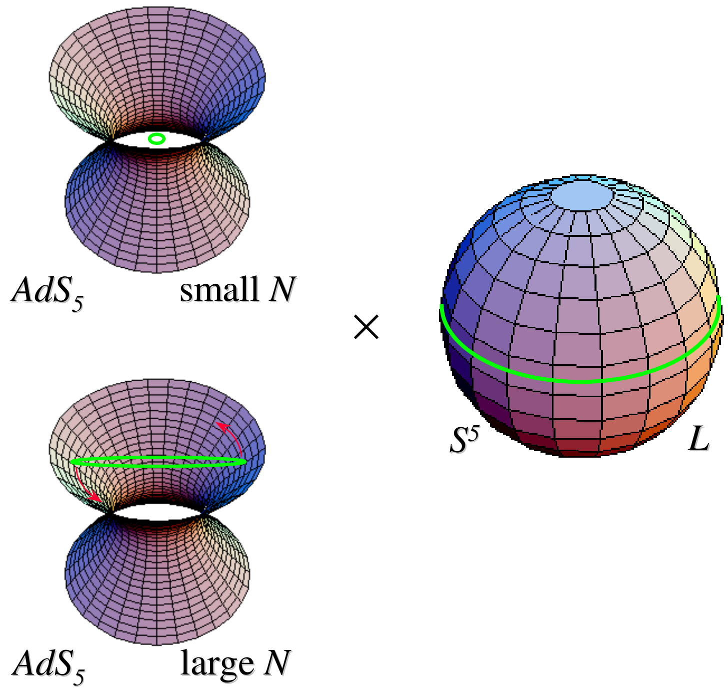
    Tseytlin et al.’02

$$\gamma = 2 g^2 \frac{1}{L} \ln^2 \frac{N}{L} + \ldots$$

  - $L \ll N$, $L \ll$ ln(N/L):

    Gubser, Klebanov, Polyakov ’02
    Kruczenski ’04

$$\gamma = 2 g \ln \frac{N}{g} + \ldots$$



non-analytic dependence in the coupling constant

Interpolating formula for long strings:

$$\gamma = L \left[\sqrt{1 + \frac{g^2}{(L/2)^2} \ln^2 \frac{N}{L}} - 1 \right] + \ldots$$

AB, Gorsky, Korchemsky’06

Strong-coupling cusp

Numerical solution of the cusp equation suggested in Benna, Benvenuti, Klebanov, Scardicchio'06

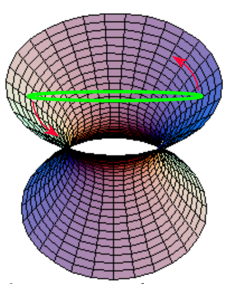
Analytical solution: Basso, Korchemsky, Kotanski '06

$$\Gamma_{\rm cusp}(g) = g - \frac{3}{2\pi}\ln 2 - \frac{K}{4\pi^2}\frac{1}{g} + O(1/g^2)$$

- Expansion coefficients are all negative for the exception of the first
- The expansion coefficients grow factorially (i.e., $1/g$ expansion is not Borel summable)

$$\Gamma_{\rm cusp}(g) = g\int_0^\infty \frac{du}{u-g\pi}\frac{e^{-u}}{\sqrt{u}} \quad \to \quad \frac{e^{-g\pi}}{\sqrt{g}}$$

String sigma-model calculation of the folded rotating string on AdS_5



AdS_5 large

$$\Gamma_{\rm cusp}(g) = g - \frac{3}{2\pi}\ln 2 - \frac{K}{4\pi^2}\frac{1}{g} + O(1/g^2)$$

Gubser, Klebanov, Polyakov '02

Frolov, Tseytlin '02

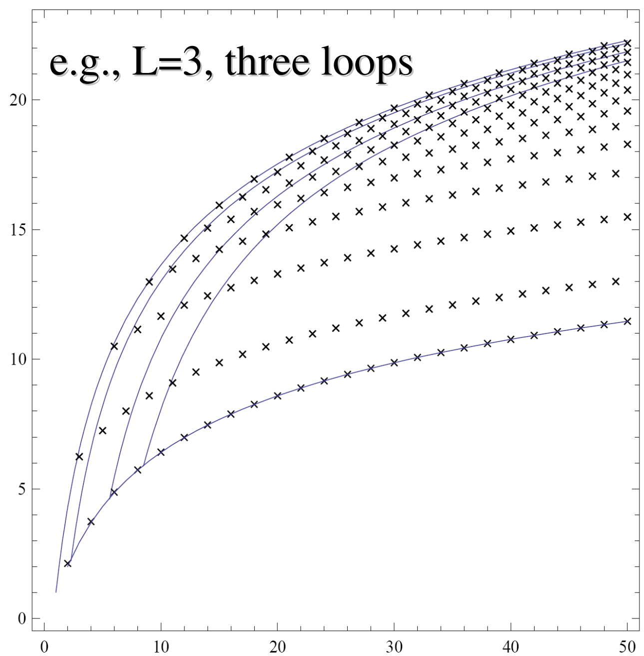
Roiban, Tseytlin '07

The O(6) model develops a mass gap: $\Gamma_{\rm cusp}(g) \sim m^2 = e^{2g/\beta_1} / g^{2\beta_2/\beta_1^2}$

Alday, Maldacena '07
Basso, Korchemsky '08
Fioravanti et al. '08

Exploring the fine structure

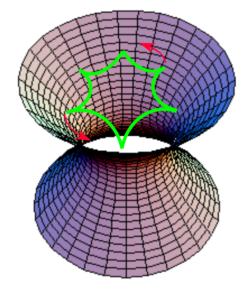
Systematic expansion of the all-order Baxter in WKB series: AB, Korchemsky, Pasechnik '08



Upper part of the spectrum, e.g.,

$$\gamma_{(2)}(N,n) = \Gamma^{(2)}_{\rm cusp}\left(\ln\frac{q^{(0)}N^3}{\sqrt{3}} - 6\psi(1)\right) - \tfrac{5}{4}\psi^{(4)}(1) - 3\psi'(1)\psi''(1) + O(1/N)$$

Spiky string for the highest trajectory:



Kruczenski '04

Lower part of the spectrum (arbitrary L):

- One-loop analysis suggested the expansion: AB, Gorsky, Korchemsky '06

$$\gamma_{(0)} = 2g^2\ln N\left[1 - \ln 2\frac{L}{\ln N} - \frac{\psi''(\frac{1}{2})}{4\ln N}\sum_{k=2}^{L-1}\left(\delta_k^{(0)}\right)^2 + \dots\right]$$

encodes fine structure

- Systematic expansion, e.g.,

$$\gamma(g) = \dots + g^6\ln N\left[\tfrac{11}{45}\pi^4 + \left[-\tfrac{11}{45}\pi^4\ln 2 - \tfrac{1}{3}\pi^2\zeta(3) - 21\zeta(5)\right]\frac{L}{\ln N} + \dots\right]$$

Agrees with a different formalism … Freyhult, Rej, Staudacher '07

Q: What is the string configuration interpolating between low and upper boundaries?

Beyond the minimal sector: Analytic Bethe Ansatz

Problem:

R-matrices are not known for the putative long-range "chains" emerging in $\mathcal{N}$=4 SYM.

Solution:

Formulation based on the Analytic Bethe Ansatz.

Reshetikhin '83

- Transfer matrices are quantum generalization of (super)characters
- Transfer matrices being traces of monodromy matrices are sums of terms, one per a component of corresponding reps
- Transfer matrices are entire functions (use Bethe Ansatz Equations to prove this)

Formalism devised for short-range (super)spin chains.

Kuniba, Tsuboi '95
Tsuboi '97

Elementary Young tableaux for $\mathcal{N}$=4 SYM

AB '08

$$\boxed{1}_u = \left(x^+\right)^L \frac{\widehat{Q}^{(1)}\left(u^+\right)}{\widehat{Q}^{(1)}\left(u^-\right)} \mathrm{e}^{\frac{1}{2}\left[\Delta_+(u^-)-\Delta_-(u^-)+\Delta_+(u^+)+\Delta_-(u^+)+\sigma_0^{(3)}(u^-)-\sigma_0^{(3)}(u^+)\right]},$$

$$\boxed{2}_u = \left(x^+\right)^L \frac{\widehat{Q}^{(1)}\left(u^+\right)}{\widehat{Q}^{(1)}\left(u^-\right)} \frac{Q^{(2)}\left(u-i\right)}{Q^{(2)}\left(u\right)} \mathrm{e}^{\frac{1}{2}\left[\Delta_+(u^+)+\Delta_-(u^+)+\sigma_0^{(3)}(u^-)-\sigma_0^{(3)}(u^+)\right]},$$

$$\boxed{3}_u = \left(x^+\right)^L \frac{Q^{(2)}\left(u+i\right)}{Q^{(2)}\left(u\right)} \frac{\widehat{Q}^{(3)}\left(u^-\right)}{\widehat{Q}^{(3)}\left(u^+\right)} \mathrm{e}^{\frac{1}{2}\left[\Delta_+(u^+)+\Delta_-(u^+)+\sigma_0^{(1)}(u^+)-\sigma_0^{(1)}(u^-)\right]},$$

$$\boxed{4}_u = \left(x^+\right)^L \frac{\widehat{Q}^{(3)}\left(u^-\right)}{\widehat{Q}^{(3)}\left(u^+\right)} \frac{Q^{(4)}\left(u+i\right)}{Q^{(4)}\left(u\right)} \mathrm{e}^{\Delta_+(u^+)+\frac{1}{2}\left[\sigma_0^{(1)}(u^+)-\sigma_0^{(1)}(u^-)\right]},$$

$$\boxed{5}_u = \left(x^-\right)^L \frac{\widehat{Q}^{(5)}\left(u^+\right)}{\widehat{Q}^{(5)}\left(u^-\right)} \frac{Q^{(4)}\left(u-i\right)}{Q^{(4)}\left(u\right)} \mathrm{e}^{\Delta_-(u^-)+\frac{1}{2}\left[\sigma_0^{(7)}(u^-)-\sigma_0^{(7)}(u^+)\right]},$$

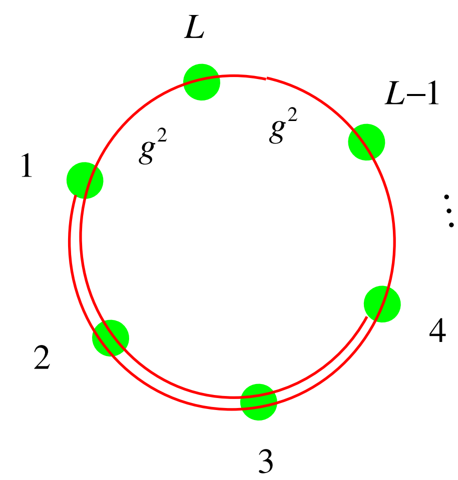
$$\boxed{6}_u = \left(x^-\right)^L \frac{Q^{(6)}\left(u-i\right)}{Q^{(6)}\left(u\right)} \frac{\widehat{Q}^{(5)}\left(u^+\right)}{\widehat{Q}^{(5)}\left(u^-\right)} \mathrm{e}^{\frac{1}{2}\left[\Delta_-(u^-)+\Delta_+(u^-)+\sigma_0^{(7)}(u^-)-\sigma_0^{(7)}(u^+)\right]},$$

$$\boxed{7}_u = \left(x^-\right)^L \frac{\widehat{Q}^{(7)}\left(u^-\right)}{\widehat{Q}^{(7)}\left(u^+\right)} \frac{Q^{(6)}\left(u+i\right)}{Q^{(6)}\left(u\right)} \mathrm{e}^{\frac{1}{2}\left[\Delta_-(u^-)+\Delta_+(u^-)+\sigma_0^{(5)}(u^+)-\sigma_0^{(5)}(u^-)\right]},$$

$$\boxed{8}_u = \left(x^-\right)^L \frac{\widehat{Q}^{(7)}\left(u^-\right)}{\widehat{Q}^{(7)}\left(u^+\right)} \mathrm{e}^{\frac{1}{2}\left[\Delta_-(u^+)-\Delta_+(u^+)+\Delta_-(u^-)+\Delta_+(u^-)+\sigma_0^{(5)}(u^+)-\sigma_0^{(5)}(u^-)\right]}.$$

Yield fused transfer matrices and T-system/Y-systems.

Wrapping



Interaction range exceeds the length of the chain.

Asymptotic Baxter equation is not valid above order g^{2L+2}.

Konishi operator:

$$O_{\text{Konishi}} = \text{tr}\, \phi_i \phi^i \quad \leftrightarrow \quad \text{tr}\, ZD^2 Z$$

Anomalous dimension:

$$\gamma_{\text{Konishi}}(g) = 3g^2 - 3g^4 + \tfrac{21}{4} g^6 - \tfrac{\gamma_4}{512} g^8 + \ldots$$

with a "diverse" predictions:

$$\gamma_4 = \begin{cases} 5307 + 1128\zeta(3) \\ 4992 - 1152\zeta(3) + 2880\zeta(5) \\ 5214 + 56\zeta(3) + 280\zeta(5) \end{cases}$$

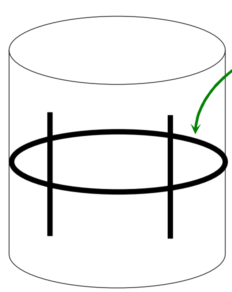
Kotikov, Lipatov, Rej, Staudacher, Velizhanin '07

Fiamberti, Santambrogio, Sieg, Zanon '08

Keeler, Mann '08

Luescher corrections to asympotic Baxter (or Bethe):

all states of su(2,2|4) propagate in the loop

$\delta\gamma_4 =$  $= -648 - 1728\zeta(3) + 2880\zeta(5)$

Bajnok, Janik '08

Conclusions

- Consistent incorporation of wrapping effects
- Operator formation of the AdS long-range spin chain
- More quantitative predictions for AdS/CFT:
  - Subleading WKB corrections at strong coupling
  - Daughter trajectories, density of states in the band
  - Etc.