Cusp Anomalous Dimension in Planar $\mathcal{N} = 4$ Super-Yang-Mills Theory

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$\mathcal{N} = 4$ SUSY and QCD, LPTHE

Outline

- Anomalous dimensions of high-spin Wilson operators and cusp anomalous dimension
- Weak and strong coupling results from gauge and string theories
- Interpolation from weak to strong coupling thanks to (conjectured) all-loop integrability
- Test of AdS/CFT correspondence

$\mathcal{N} = 4$ Super-Yang-Mills Theory

Definition

Maximally supersymmetric extension of Yang-Mills theory

$$\mathcal{L} = -\mathrm{Tr}\{F_{\mu\nu}F^{\mu\nu}\}/2g_{\mathrm{YM}}^2 + \mathrm{fermions} + \mathrm{scalars}$$

where all fields are in the adjoint representation of the gauge group $SU(N_c)$

- Conformal invariance

  The coupling constant g_{YM}^2 does not run $\qquad \beta(g_{\mathrm{YM}}^2) = 0$

- 't Hooft planar limit ['t Hooft'74]

$$N_c \to \infty \qquad \text{with} \qquad g^2 = g_{\mathrm{YM}}^2 N_c / 16\pi^2 \qquad \text{fixed}$$

  corresponds to summing over planar Feynman diagrams

AdS/CFT correspondence [Maldacena'97]

(Planar) $\mathcal{N} = 4$ $SU(N_c)$ Super-Yang-Mills is dual to type IIB (non-interacting) string theory on $AdS_5 \times S^5$ background geometry

$$\text{string tension} \qquad \frac{1}{2\pi\alpha'} = \frac{\sqrt{g_{\mathrm{YM}}^2 N_c}}{2\pi} = 2g$$

Anomalous Dimensions of Wilson Operators

Wilson operators

Single-trace operators built from L complex scalars $\mathcal{Z}(0)$ and N lightcone derivatives D_+

$$\mathcal{O}_{\mathbf{n}}(0) = \mathrm{Tr}\{D_+^{n_1}\mathcal{Z}(0)...D_+^{n_L}\mathcal{Z}(0)\} \qquad \mathbf{n} = (n_1, ..., n_L) \in \mathbb{N}^L$$

Quantum numbers : Twist L and Lorentz spin $N = n_1 + ... + n_L$

Dilatation operator

Wilson operators with same quantum numbers mix under a change of the renormalization scale according to the Callan-Symanzik equation

$$\mu\frac{\partial}{\partial\mu}\mathcal{O}_{\mathbf{n}}(0) = \left(\mathbb{H}\cdot\mathcal{O}\right)_{\mathbf{n}}(0) \qquad \mathbb{H} = \text{dilatation operator}$$

Operators $\mathcal{O}_{\mathbf{n}}(0)$ belong to a closed $SL(2)$ sector

Anomalous dimensions

Spectrum of the dilatation operator = anomalous dimensions of Wilson operators

$$\delta_{L,N}(g) = \Delta_{L,N}(g) - N - L \qquad (= \mathcal{O}(g^2))$$

with the scaling dimensions $\Delta_{L,N}(g)$

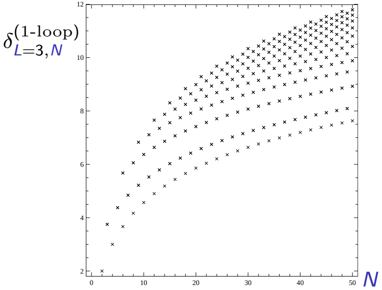
$$\langle\mathcal{O}(x)\mathcal{O}(y)\rangle \sim \left[(x-y)^2\right]^{-\Delta_{L,N}(g)}$$

Logarithmic Scaling for High-Spin Operators

Anomalous dimensions of Wilson operators $\delta_{L,N}(g)$ with large spin $N \gg 1$ and fixed twist L occupy the band [Korchemsky'95],[Belitsky,Gorsky,Korchemsky'03],

$$2\ \Gamma_{\text{cusp}}(g) \ln N \leqslant \delta_{L,N}(g) \leqslant L\ \Gamma_{\text{cusp}}(g) \ln N$$

with $\Gamma_{\text{cusp}}(g)$ the cusp anomalous dimension



Minimal anomalous dimension has universal (L independent) scaling behavior in a generic planar Yang-Mills theory

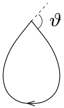
$$\delta^{\min}_{L,N}(g) = 2\ \Gamma_{\text{cusp}}(g) \ln N + \mathcal{O}(N^0)$$

Cusp Anomalous Dimension

Definition

Cusp anomalous dimension governs the renormalization of Wilson loops evaluated over a closed euclidean contour with a cusp [Polyakov'80]

$$\left\langle \mathrm{Tr}\ \mathrm{P}\exp\left(\imath \oint_{\mathrm{C}} dx \cdot A(x)\right)\right\rangle \sim (\Lambda_{\mathrm{UV}})^{\Gamma_{\mathrm{cusp}}(g,\vartheta)}$$

$\mathrm{C} =$ 

Controls infrared asymptotics of scattering amplitudes in gauge theories [Korchemsky,Radyushkin'86]

- An integration contour C is defined by the particle momenta
- The cusp angle ϑ is related to the scattering angles in Minkowski space

$$\Gamma_{\mathrm{cusp}}(g,\vartheta) = \vartheta\ \Gamma_{\mathrm{cusp}}(g) + \mathcal{O}(\vartheta^0) \qquad |\vartheta| \gg 1$$

Ubiquitous observable of gauge theories

- Logarithmic scaling of anomalous dimensions of high-spin Wilson operators
- IR singularities of on-shell gluon scattering amplitudes
- ...

Weak and Strong Coupling Expansion of Cusp in Planar $\mathcal{N} = 4$ SYM

Weak coupling expansion

[Belitsky,Gorsky,Korchemsky'03],[Kotikov,Lipatov,Onishchenko,Velizhanin'04]
[Bern,Czakon,Dixon,Kosower,Smirnov'06],[Cachazo,Spradlin,Volovich'06]

$$\Gamma_{\text{cusp}}(g) = 4g^2 - \frac{4}{3}\pi^2 g^4 + \frac{44}{45}\pi^4 g^6 - 8\left(\frac{73}{630}\pi^6 + 4\zeta_3^2\right) g^8 + \mathcal{O}(g^{10})$$

Fulfills the Kotikov-Lipatov maximal transcendentality principle :

Expansion coefficient at order g^{2k} has given degree of transcendentality $= 2k - 2$

$$\begin{aligned}
&\text{one-loop} \leftrightarrow 1\ (\sim \zeta(0)) \\
&\text{two-loop} \leftrightarrow \pi^2\ (\sim \zeta(2)) \\
&\text{three-loop} \leftrightarrow \pi^4\ (\sim \zeta(2)^2 \text{ and/or } \zeta(4)) \\
&\text{four-loop} \leftrightarrow \pi^6, \zeta(3)^2 \\
&\text{five-loop prediction} \leftrightarrow \pi^8, \zeta(3)^2\pi^2, \zeta(5)\zeta(3)
\end{aligned}$$

Weak and Strong Coupling Expansion of Cusp in Planar $\mathcal{N} = 4$ SYM

Strong coupling expansion from AdS/CFT correspondence

- Semiclassical string [Gubser,Klebanov,Polyakov'02],[Frolov,Tseytlin'02]

Wilson operator with twist L, spin N and minimal scaling dimension $\Delta_{L,N}(g) = N + L + \delta_{L,N}(g)$ $=$ Folded string spinning in $AdS_3 \times S^1$ with angular momenta N, L and energy $E_{L,N}(g) = \Delta_{L,N}(g)$

Semiclassical string expansion reproduces the logarithmic scaling

$$E_{L,N}(g) = N + 2\Gamma_{\text{cusp}}(g) \ln N + \mathcal{O}(N^0) \qquad \text{for} \qquad N \gg g \qquad \text{and} \qquad L \sim g$$

- Vacuum expectation value of a Wilson loop with a cusp (minimal surface) [Kruczenski'02],[Makeenko'02]

String theory prediction for the strong coupling expansion of the cusp anomalous dimension

$$\Gamma_{\text{cusp}}(g) = 2g - \frac{3 \ln 2}{2\pi} + \mathcal{O}(1/g)$$

Weak and Strong Coupling Expansion of Cusp in Planar $\mathcal{N} = 4$ SYM

Cusp anomalous dimension in the 't Hooft planar limit

- Weak coupling expansion from the gauge theory

$$\Gamma_{\text{cusp}}(g) = 4g^2 - \frac{4}{3}\pi^2 g^4 + \frac{44}{45}\pi^4 g^6 - 8\left(\frac{73}{630}\pi^6 + 4\zeta_3^2\right) g^8 + \mathcal{O}(g^{10})$$

- Strong coupling expansion from AdS/CFT correspondence

$$\Gamma_{\text{cusp}}(g) = 2g - \frac{3\ln 2}{2\pi} + \mathcal{O}(1/g)$$

Interpolation?

- A proposal was put forward (Beisert-Eden-Staudacher equation) about the form of the all-loop cusp anomalous dimension derived from the conjectured all-loop integrability of the gauge/string theory in the planar limit

Spin Chain Representation

Kinematics

Single-trace Wilson composite operators built from L complex scalar fields $\mathcal{Z}(0)$ and N lightcone derivatives D_+

$$\mathcal{O}_{\mathbf{n}}(0) = \mathrm{Tr}\{D_+^{n_1}\mathcal{Z}(0)...D_+^{n_L}\mathcal{Z}(0)\} \qquad \mathbf{n} = (n_1, ..., n_L) \in \mathbb{N}^L$$

- $\mathrm{Tr}\{\mathcal{Z}(0)...\mathcal{Z}(0)...\mathcal{Z}(0)\} \rightarrow$ vacuum state of the spin chain
- $\mathrm{Tr}\{\mathcal{Z}(0)...D_+\mathcal{Z}(0)...\mathcal{Z}(0)\} \rightarrow$ one-particle state of the spin chain (magnon)

Quantum numbers :

- Twist L $\rightarrow$ spin chain length
- Lorentz spin $N = n_1 + ... + n_L$ $\rightarrow$ number of excitations (magnons) over the vacuum

Dynamics

Wilson operators with same quantum numbers mix under a change of the renormalization scale according to the Callan-Symanzik equation

$$\mu\frac{\partial}{\partial\mu}\mathcal{O}_{\mathbf{n}}(0) = (\mathbb{H}\cdot\mathcal{O})_{\mathbf{n}}(0)$$

- $\mathbb{H}$ dilatation operator of the conformal $\mathcal{N} = 4$ gauge theory $\rightarrow$Hamiltonian of the spin chain
- Spectrum of scaling dimensions $\Delta_{L,N}(g)$ $\rightarrow$ spectrum of energies of the spin chain

$\mathbb{H}$ is integrable and can be solved by means of the Bethe ansatz

[Lipatov'97],[Braun,Derkachov,Manashov'98],[Belitsky'98],[Korchemsky'98],
[Minahan,Zarembo'02],[Beisert,Staudacher'03],[Beisert,Kristjansen,Staudacher'03],[Beisert'04]

One-loop (planar) dilatation operator

- $\mathbb{H}$ = Hamiltonian of $\mathrm{XXX}_{s=-1/2}$ Heisenberg spin chain

  Generalization to $SL(2)$-representation of the original Heisenberg magnet with $SU(2)$ spins

- System with L degrees of freedom... and L commuting conserved charges!

  Liouville definition of a completely integrable system

- The complete family of conserved charges can be diagonalized simultaneously with $\mathbb{H}$ by means of the Bethe ansatz

  Extension (here to $SL(2)$) of the original Bethe solution for diagonalization of the Hamiltonian of the Heisenberg magnet

Bethe ansatz for (planar) dilatation operator (one-loop example)

- Bethe equations for integrable $SL(2)$ Heisenberg spin chain

$$\left(\frac{u_k + \imath/2}{u_k - \imath/2}\right)^L = \prod_{j\neq k}^{N} \frac{u_k - u_j - \imath}{u_k - u_j + \imath} \quad \text{(quantization conditions)}$$

- One-loop anomalous dimension

$$\delta_{L,N}(g) = g^2 \sum_{k=1}^{N} \frac{1}{u_k^2 + 1/4} \quad \text{(energy)}$$

- Extra constraint on physical state

$$\prod_{k=1}^{N} \frac{u_k + \imath/2}{u_k - \imath/2} = 1 \quad \text{(cyclicity condition)}$$

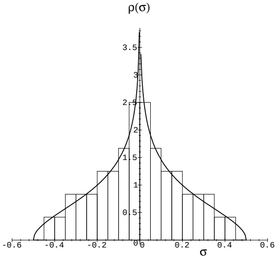
Simplification in the large spin limit $N \gg 1$: condensation of the Bethe roots $\{u_k\} \to$ continuum limit of the Bethe Ansatz equations [Korchemsky'95]

Cusp from Bethe Ansatz Equations

Large spin limit $N \gg 1$: continuum limit of the Bethe Ansatz equations

- Distribution density of Bethe roots for minimal anomalous dimensions [Korchemsky'95],[Eden,Staudacher'06]

$$\rho(\sigma) = \frac{1}{N}\sum_{k=1}^{N}\delta\left(\sigma - \frac{u_k}{N}\right) \qquad \text{solution to} \qquad \int_{-1/2}^{+1/2} d\sigma \frac{\rho(\sigma)}{\tau^2-\sigma^2} = 2\pi\delta(\tau)$$



- Bethe roots condense at the origin → logarithmic scaling $\sim 2\,\Gamma_{\text{cusp}}(g)\ln N$

$$\rho(\sigma) = \frac{1}{\pi}\ln\frac{1+\sqrt{1-4\sigma^2}}{1-\sqrt{1-4\sigma^2}} \quad \rightarrow \quad \delta^{\text{min}}_{L,N}(g) = g^2\int_{-1/2}^{+1/2} d\sigma \frac{\rho(\sigma)/N}{\sigma^2+(1/2N)^2} \sim 8g^2\ln N$$

All-Loop Asymptotic Bethe Ansatz

Ingredients

- Exploration and understanding of integrable structures in planar $\mathcal{N} = 4$ SYM at higher loop
  [Serban,Staudacher'04],[Eden,Jarczak,Sokatchev'04],[Beisert,Dippel,Staudacher'04],[Staudacher'04]
- Comparison with integrable structures on the stringy side of AdS/CFT correspondence
  [Bena,Polchinski,Roiban'03],[Kazakov,Marshakov,Minahan,Zarembo'04]
- Necessity and presence of a dressing phase [Arutyunov,Frolov,Staudacher'04],[Beisert,Tseytlin'05],[Hernández,López'06]
- Constraint on the dressing phase: crossing-symmetry [Janik'06],[Arutyunov,Frolov'06]

Proposal for $SL(2)$ sector

- All-loop asymptotic Bethe ansatz [Beisert,Staudacher'05],[Beisert'05]

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{j\neq k}^{N} \frac{x_k^- - x_j^+}{x_k^+ - x_j^-} \frac{1 - g^2/x_k^+ x_j^-}{1 - g^2/x_k^- x_j^+} \exp\left(2\imath\theta(u_k, u_j)\right) \qquad u_k \pm \imath/2 = x_k^\pm + g^2/x_k^\pm$$

  with the dressing phase $\theta(u_k, u_j)$ $\left(= \mathcal{O}\left(g^6\right)\right)$ [Beisert,Hernández,López'06],[Beisert,Eden,Staudacher'06]

- All-loop 'asymptotic' anomalous dimension

$$\delta_{L,N}(g) = g^2 \sum_{j=1}^{N} \left[\frac{\imath}{x_j^+} - \frac{\imath}{x_j^-}\right]$$

- Wrapping effect $\rightarrow$ accuracy $= \mathcal{O}(g^{2L})$ when compared with gauge perturbation theory

Large Spin Continuum Limit : BES equation

The large spin limit of the all-loop asymptotic Bethe ansatz is consistent with the logarithmic scaling and leads to the...

BES equation for the distribution density of roots [Eden,Staudacher'06],[Beisert,Eden,Staudacher'06]

$$\delta\rho(t) = \frac{t}{e^t - 1}\left(K(2gt,0) - 4g^2\int_0^\infty dt' K(2gt,2gt')\delta\rho(t')\right)$$

with the kernel

$$K(t,t') = K^{(m)}(t,t') + 2\ K^{(d)}(t,t')$$

sum of the main scattering kernel and of the dressing kernel

$$K^{(m)}(t,t') = K_0(t,t') + K_1(t,t')$$

$$K^{(d)}(t,t') = 4g^2\int_0^\infty dt'' K_1(t,2gt'')\frac{t''}{e^{t''}-1}K_0(2gt'',t')$$

where

$$K_0(t,t') = \frac{tJ_1(t)J_0(t') - t'J_0(t)J_1(t')}{t^2 - t'^2} = \frac{2}{tt'}\sum_{n\geqslant 1}(2n-1)J_{2n-1}(t)J_{2n-1}(t')$$

$$K_1(t,t') = \frac{t'J_1(t)J_0(t') - tJ_0(t)J_1(t')}{t^2 - t'^2} = \frac{2}{tt'}\sum_{n\geqslant 1}(2n)J_{2n}(t)J_{2n}(t')$$

Relation to the cusp : $\Gamma_{\text{cusp}}(g) = 8g^2\delta\rho(0)$

Weak Coupling Expansion of Cusp from BES Equation

BES equation

$$\delta\rho(t) = \frac{t}{e^t - 1}\left(K(2gt, 0) - 4g^2 \int_0^{+\infty} dt' K(2gt, 2gt')\delta\rho(t')\right)$$

Solution at weak coupling

[Eden,Staudacher'06],[Beisert,Eden,Staudacher'06]
[Belitsky'06]

$$\delta\rho(t) = \frac{t}{e^t - 1}\left[K(2gt, 0) - 4g^2 \int_0^{\infty} dt' K(2gt, 2gt') \frac{t'}{e^{t'} - 1} K(2gt', 0) + O(g^4)\right]$$

Weak coupling expansion of the cusp anomaly

$$\Gamma_{\mathrm{cusp}}(g) = 8g^2\delta\rho(0) = 4g^2 - \frac{4}{3}\pi^2 g^4 + \frac{44}{45}\pi^4 g^6 - 8\left(\frac{73}{630}\pi^6 + 4\zeta_3^2\right) g^8$$

$$+32\left(\frac{887}{14175}\pi^8 + \frac{4}{3}\pi^2\zeta_3^2 + 40\zeta_3\zeta_5\right) g^{10} + \mathcal{O}(g^{12})$$

- Reproduces the known four-loop result
- Verifies the KL maximal transcendentality principle to all loop [Kotikov,Lipatov'06]
- Numerical analysis indicates that the weak coupling expansion is convergent

Strong Coupling Expansion : Numerical Approach

Strategy

[Benna,Benvenuti,Klebanov,Scardicchio'06]

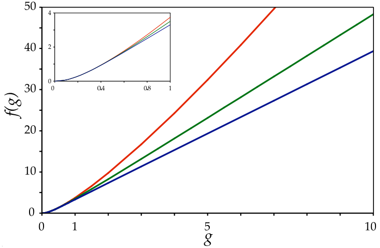
- Expand the solution $s(t) = (e^t - 1)\delta\rho(t)/t$ over the Bessel functions
- Truncate the series at sufficiently large number of terms $M \sim g$

$$s(t) = \sum_{n=1}^{M} s_n(g)\, \frac{J_n(2gt)}{2gt} \qquad s_{n>M}(g) = 0$$

The integral equation becomes a finite-dimensional matrix equation for the coefficients $s_n(g)$

- Solve numerically the matrix equation and extract the cusp anomaly $\Gamma_{\rm cusp}(g) = 4g^2 s_1(g)$

Result



$$f(g) = 2\Gamma_{\rm cusp}(g) = (4.000000 \pm 0.000001)g$$
$$-(0.661907 \pm 0.000002) - \frac{0.0232 \pm 0.0001}{g} + \ldots$$

The first two terms are in remarkable agreement with the string theory result!

$$0.661907 = \frac{3\ln 2}{\pi}, \qquad 0.0232 = ?$$

Strong Coupling Expansion of Cusp from BES Equation I

Analytically, the strong coupling solution was first analyzed at leading order

[Kotikov,Lipatov'06],[Benna,Benvenuti,Klebanov,Scardicchio'07], [Kostov,Serban,Volin'07], [Alday,Arutyunov,Benna,Eden,Klebanov'07],[Beccaria,De Angelis,Forini'07]

and then in a more systematic approach [B.,Korchemsky,Kotański'07],[Kostov,Serban,Volin'08]

Result

[B.,Korchemsky,Kotański'07]

$$\Gamma_{\rm cusp}(g+c_1) = 2g\Big[1 - c_2 g^{-2} - c_3 g^{-3} - \left(c_4 + 2\,{c_2}^2\right) g^{-4} - \left(c_5 + 23\,c_2 c_3\right) g^{-5} - \left(c_6 + \frac{166}{7}\,c_2 c_4 + 54\,{c_3}^2 + 25\,{c_2}^3\right) g^{-6} + O\left(g^{-7}\right)\Big]$$

where

$$c_1 = \frac{3\ln 2}{4\pi} \qquad c_2 = \frac{1}{16\pi^2}\mathrm{K} \qquad c_3 = \frac{27}{2^{11}\pi^3}\zeta(3)$$

$$c_4 = \frac{21}{2^{10}\pi^4}\beta(4) \qquad c_5 = \frac{43065}{2^{21}\pi^5}\zeta(5) \qquad c_6 = \frac{1605}{2^{15}\pi^6}\beta(6)$$

with the special functions

$$\zeta(x) = \sum_{n\geq 1} n^{-x} = \text{Riemann zeta function}$$

$$\beta(x) = \sum_{n\geq 0} (-1)^n (2n+1)^{-x} = \text{Dirichlet zeta function}$$

$$\mathrm{K} = \beta(2) = \text{Catalan's constant}$$

Strong Coupling Expansion of Cusp from BES Equation II

Result

$$\Gamma_{\rm cusp}\,(g+c_1) = 2g\Big[1 - c_2 g^{-2} - c_3 g^{-3} - \left(c_4 + 2\,{c_2}^2\right) g^{-4}$$
$$- \left(c_5 + 23\,c_2 c_3\right) g^{-5} - \left(c_6 + \frac{166}{7}\,c_2 c_4 + 54\,{c_3}^2 + 25\,c_2^3\right) g^{-6} + O\left(g^{-7}\right)\Big]$$

Features

- Agreement with numerical values obtained within [Benna,Benvenuti,Klebanov,Scardicchio'07] approach
- Maximal transcendentality principle at strong coupling

$$\begin{array}{rcl} \text{weak coupling} & \longrightarrow & \text{strong coupling} \\ \zeta(2n) & \longrightarrow & \beta(2n) \\ \zeta(2n-1) & \longrightarrow & \zeta(2n-1) \end{array}$$

- c_1-dependent terms inside $\Gamma_{\rm cusp}(g)$ can be resummed by shifting $g \longrightarrow g + c_1$

AdS/CFT Correspondence

$\Gamma_{\rm cusp}(g) \leftrightarrow \begin{cases} \text{semiclassical energy of string spinning on AdS3} \\ \text{v.e.v. of a Wilson loop with a cusp} \end{cases}$ [Gubser,Klebanov,Polyakov'02],[Frolov,Tseytlin'02] [Kruczenski'02],[Makeenko'02]

Perturbative string theory prediction

$$\Gamma_{\rm cusp}(g) = 2g\left[a_0 - a_1 g^{-1} - a_2 g^{-2} - a_3 g^{-3} - ...\right]$$

with $a_0 = 1$:= classical solution, $a_1 = 3\ln 2/4\pi$:= 1-loop correction, a_2 := 2-loop correction, ...

$a_2 = \mathrm{K}/(4\pi)^2$ was confirmed by 2-loop superstring computation [Roiban,Tirziu,Tseytlin'07],[Roiban,Tseytlin'07]

AdS/CFT correspondence works fine to two-loop accuracy!

Verification of the prediction for higher order coefficients a_k remains a challenge for the string theory

Large-order behavior from BES equation?

- All coefficients a_k are positive
- The expansion coefficients grow factorially at large orders

$$a_k \propto \frac{\Gamma(k-1/2)}{(2\pi)^k} \quad \text{for} \quad k \gg 1 \quad \longrightarrow \quad \text{radius of convergence} = 0$$

Strong coupling expansion of the cusp anomalous dimension is only asymptotic!

Does perturbative string theory make sense?

Borel improved expansion

$$\Gamma_{\rm cusp}(g) \sim -g \sum_k \frac{\Gamma(k-\frac{1}{2})}{(2\pi g)^k} = g \int_0^\infty \frac{du\, u^{-1/2} e^{-u}}{u - 2\pi g}$$

... but it is not well-defined due to a pole at $u = 2\pi g$

Strong coupling expansion of the cusp anomalous dimension is only asymptotic...
and not Borel summable!

Interpretation

The cusp anomalous dimension (= energy density of quantum spinning folded string) receives 'non-perturbative' contribution at large g

$$\Delta\Gamma_{\rm cusp}(g) \sim g^{1/2} e^{-2\pi g}$$

What could be the origin of such correction on the string theory side?

Relation with O(6) Sigma-Model?

- The cusp anomalous dimension (= energy density of quantum spinning folded string) receives 'non-perturbative' contribution at large g

$$\Delta\Gamma_{\text{cusp}}(g) \sim g^{1/2} e^{-2\pi g}$$

Physical origin of such corrections?

- Proposal [Alday,Maldacena'07]

A non-perturbative scale m emerges from the low-energy effective dynamics of (world-sheet) excitations around the classical spinning string solution

  - Presence of massless excitations described by a noncritical $O(6)$ sigma-model with a UV cut-off set by the masses of massive (fermions + bosons) excitations
  - The $O(6)$ sigma-model develops a mass gap which affects the cusp anomalous dimension

$$\Delta\Gamma_{\text{cusp}}(g) \sim m^2 \propto g^{-2\beta_2/\beta_1^2} e^{4g/\beta_1}$$

Two-loop beta function for the $O(6)$ sigma-model $\beta_1 = -2/\pi$, $\beta_2 = -1/\pi^2$

Relation with O(6) Sigma-Model?

- The cusp anomalous dimension (= energy density of quantum spinning folded string) receives 'non-perturbative' contribution at large g

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perfect agreement with

Two-loop beta function for the $O(6)$ sigma-model $\beta_1 = -2/\pi$, $\beta_2 = -1/\pi^2$

Conclusion

- Anomalous dimensions of high-spin Wilson operators scale logarithmically with the Lorentz spin
- Logarithmic scaling is controlled to all-loop by the cusp anomalous dimension
- Conjectured integrability of the gauge/string theory leads to an integral equation that predicts the cusp anomalous dimension at any value of the coupling constant
- Both at weak and strong coupling the BES interpolation agrees with the known gauge and string results
  - At weak coupling the cusp anomalous dimension is given by a convergent series in g^2 that satisfies the maximal transcendentality principle
  - At strong coupling the cusp anomalous dimension is given by an asymptotic series in $1/g$ and suffers from non-perturbative ambiguities...
  - ... that admit interpretation on the string theory side
- Altogether these results provide a non-trivial test of the AdS/CFT correspondence