Algebraic entropy for lattice equations

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Abstract. We give the basic definition of algebraic entropy for lattice equations. The entropy is a canonical measure of the complexity of the dynamics they define. Its vanishing is a signal of integrability, and can be used as a powerful integrability detector. It is also conjectured to take remarkable values (algebraic integers).

The analysis of discrete dynamical systems, in particular the measure of their complexity, and possibly the detection of their integrability is a huge subject†, originating in the work of Poincaré. It contains the study of the dynamics of rational maps, already a vast topic of research. It also contains the study of lattice equations, which are to maps what partial differential equations are to ordinary differential equations. Our purpose here is to extend the notion of algebraic entropy, already widely used for maps [1, 2] [3, 4], and recognized as an unmatched integrability detector [5], to lattice equations as in [6], thus introducing a measure of complexity for higher dimensional discrete dynamics.

We briefly describe the setting, the space of initial data, the evolutions, and define the related entropies. We give examples, some integrable, some not integrable, showing how to extract information about the global (and asymptotic) behaviour of the system from a few iterates. We formalize and confirm the results of [6]. We also present some conjectures and perspectives.

1. The setting

Consider a cubic lattice of dimension D. The vertices of the lattice are labeled by D relative integers $[n_1, n_2, \ldots, n_D]$. To each vertex is associated the variable $y_{[n_1,n_2,\ldots,n_D]}$. We are given a defining relation, which links the values of y at each corner of the elementary cells (square for $D = 2$, cube for $D = 3$ and so on).

We will suppose that the defining relations allow to calculate any corner values on a cell from the $2^D - 1$ remaining ones, and that the value is given by a rational expression. This implies that our defining relations are multilinear, which covers a large number of interesting cases (see for example [7, 8] and more in the last sections). This restriction may be partially lifted.

As an illustration, consider the two-dimensional case of the square plane lattice. The elementary cell is a plaquette shown in Figure (1).

† We will not dwell here upon the definition of integrability

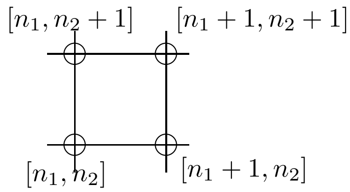

Figure 1: elementary cell in two dimensions

The defining relation is a constraint of the form

$$f(y_{[n_1,n_2]}, y_{[n_1+1,n_2]}, y_{[n_1,n_2+1]}, y_{[n_1+1,n_2+1]}) = 0. \tag{1}$$

We will use some specific examples later.

2. Initial conditions

For the sake of clarity, we will concentrate on the $D = 2$ case, but everything we will say generalizes straightforwardly to higher dimensions.

In order to define an evolution we have to specify initial conditions. From the form of the defining relation, it appears that the values of y have to be given on some "diagonal" of the lattice.

The space of initial conditions is infinite dimensional.

When $D = 2$, the diagonals need to go from $[n_1 = -\infty, n_2 = -\infty]$ to $[n_1 = \infty, n_2 = \infty]$, or from $[n_1 = -\infty, n_2 = +\infty]$ to $[n_1 = \infty, n_2 = -\infty]$. We will restrict ourselves to regular diagonals which are staircases with steps of *constant* horizontal length, and *constant* height. Figure (2) shows four diagonals. The ones labeled (1) and (2) are regular. The one labeled (3) would be acceptable, but we will not consider such diagonals. Line (4) is excluded since it may lead to incompatibilities.

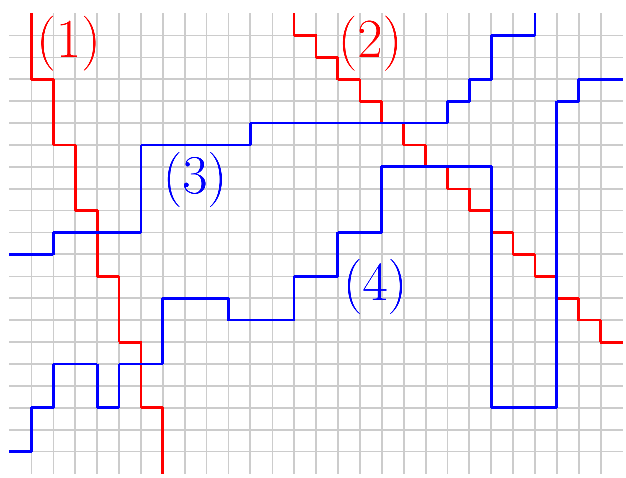

Figure 2: diagonals

3. Restricted initial conditions

Here again we use the $D = 2$ example, to make things simple. Given a line of initial conditions, it is possible to calculate the values y all over the D-dimensional space.

We have a well defined evolution, since we restrict ourselves to regular diagonals. Moreover, and this is a crucial point, *if we want to evaluate the transformation formula for a finite number of iterations, we only need a diagonal of initial conditions with finite extent.*

For any positive integer N, and each pair of relative integers $[\lambda_1, \lambda_2]$, we denote by $\Delta^{(N)}_{[\lambda_1,\lambda_2]}$, a regular diagonal consisting of N steps, each having horizontal size $l_1 = |\lambda_1|$, height $l_2 = |\lambda_2|$, and going in the direction of positive (resp. negative) n_k, if $\lambda_k > 0$ (resp. $\lambda_k < 0$), for $k = 1..D = 2$. See Figure (3).

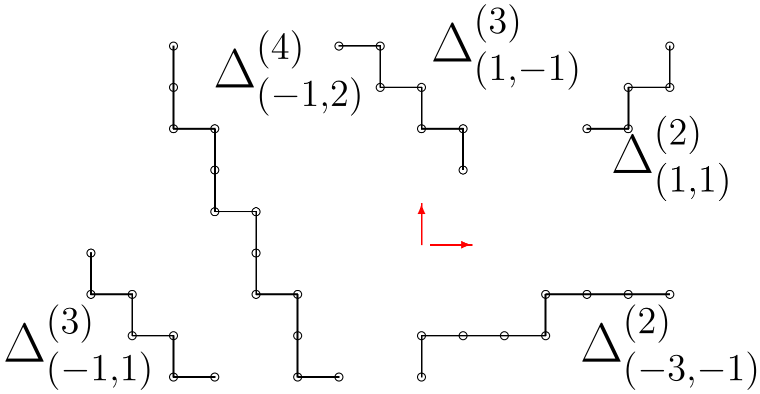

Figure 3: various choices of restricted initial conditions

Suppose we fix the initial conditions on $\Delta^{(N)}_{[\lambda_1,\lambda_2]}$. We may calculate y over a rectangle of size $(Nl_1 + 1) \times (Nl_2 + 1)$. The diagonal cuts the rectangle in two halves. One of them uses all initial values, and we will calculate the evolution only on that part. See Figure(4).

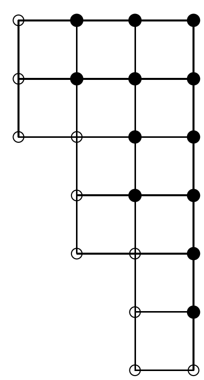

Figure 4: range of initial conditions $\Delta^{(3)}_{[-1,2]}$

4. Fundamental entropies

We are now in position to calculate "iterates" of the evolution. Choose some restricted diagonal $\Delta^{(N)}_{[\lambda_1,\lambda_2]}$. The total number of initial points is $q = N(l_1 + l_2) + 1$.

For such restricted initial data, the natural space where the evolution acts is the projective space P_q of dimension q. We may calculate the iterates and fill Figure (4), considering the q initial values as inhomogeneous coordinates of P_q.

Evaluating the degrees of the successive iterates, we will produce double sequences of degrees.

The simplest possible choice is to apply this construction to the restricted diagonals $\Delta^{(N)}_{[\pm 1, \pm 1]}$, which we will denote $\Delta^{(N)}_{++}$, $\Delta^{(N)}_{+-}$, …, and call them fundamental diagonals (the upper index (N) is omitted for infinite lines).

The pattern of degrees is then of the form

$$\begin{array}{cccccc} 1 & d^{(1)} & d^{(2)} & \dots & d^{(N-1)} & d^{(N)} \\ 1 & 1 & d^{(1)} & d^{(2)} & \dots & d^{(N-1)} \\ & 1 & 1 & d^{(1)} & d^{(2)} & \dots \\ & & 1 & 1 & d^{(1)} & d^{(2)} \\ & & & 1 & 1 & d^{(1)} \\ & & & & 1 & 1 \end{array} \tag{2}$$

To each choice of indices $[\pm 1, \pm 1]$ we associate a sequence of degrees $d^{(n)}_{\pm\pm}$.

Definition: The fundamental entropies of the lattice equation are given by

$$\epsilon_{\pm\pm} = \lim_{n \to \infty} \frac{1}{n} \, log(d^{(n)}_{\pm\pm}). \tag{3}$$

Claim: These entropies always exist [2], because of the subadditivity property of the logarithm of the degree of composed maps.

The fundamental entropies correspond to initial data given on diagonals with slope $+1$ or -1, and evolutions towards the four corners of the lattice, as shown in Figure (5).

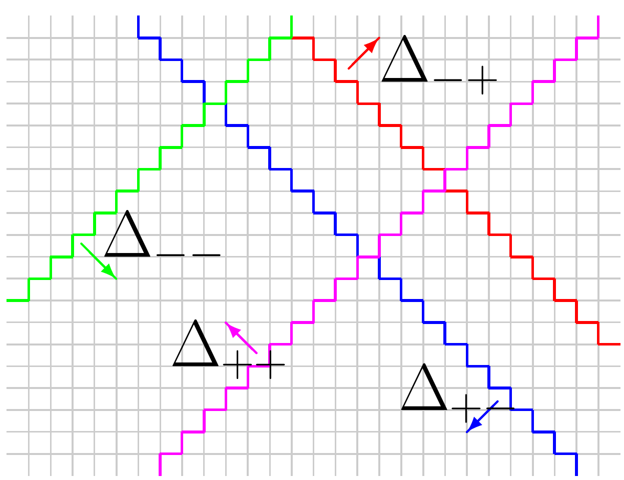

Figure 5: fundamental evolutions on a square lattice

These four entropies do not have to be identical (see section (10)).

When the entropy vanishes, the growth of the degree is polynomial, and the degree of that polynomial is a secondary characterization of the complexity.

5. Subsidiary entropies

We may also define entropies for the other regular diagonals. They correspond to initial data given on a line with a slope different from ± 1. They are useful in view of the various finite dimensional reductions presented for example in [9].

As an example, the pattern of degrees for $\Delta^{(3)}_{[-1,2]}$ looks like

$$\begin{bmatrix} 1 & d^{(1)}[1] & d^{(2)}[1] & d^{(3)}[1] = d^{(6)}[2] \\ 1 & \dots & \dots & d^{(5)}[2] \\ 1 & 1 & \dots & d^{(4)}[2] \\ & 1 & \dots & d^{(3)}[2] \\ & 1 & 1 & d^{(2)}[2] \\ & & 1 & d^{(1)}[2] \\ & & 1 & 1 \end{bmatrix} \tag{4}$$

The sequences of degrees we will retain are the border sequences $\{d^{(n)}[\nu]\}$, $\nu = 1, 2$, seen on the edges of the domain. There are as many such border sequences as there are dimensions in the lattice (here $D{=}2$). The index in bracket refers to the direction of the edge considered. This leads to subsidiary entropies $\epsilon_{[\lambda_1,\dots,\lambda_D]}[\nu]$, $\nu = 1 \dots D$:

$$\epsilon_{[\lambda_1,\dots,\lambda_D]}[\nu] = \lim_{n\to\infty} \frac{1}{n} \log(d^{(n)}_{[\lambda_1,\dots,\lambda_D]}[\nu]). \tag{5}$$

6. Explicit calculation

A full calculation of iterates is usually beyond reach. We can however get explicit sequences of degrees by considering the images of a generic projective line in P_q, as was introduced in [1], making a link with the geometrical picture of [3]:

Suppose we start from a restricted diagonal $\Delta^{(N)}_{[\lambda_1,\lambda_2]}$. It contains $q = N(l_1+l_2)+1$ vertices $V_1, \dots, V_q$. For each of these q vertices, we assign to y an initial value of the form:

$$y_{[V_k]} = \frac{\alpha_k + \beta_k\, x}{\alpha_0 + \beta_0\, x}, \qquad k = 1 \dots q \tag{6}$$

where α_0, β_0 and α_k, β_k, $(k = 1..q)$ are arbitrary constants, and x is some unknown. We then calculate the values of y at the vertices which are within the range of $\Delta^{(N)}_{[\lambda_1,\lambda_2]}$. These values are rational fractions of x, whose numerator and denominator are of the same degree, and that is the degree we are looking for.

The next step is then to evaluate the growth of degrees. One very fruitful method is to introduce the generating function of the sequence of degrees

$$g(s) = \sum_{k=0}^{\infty} s^k\, d^{(k)} \tag{7}$$

and try to fit it with a rational fraction.

The remarkable fact is that it again works surprisingly well, as it did for maps, although we know that it may not always be the case [10]. This means that we can often extract the asymptotic behaviour measured by (3) and (5) just by looking at a finite part of the sequence of degrees.

The existence of a rational generating function with integer coefficients for the sequence of degrees implies that it verifies a finite recurrence relation. For maps this can sometimes be proved through a singularity analysis. A similar singularity analysis should be done here, but it is beyond the scope of this letter.

We have explored a number of examples. We found examples with non zero entropy, and examples with vanishing entropy either with linear growth either with quadratic growth, up to now.

7. Example 1: the deformed cross-ratio relation

Take as a defining relation:

$$f_{dcr} = \frac{\left(y_{[n_1,n_2]} - a\, y_{[n_1+1,n_2]}\right)\left(y_{[n_1,n_2+1]} - b\, y_{[n_1+1,n_2+1]}\right)}{\left(y_{[n_1,n_2]} - c\, y_{[n_1,n_2+1]}\right)\left(y_{[n_1+1,n_2]} - d\, y_{[n_1+1,n_2+1]}\right)} - s = 0 \tag{8}$$

This relation is based on a deformed version of the cross ratio of the four corner values. It is known to define an integrable lattice equation for $a = b = c = d$ [11].

At generic values of the parameters we get the following explicit values for $d_{\pm\pm}^{(n)}$ (the defining relation being very symmetric, the four sequences are the identical):

$$\{d_{\pm\pm}^{(n)}\} = \{1, 2, 4, 9, 21, 50, 120, 289, \ldots\}. \tag{9}$$

This sequence is fitted by the generating function

$$g_{\pm\pm}^{dcr}(s) = \frac{1 - s - s^2}{(1-s)\,(1 - 2\,s - s^2)}. \tag{10}$$

The entropy is the logarithm of the inverse of the modulus of the smallest pole of $g(s)$

$$\epsilon_{\pm\pm}^{dcr} = \log(1 + \sqrt{2}). \tag{11}$$

We have calculated a number of subsidiary entropies for various values of $[\lambda_1, \lambda_2]$. They all give sequences which can be fitted with rational generating functions. The entropies depend on the "slope" $\sigma = \lambda_2/\lambda_1$: large σ give larger $\epsilon_{[\lambda_1,\lambda_2]}[1]$ and smaller $\epsilon_{[\lambda_1,\lambda_2]}[2]$, and conversely. As an example, we see that $\epsilon_{[p,1]}[1]$ is the inverse of the logarithm of the smallest modulus of the roots of $1 - s - s^p - s^{p+1}$. There is an interesting interplay between the various fundamental and subsidiary entropies.

For the known integrable case (parameters $a = b = c = d$) [11], we find the same sequence for all four $\Delta_{\pm\pm}$:

$$\{d_{\pm\pm}^{(n)}\} = \{1, 2, 4, 7, 11, 16, 22, 29, 37, 46, 56, \ldots\}, \tag{12}$$

fitted by

$$g_{\pm\pm}^{int}(s) = \frac{1 - s + s^2}{(1-s)^3} \tag{13}$$

The growth of the degree is quadratic

$$d_{\pm\pm}^{(n)} = 1 + n\,(n+1)\,/2 \tag{14}$$

and the entropy vanishes.

The few subsidiary entropies we have calculated when $a = b = c = d$ also vanish, and the degree growth is quadratic.

8. Example 2: Q_4

We have analysed the so-called Q_4 lattice equation [12, 13, 14]. The defining relation is given by:

$$\begin{aligned} &A\left(\left(y_{[n_1,n_2]}-b\right)\left(y_{[n_1,n_2+1]}-b\right)-d\right)\left(\left(y_{[n_1+1,n_2]}-b\right)\left(y_{[n_1+1,n_2+1]}-b\right)-d\right)\\ &+B\left(\left(y_{[n_1,n_2]}-a\right)\left(y_{[n_1+1,n_2]}-a\right)-e\right)\left(\left(y_{[n_1,n_2+1]}-a\right)\left(y_{[n_1+1,n_2+1]}-a\right)-e\right)=f \end{aligned} \quad (15)$$

with

$$\begin{aligned} &d=(a-b)\,(c-b)\\ &e=(b-a)\,(c-a)\\ &f=A\,B\,C\,(a-b)\\ &A\,(c-b)+B\,(c-a)=C\,(a-b) \end{aligned} \quad (16)$$

We have used our approach for generic values of the parameters, that is to say *without* (16). We find for the fundamental evolutions:

$$\{d_{\pm\pm}^{(n)}\}=\{1,3,7,13,21,31,43,57,73,91,111,\ldots\} \quad (17)$$

fitted with the generating function

$$g_{\pm\pm}(s)=\frac{1+s^2}{(1-s)^3}. \quad (18)$$

The growth of the degree is quadratic,

$$d_{\pm\pm}^{(n)}=1+n(n-1) \quad (19)$$

This indicates integrability of the form (15) !

It is interesting to calculate more of the entropies, related to initial conditions with a different slope, still for unconstrained parameters. For example

$$\{d_{[1,2]}^{(n)}[1]\}=\{1,5,13,25,41,61,85,113,\ldots\} \quad (20)$$

$$\{d_{[1,2]}^{(n)}[2]\}=\{1,3,5,9,13,19,25,33,41,51,61,73,85,99,113,\ldots\} \quad (21)$$

Both give zero entropy and quadratic growth, as for the fundamental values.

9. Example 3: Discrete Sine-Gordon

A multilinear defining relation for the discrete Sine-Gordon equation can be found in [15, 9]:

$$\begin{aligned} &y_{[n_1,n_2]}\,y_{[n_1+1,n_2]}\,y_{[n_1,n_2+1]}\,y_{[n_1+1,n_2+1]}\\ &\qquad -a\,\left(y_{[n_1,n_2]}\,y_{[n_1+1,n_2+1]}-y_{[n_1+1,n_2]}\,y_{[n_1,n_2+1]}\right)-1=0 \end{aligned} \quad (22)$$

For this lattice equation, we find

$$\{d_{\pm\pm}^{(n)}\}=\{1,3,7,13,21,31,43,57,73,91,111,\ldots\} \quad (23)$$

as in the previous case (the same quadratic growth, vanishing $\epsilon_{\pm\pm}$).

The subsequent calculation of $\epsilon_{[1,2]}[1]$ and $\epsilon_{[1,2]}[2]$ yields

$$\{d_{[1,2]}^{(n)}[1]\}=\{1,4,11,21,34,51,71,94,121,151\ldots\} \quad (24)$$

$$\{d_{[1,2]}^{(n)}[2]\}=\{1,3,4,8,11,16,21,28,34,43,51,61,71\ldots\} \quad (25)$$

fitted respectively by

$$g^{sg}_{[1,2]}[1] = \frac{1+2\,s+4\,s^2+2\,s^3+s^4}{(s^2+s+1)\,(1-s)^3} \tag{26}$$

$$g^{sg}_{[1,2]}[2] = \frac{1+2\,s+s^3+s^5}{(s+1)\,(s^2+s+1)\,(1-s)^3} \tag{27}$$

Which mean vanishing entropies and quadratic growth.

10. Example 4: Non-isotropic model

There are cases where the various directions of evolution are not equivalent. The entropies $\epsilon_{\pm\pm}$ are not all equal. Take the simple defining relation (see also [16, 17]):

$$y_{[n_1,n_2+1]}y_{[n_1,n_2]}y_{[n_1+1,n_2]} + y_{[n_1,n_2+1]}y_{[n_1+1,n_2+1]} + y_{[n_1+1,n_2]} = 0 \tag{28}$$

The sequences of degrees for the fundamental evolutions differ:

$$\{d^{(n)}_{[-+]}\} = \{1, 3, 7, 17, 41, 99, 239, \ldots\} \tag{29}$$

$$\{d^{(n)}_{[++]}\} = \{1, 2, 4, 7, 14, 28, 56, \ldots\} \tag{30}$$

$$\{d^{(n)}_{[+-]}\} = \{1, 2, 5, 10, 20, 40, 80, \ldots\} \tag{31}$$

$$\{d^{(n)}_{[--]}\} = \{1, 2, 4, 8, 16, 32, 64, \ldots\} \tag{32}$$

They fit with the generating functions

$$g_{[-+]} = \frac{1+s}{1-2\,s-s^2}, \qquad g_{[++]} = \frac{(1-s)(1+s+s^2)}{1-2s}, \tag{33}$$

$$g_{[+-]} = \frac{1+s^2}{1-2\,s}, \qquad g_{[--]} = \frac{1}{1-2\,s}, \tag{34}$$

so that

$$\epsilon_{++} = \epsilon_{+-} = \epsilon_{--} = \log(2) \tag{35}$$

but

$$\epsilon_{-+} = \log(2.414...) \tag{36}$$

It does not seem excluded a priori to have vanishing entropy in some direction and non-vanishing entropy in some other direction, but we have not exhibited any explicit example of that yet.

11. Conclusion and perspectives

The definitions presented here extend to all dimensions ($D > 2$), and apply to non-autonomous equations, as well as multicomponent systems, provided the evolutions are rational.

We may use a defining relation which is not multilinear if we do not insist on having all of the 2^D evolutions described in section (5). For $D = 2$ we may for example accept a defining relation which is of higher degree in $y_{[n_1+1,n_2]}$ and $y_{[n_1,n_2+1]}$. The price to pay is to consider only initial conditions with $\lambda_1\lambda_2 < 0$.

We may also consider defining relations extending over more than one elementary cell: this means considering equations of higher order [18].

We have done a number of explicit calculations of the fundamental (and subsidiary) entropies, beyond the ones presented here. All the lattice equations which are known to be integrable have vanishing entropies. These results, and what is already known for maps, suggest that the vanishing of entropy, that is to say the drastic drop of the complexity of the evolution, is indeed always a sign of integrability.

Of course the drop of the degree is related to the singularity content of the evolution, as it was for maps. A systematic singularity/factorization analysis is one of the tracks to follow. In this spirit, one can undertake a description of all lattice equations of a given degree, and for a given dimension. This is the purpose of [16, 17].

One should also determine which of the properties are canonical, that is to say independent of any change of coordinates one may perform on the variables. For maps this was just invariance by birational changes of coordinates on a finite dimensional projective space. Here the situation is made much more intricate by the infinite number of dimensions of the space of initial conditions.

All the entropies we have calculated explicitly are the logarithm of an algebraic integer. There is a conjecture that this is always the case for maps [2, 10], and we are lead to the same conjecture here.

Another line of research touches upon arithmetic: a link has been established between the algebraic entropy and the growth of the height of iterates, when the parameters and the values of $y_{[n_1,\dots,n_D]}$ are rational numbers [19, 20, 21]. This applies here as well, and will be the subject of further analysis.

Acknowledgment: I would like to thank M. Talon for constructive discussions.

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