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Discrete Painlevé I and singularity confinement in projective space

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Abstract

We write the discrete Painlevé I equation as a polynomial map in projective space and follow the development of the singularity and the fate of the other initial value. Most of the time the initial value is at the next to leading term and we can recover it (and confine the singularity) at the $(3m+1)$th iteration by imposing a condition on the de-autonomization. For $m=1$ one gets the usual d-P_I.

The integrability of a dynamical system is a strong property with many implications on its evolution, including regularity and long time predictability. When one is faced with a new dynamical system it is therefore of interest to find out whether the system is integrable or not. First indications can be obtained numerically, but usually such indications are negative: the only thing one can say that is if one can see numerical chaos the system is probably not integrable. It is in fact a demanding task to actually prove that a system is integrable, so much so that before starting the job one would like to have some hint that the system might be integrable.

For continuous systems there is the "Painlevé test" [1], which is algorithmic and very accurate (although known to fail in some cases). For discrete systems an analogous test was proposed in [2] and it has since then been used productively to identify integrable maps (see, e.g, [3]). In this *singularity confinement test* one again analyses behavior around a movable singularity of the map. But what is a singularity in this case? As has been stressed by Kruskal in many occasions, infinity itself is not a singularity (and this will be obvious if one uses projective variables, as will be done below). The singularities one is faced with in discrete systems, are ambiguous quantities like $\infty - \infty$, $0 \cdot \infty$. If the map in question leads to such an ambiguity, one should next study the behavior around the singularity, and if the map can be continued in a way which allows one to exit from the singularity, after a finite number of steps and without loss of information, then the system is said to pass the test.

Here we will show that for d-P_I type equations one can control *when* one exits the singularity by the type of nonautonomity. However, it turns out that if the nonautonomity does not allow the exit at the first opportunity then the map is not integrable, although it passes the test.

(After this conference we have studied the problem further and we have found an example which is autonomous and passes the test, but nevertheless shows numerical chaos [4]).

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Let us consider a discrete version of Painlevé's first equation, the d-P_I [3]. One version is

$$x_{n+1} + x_n + x_{n-1} = \frac{a_n}{x_n} + b. \tag{1}$$

Question: For which function(s) a_n does the equation pass the singularity confinement test?

In order to answer this question we first write the system as a polynomial map in the projective space $\mathbb{P}^3$. We start by writing (1) as a first order system

$$\begin{aligned} x_{n+1} &= -x_n - y_n + \frac{a_n}{x_n} + b, \\ y_{n+1} &= x_n, \end{aligned} \tag{2}$$

and then homogenizing by substituting $x_n = u_n/f_n,\ y_n = v_n/f_n$ (and using the least common multiplier of the denominators of the right-hand side for defining the map for f) we get the polynomial map

$$\begin{aligned} u_{n+1} &= -u_n(u_n + v_n) + f_n(a_n f_n + b u_n), \\ v_{n+1} &= u_n^2, \\ f_{n+1} &= f_n u_n. \end{aligned} \tag{3}$$

The inverse map is

$$\begin{aligned} u_{n-1} &= v_n^2, \\ v_{n-1} &= -v_n(v_n + u_n) + f_n(a_{n-1} f_n + b v_n), \\ f_{n-1} &= f_n v_n. \end{aligned} \tag{4}$$

For the original map (1) the sequence that leads to a singularity is

$$x_{-1} = x,\ x_0 = 0,\ x_1 = \infty,\ x_2 = \infty - \infty = \ ? \tag{5}$$

where the initial value x is free. The corresponding sequence in projective space $\mathbb{P}^3$ is

$$\begin{pmatrix} 0 \\ x \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{6}$$

where the last term is a true singularity, since it is not in $\mathbb{P}^3$. Note that here an infinite value of x_n corresponds to a zero in the third component of the vector. The ambiguity appears later in (6) than in (5), and manifests itself by taking us outside $\mathbb{P}^3$.

To study in detail the character of the singularity let us start with $x_{-1} = x,\ x_0 = \epsilon$, which means the initial configuration

$$\begin{pmatrix} u_0 \\ v_0 \\ f_0 \end{pmatrix} = \begin{pmatrix} \epsilon \\ x \\ 1 \end{pmatrix}.$$

During computations it is necessary to keep terms up to order ϵ^6 although we display only the leading term or two.

$$\begin{pmatrix} u_1 \\ v_1 \\ f_1 \end{pmatrix} = \begin{pmatrix} a_0 + (-x + b)\epsilon + \cdots \\ \epsilon^2 \\ \epsilon \end{pmatrix}$$

Note how the information about the initial value has already moved into the non-leading term.

$$\begin{pmatrix} u_2 \\ v_2 \\ f_2 \end{pmatrix} = \begin{pmatrix} -a_0^2 + \epsilon a_0(2x - b) + \cdots \\ a_0^2 + 2\epsilon a_0(-x + b) + \cdots \\ \epsilon a_0 + \epsilon^2(-x + b) + \cdots \end{pmatrix}$$

$$\begin{pmatrix} u_3 \\ v_3 \\ f_3 \end{pmatrix} = \begin{pmatrix} \epsilon^2 a_0^2(-a_0 + a_1 + a_2) + \cdots \\ a_0^4 + 2\epsilon a_0^3(-2x + b) \cdots \\ -\epsilon a_0^3 + \epsilon^2 a_0^2(3x - 2b) + \cdots \end{pmatrix}.$$

Next we finally reach the potential singularity $(0, 0, 0)$:

$$\begin{pmatrix} u_4 \\ v_4 \\ f_4 \end{pmatrix} = \begin{pmatrix} \epsilon^2 a_0^6 A_3 + \epsilon^3 a_0^5 (b(4A_3 + a_0 - a_2) - x(6A_3 + a_0)) + \cdots \\ \epsilon^4 a_0^4 A_2^2 + \cdots \\ -\epsilon^3 a_0^5 A_2 + \cdots \end{pmatrix}, \tag{7}$$

where we have used the shorthand notation $A_2 = a_2 + a_1 - a_0$ and $A_3 = a_0 - a_1 - a_2 + a_3$.

This is the crucial point of singularity confinement. We can see from (7), that *if* $A_3 = 0,\ A_2 \neq 0$ then ϵ^3 is a common factor and can be divided out. Then we can take the $\epsilon \to 0$ limit and get (after factoring out $-a_0^5$)

$$\begin{pmatrix} u_4 \\ v_4 \\ f_4 \end{pmatrix} = \begin{pmatrix} (a_0(x - b) + a_2 b) \\ 0 \\ a_3 \end{pmatrix}.$$

Thus we have emerged from the singularity and in particular recovered the initial data x. The condition $A_3 = 0$ means

$$a_{n+3} - a_{n+2} - a_{n+1} + a_n = 0,$$

which has the solution

$$a_n = \alpha + \beta n + \gamma(-1)^n. \tag{8}$$

However, it is not necessary that we exit from the singularity at the fourth iteration. If we assume that $A_3 \neq 0$ we may still divide out $\epsilon^2 a_0^4$ and iterate further. We get fairly complicated intermediate expressions and then at the seventh iteration there is another possibility of confining the singularity, we get first

$$\begin{pmatrix} u_7 \\ v_7 \\ f_7 \end{pmatrix} = \begin{pmatrix} \epsilon^2 A_0^{14} A_2^2 A_3^6 A_6 + \epsilon^3 A_0^{13} A_2 A_3^5 (bM + xN + A_6(\ldots)) + \cdots \\ \epsilon^4 A_0^{12} A_2^4 A_3^4 A_5^2 + \cdots \\ \epsilon^3 A_0^{13} A_2^3 A_3^5 A_5 + \cdots \end{pmatrix},$$

where
$M = -4A_2A_3^2 + A_2^2A_5 - A_2^2A_4 - A_2^2A_3 + 2A_1A_3^2 + A_1A_2A_3 + 2A_0A_3^2 + A_0A_2A_3,\ N = A_3(4A_2A_3 - 2A_0A_3 - A_0A_2)$.

If now $A_6 = 0$ we can divide out ϵ^3 and again confine the singularity. The condition is

$$A_6 \equiv a_6 - a_5 - a_4 + a_3 - a_2 - a_1 + a_0 = 0, \tag{9}$$

and it can be solved by assuming $a_n = a_0 X^n$ which leads to the characteristic equation

$$X^6 - X^5 - X^4 + X^3 - X^2 - X + 1 = 0.$$

Note that 1 is not a root of this equation.

This can be generalized, the singularity can be confined at any $(3k + 1)$th step, and the confinement condition leads to the characteristic equation

$$X^{3k} - X^{3k-1} - X^{3k-2} + \cdots + X^3 - X^2 - X + 1 = 0.$$

Unfortunately it turns out that the non-autonomous systems that confine later are in fact *not* integrable. For these systems the degree of the map grows exponentially (as opposed to polynomially with confinement at the first possibility).

Notice also that the solution a_n of (9) cannot be considered as a discretization of continuous Painlevé equation, since it does not have a continuum limit.

If we analyze the degree growth and define the ''algebraic entropy'' along the lines described in [4,5], we get for the two first solutions (8) and (9) the following sequences of degrees, respectively:

$$\sigma_1 = 1, 2, 4, 8, 13, 20, 28, 38, 49, 62, 76, \ldots$$
$$\sigma_2 = 1, 2, 4, 8, 14, 24, 40, 65, 104, 164, 258, 404, 632, 986, 1537, 2394, 3728, 5804, 9034, \ldots$$

Sequence σ_1 corresponds to a polynomial growth of the degree: $d_n = \frac{1}{8}(6n^2 + 9 - (-1)^n)$, and vanishing entropy (integrable case). On the contrary, sequence σ_2 has a generating functional

$$\frac{1 + 2z^3 + 2z^6}{(1 - z)(1 - z - z^2 + z^3 - z^4 - z^5 + z^6)}. \tag{10}$$

It has non-vanishing entropy $\mathscr{E} \equiv \lim_{n\to\infty} 1/n \log(d_n) = \log(\alpha^{-1})$, where α is the root of the denominator of (10) with the smallest modulus. [In this case we have, approximately, $\epsilon = 0.44214\ldots$, i.e., $d_n \sim (1.55603\ldots)^n$.]

Thus the singularity confinement test does impose conditions on discrete maps, but these examples (and further results in [4]) show that it is not restrictive enough. We propose the degree growth of the system as a better indication of integrability: If the growth is polynomial then the system is likely to be integrable.

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