Algebraic entropy: measuring the complexity of rational dynamics

Claude Viallet
Centre National de la Recherche Scientifique
Universités de Paris 6 et Paris 7.

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We will deal with discrete-time systems with a finite number of degrees of freedom and a (bi)-rational evolution. We will associate a canonical invariant to any (bi)-rational map of projective space of any dimension. This invariant characterizes the complexity of the map. It may be used as an “integrability detector”. It also has some very interesting properties by itself.

- Examples and typical construction
- Definition of the entropy
- How to calculate the entropy in practice
- Two geometrical descriptions: intersections and blowdown-blowup
- Perspectives, open questions
Two examples in the 2-plane

Consider the basic following plane (bi)-rational involutions \((i_5, i_7, j)\):

\[
i_5 : (u, v) \longrightarrow \left( u' = \frac{u^2 - v^2 + u - uv}{-1 - u - v + u^2 + v^2 + uv}, v' = \frac{v^2 - u^2 + v - uv}{-1 - u - v + u^2 + v^2 + uv} \right)
\]

\[
i_7 : (u, v) \longrightarrow \left( u' = \frac{2v^2 - u - u^2}{1 + 2u + 2v - u^2 - v^2 - 3uv}, v' = \frac{2u^2 - v - v^2}{1 + 2u + 2v - u^2 - v^2 - 3uv} \right)
\]

\[
j : (u' = 1/u, v' = 1/v)
\]

And construct the two birational infinite order maps

\[
\varphi_5 = i_5 \cdot j \quad \text{and} \quad \varphi_7 = i_7 \cdot j
\]

Draw a few generic orbits of respectively \(\varphi_5 = i_5 \cdot j\) and \(\varphi_7 = i_7 \cdot j\).
Graphical analysis

Although the two involutions are constructed in a very similar way (inversions of cyclic matrices) and come from comparable algebraic structures of bialgebras, the behaviour of the iterates is completely different.
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One of our goals is to detect the integrability of the map: a priori just the existence of invariants, since we do not have symplectic structure given with the map.
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Even if it sometimes works beyond expectations, $\mathbb{P}^{15}$, $\mathbb{P}^9$, graphical analysis does not take us very far, in particular if the number of parameters gets bigger. We will define a canonical object associated to any (bi)-rational map of projective space of any dimension: the algebraic entropy.
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Even if it sometimes works beyond expectations, $P_{15}, P_9$, graphical analysis does not take us very far, in particular if the number of parameters gets bigger. We will define a canonical object associated to any (bi)-rational map of projective space of any dimension: the algebraic entropy.

The idea is to look at the growth of the degree of the iterates of the evolution map.

The first step is to write the transformation in a more canonical way, using projective space, and the $(n + 1)$ homogeneous coordinates $\{x_0, x_1, \ldots, x_n\}$ for $n$-dimensional projective space.

A rational transformation $\varphi$ may then be written as a polynomial transformation in the homogeneous coordinates. If one factors out any common polynomial factors, the degree is well defined, in a given system of coordinates, but is not invariant by changes of coordinates.

\[
\begin{align*}
  x_0 & \longrightarrow \phi_0(x_0, x_1, \ldots, x_n) \\
  x_1 & \longrightarrow \phi_1(x_0, x_1, \ldots, x_n) \\
  \vdots & \\
  x_n & \longrightarrow \phi_n(x_0, x_1, \ldots, x_n)
\end{align*}
\]
Definition

Let $d_n$ the degree of the $n$-th iterate of $\varphi$. Define the entropy

$$
\epsilon = \lim_{n \to \infty} \frac{1}{n} \log(d_n).
$$

Since for any pair of birational maps $\varphi$ and $\psi$,

$$
d_{\psi \circ \varphi} \leq d_{\psi} \cdot d_{\varphi},
$$

**Proposition:** The entropy is always defined and is invariant under changes of coordinates: it is a birational invariant associated to the transformation $\varphi$. 
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The interest is that it is calculable exactly for a number of maps, sometimes heuristically, sometimes with a complete proof.
Images of a line, Number of intersections

We may try to calculate the iterates directly, but this a rapidly growing process, and your favorite formal calculation software will not help. It is equivalent, and much more efficient to calculate the successive images of a generic line and look at the degrees of the coordinates of the successive iterates. One gets in this way the sequence \( \{d_n\} \).
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The geometrical meaning of this sequence is simple: one is counting the number of intersections of the images with a fixed generic hyperplane, which coincides with the degree of the images since we work with complex projective space. This is also a particular case of a notion of complexity of maps that was introduced by Arnold for diffeomorphisms.
Take for example the case of the Hénon map:

\[ [x, y, z] \longrightarrow [x^2, x^2 + xz - \alpha y^2, \beta xy] \]

We get the sequence

\[ 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, ??? \]

The game is to complete the list in a reasonable way.
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One first try is to calculate the discrete derivatives of the sequence.

\[ d'_n = d_{n+1} - d_n, \quad d''_n = d'_{n+1} - d'_n, \quad \ldots \]
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Another possibility is to write down the generating function of the sequence

\[g(s) = \sum_{k=0}^{\infty} d_k s^k\]

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That is easy for Hénon, where you have guessed :

\[
d_n = 2^n, \quad d'_n = d_n, \quad g_{\text{Hénon}} = \frac{1}{1 - 2s}.
\]
The remarkable result is that this method works in many cases and the generating function we find is a rational fraction with integer coefficients!

- $Z_5$: iterates of alternating $j$ and $i_5$
  1, 2, 4, 7, 12, 18, 25, 34, 44, 55, 68, 82, 97, 114, ...

- $Z_7$: iterates of alternating $j$ and $i_7$
  1, 2, 4, 7, 12, 20, 33, 54, 88, 143, 232, 376, 609, 986, 1596, 2583 ...

- Another map in $P_5$ again from matrix inversions
  1, 3, 15, 25, 65, 83, 167, 193, 337, 371, 591, 633, 945, 995, 1415, 1473, ...

For these examples:

\[
g_{Z_5} = \frac{1 + s^2 + 2 s^4}{(s^2 + s + 1) (1 - s)^3}, \quad g_{Z_7} = \frac{1}{(1 - s)(1 - s - s^2)},
\]

\[
g_3 = \frac{1 + 2 s + 9 s^2 + 4 s^3 + 7 s^4 - 6 s^5 - s^6}{(1 + s)^3 (1 - s)^4}
\]
but if we take a well chosen monomial map (J. Propp):

\[ [x, y, z, t] \longrightarrow [yt, zt, x^2, xt] \]

we get the list:

1, 2, 3, 4, 6, 9, 12, 17, 25, 33, 45, 65, 85, 112, 159, 215, 262, 365, 524, 627, 833, 1198, 1404, 1760, 2537, 3415, 3937, 5507, 8481, 11455, 16881, 25281, 33681, 47426, 69571, 91716, 124470, 179369, 234268, 307249, 435129, 593006,...

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\[g_{\text{Propp}} = ?\]

It is not possible to fit the sequence with a rational fraction. It is however possible to show that the entropy is the log of the smallest modulus of the roots of \(1 + s + s^2 - s^3\), i.e. the log an algebraic integer.

In all known cases the entropy, either inferred form the rational generating function, either calculated directly (as in the case of the above toral automorphism) is the log of an algebraic integer.
The last example came from the following question: is the “forward entropy” (i.e. the entropy of the map $\varphi$) the same as the “backward entropy”, (i.e. the one of the inverse map $\varphi^{-1}$). The answer is no, since the sequence of degrees for the inverse map is 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, ... and it can be fitted with the rational generating function

$$g_{\text{inverse}} := \frac{(1 + s + s^2)}{(1 - s - s^2 - s^3)}.$$ 

The nature of the generating functions, as well as the location of the smallest of its poles differ.
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This property comes more than often from the stronger property that the generating function of the sequence of degrees of the iterates is a rational fraction.

The latter means that the sequence of degree verifies a finite recurrence relation with integer coefficients, and it is possible to prove this property in some cases (see later).
The geometrical meaning of the drop of the degree

We have a natural bound on the degrees of the iterates:

\[ d_n \leq d_1^m \]

We are working with projective space: we have to factor out any common polynomial factor. If this happens, it yields a drop of the degree, and the bound is not saturated anymore.
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We are working with projective space: we have to factor out any common polynomial factor. If this happens, it yields a drop of the degree, and the bound is not saturated anymore.

This has to do with the singularity structure of the map: indeed a birational map is not one to one everywhere: some points are singular and the map is not invertible along some varieties. The singular varieties are at least of codimension 2 (at most points in \( P_2 \), lines in \( P_3 \), etc.). We may have something like:
Write $\varphi$ and its inverse $\varphi^{-1}$ as homogeneous polynomial maps. The composed maps $\varphi \cdot \varphi^{-1}$ and $\varphi^{-1} \cdot \varphi$ are then just multiplication of all coordinates by some polynomial $\kappa_\varphi$ and $\kappa_{\varphi^{-1}}$

$$\varphi^{-1} \cdot \varphi(m) = \kappa_\varphi(m).id(m) \quad \text{and} \quad \varphi \cdot \varphi^{-1}(m) = \kappa_{\varphi^{-1}}(m).id(m)$$
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The equation of the surface $\Sigma$, (resp. $\Sigma'$), is $\kappa_\varphi(m) = 0$ (resp. $\kappa_{\varphi^{-1}}(m) = 0$). A typical scheme is the one of the above picture.
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It is easy to convince oneself that, if the singularity pattern is the one shown above, some factors will appear at the fourth and fifth iteration of $\varphi$, and those are powers of $\Sigma$ (or of pieces of $\Sigma$ if it is decomposable): indeed all the points of $\Sigma$ give $[0, 0, \ldots, 0]$ when $\varphi$ is iterated four times, and again at the next step.
Vanishing and non-vanishing entropy

The drop of the degree may be so important that the growth of $\{d_n\}$ is not exponential anymore. The sequence may be bounded, or the growth may be polynomial.

Conjecture: The degree can only have exponential or polynomial growth, or be bounded.

In the case the growth is polynomial, $\epsilon = 0$, and the degree of this polynomial is another birational invariant of the map.

We have explicit examples with linear, quadratic, cubic growth.

It has been shown that if the orbits are confined to one-dimensional varieties, the growth is quadratic: this will be the case for $N$-dimensional maps having $N - 1$ independent rational invariants (we have explicit examples for various values of $N$, and the curves shown above are among two such examples).
Removal of singularities

It is known that given a birational map $\varphi : X \to Y$, it is possible to remove the singularities and the non invertibility by blowing up a number of sub-varieties of $X$ and $Y$ respectively.

\[
\begin{array}{c}
\tilde{X} \\
\pi_X
\end{array}
\quad
\xrightarrow{\tilde{\varphi}}
\quad
\begin{array}{c}
\tilde{Y} \\
\pi_Y
\end{array}
\]

\[
\begin{array}{c}
X \\
\varphi
\end{array}
\quad
\longrightarrow
\quad
\begin{array}{c}
Y
\end{array}
\]

The problem is that we want to iterate the map, i.e. we need to have $\tilde{X} = \tilde{Y}$. 
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$$
\begin{array}{ccc}
\tilde{\varphi} & \tilde{X} & \rightarrow & \tilde{Y} \\
\pi_X & \downarrow & & \downarrow & \pi_Y \\
X & \rightarrow & Y & \varphi
\end{array}
$$

The problem is that we want to iterate the map, i.e. we need to have $\tilde{X} = \tilde{Y}$.

This leads us in general to an infinite sequence of successive blow-ups. Fortunately it is sometimes possible to realize this regularization with a finite number of blow ups\(^1\). We have in this case

$$
\begin{array}{ccc}
\tilde{\varphi} & \tilde{X} & \rightarrow & \mathcal{P} \\
\Pi & \downarrow & & \downarrow & \Pi \\
P & \rightarrow & P & \varphi
\end{array}
$$

\(^1\)This is related to what has been called “discrete singularity confinement”
The projection $\Pi$ is a product of a finite number of blow-ups. The resulting map $\tilde{\varphi}$ is a smooth map on a rational variety $\tilde{X}$.

The two-dimensional case is particularly interesting, because we can use intersection theory of curves drawn on two-dimensional varieties.
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The two-dimensional case is particularly interesting, because we can use intersection theory of curves drawn on two-dimensional varieties.

The Picard group of $P_2$ has one generator. Since the singular varieties always have codimension at least 2, we have to blow-up only points, and each blow up adds one generator to the Picard group.

There exists a (non-positive) scalar product on the Picard group $\text{Pic}(\tilde{X})$ of $\tilde{X}$. 
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The map $\tilde{\varphi}$ induces an isometry $\Phi^*$ on $Pic(\tilde{X})$. It is possible to represent the isometry $\Phi^*$ with a matrix $\mu(\varphi)$.

It is then possible to read the number of intersections of the images of a generic line under $\varphi$ from the powers of $\mu(\varphi)$. This proves the existence of a finite recurrence relation on the successive degrees: it is just the characteristic polynomial of $\mu(\varphi)$. 
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Remark: We also have examples in higher dimensions, where is is possible to prove that the sequence of degrees verifies a finite recurrence relation with integer coefficients.
Arithmetical approach

We have described two approaches:

- a heuristic method, where one calculates the first degrees of the sequence \( \{d_n\} \), and tries to infer the sequel. We do not get a proof, but a strong plausibility when the sequence is fitted by a reasonably simple rational generating function

- a serious singularity analysis, with a limitation to 2-dimensional maps, or some exceptional higher dimensional cases

How can we go further, in particular to high dimensions?
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How can we go further, in particular to high dimensions?

We have tried the following “arithmetical” approach: start from a rational point, and evaluate the iterates. In homogeneous coordinates, this means calculating with integers. We evaluate the size of the iterates, i.e. the number of digits one needs to write down the images.

The outcome is that the growth of this length is the same as the growth of the degree!
Perspectives, open questions

Specific

- Prove that $\exp(\epsilon)$ is an algebraic integer.
- Determine what generating functions are possible, beyond rational fractions.
- Characterize which values are possible for $\epsilon$.
- Characterize which values are possible for the secondary invariant $N$ when $\epsilon = 0$.
- Explore the arithmetical approach.
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• Explore the arithmetical approach.

General

• There are many more entropies: metric, topological, ....

  When they are calculable, how do they compare with algebraic entropy?

As an example: there are some preliminary results on the topological entropy calculated as the growth of the number of periodic orbits as a function of their length. Here again one should explore the arithmetical approach.

---

2 Beware of singularities


“Dynamics of complexity of intersections”, V.I. Arnold, Bol. Soc. Bras. Mat. 21 (1990), pp 1–10


“Monomial maps and algebraic entropy” B. Hasselblatt and J. Propp, (2005) to be submitted to Ergodic Theory and Dynamical Systems


An example in $P_{15}$ with 17 algebraic invariants (and 3 relations)
An example in $P_9$ with 7 invariant