

## THE GEOMETRICAL INTERPRETATION OF THE FADDEEV–POPOV DETERMINANT

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We give a geometrical interpretation of the Faddeev–Popov determinant in Yang–Mills theories by comparing it to the natural metric on the orbit space.

### Introduction

The fundamental objects in pure Yang–Mills theories are at first the gauge potentials. However, as a principle of gauge invariance, only the classes of gauge equivalent potentials are physically relevant. The space of these classes is called the *orbit space*.

The usual Feynman path integral, which one uses to “quantize” the theory is essentially an integral over the orbit space. The point is that this space is equipped with a natural riemannian structure (as we will see) and consequently we know how to define (formally) a volume element on this space.

The purpose of this letter is to show what the Faddeev–Popov determinant is, in terms of this metric. We show that in the non-covariant formalism, when the path integral is written in the true configuration space, the determinant is nothing but the natural metric on the configuration space. We also show that in the covariant formalism where paths are replaced by one point of the orbit space, the determinant contains an extra factor interpreted as a part of the volume of the group of gauge transformations.

### 1. The mathematical setting

We use the following appropriate objects [1]:  $P(M, G)$  is a  $G$ -bundle over the compact euclidean space–time  $M$  (e.g.,  $M = S^4$ ), respectively over euclidean space (e.g.,  $M = S^3$ ). We shall use the associated bundle  $E = P \times_{\text{Ad}} \mathcal{A}(G)$  (the vector bundle with standard fibre  $\mathcal{A}(G)$ , the Lie algebra of  $G$ , with the ad-

joint action of  $G$  on  $\mathcal{A}(G)$ ).  $A^p = \Gamma(E \otimes \Lambda^p)$  is the space of  $p$ -forms on  $M$  with values in  $E$ .  $\mathcal{C}$  is the set of irreducible connection forms on  $P$  (i.e. all connection forms on  $P$  if  $M = S^4$  and  $G = \text{SU}(2)$ , and  $k = \text{instanton number different from zero}$ ; in any case almost all connections). If  $\mathcal{G}$  is the group of gauge transformations (reduced by the center as in ref. [2]), then  $\mathcal{G}$  acts on  $\mathcal{C}$  and defines a fibration  $\mathcal{C} \rightarrow^p \eta$  with projection  $p$ . The quotient space  $\eta = \mathcal{C}/\mathcal{G}$  is a  $C^\infty$  riemannian manifold [2].  $\eta$  is the *orbit space*. Notice that if  $M = S^3$ ,  $\eta$  is the *true configuration space*.

The tangent space to  $\mathcal{C}$  at  $\omega$  is  $T_\omega(\mathcal{C}) = A^1$  and  $\mathcal{C}$  is an affine space modelled on  $A^1$ . We have a notion of horizontality in  $T_\omega(\mathcal{C})$  given by the connection form on  $\mathcal{C}$ :  $\chi_\omega = G_\omega \nabla_\omega^*$  (with the convention that  $\nabla_\omega^*$  is the adjoint of the covariant derivative  $\nabla_\omega$  associated to  $\omega$  with respect to the natural scalar product  $(,)$  in  $A^1$ , and  $G_\omega$  is the inverse of the laplacian  $\square_\omega = \nabla_\omega^* \nabla_\omega$ ). Notice that  $G_\omega$  is well defined since  $\omega$  is irreducible [3,2]. This horizontality gives a local gauge fixing condition (the covariant background gauge condition [4], the generalized Coulomb gauge [3]).

### 2. The riemannian structure on $\eta$ [2]

**2.1. Definition.**  $A^1$  is equipped with a natural scalar product denoted by  $(,)$ , which provides a riemannian structure on  $\mathcal{C}$ . This riemannian structure, which uses extensively both the riemannian structure on  $M$  and the inner product in  $\mathcal{A}(G)$  is gauge invariant. There exists an induced metric  $g$  on  $\eta$  defined as follows: Suppose  $A$  and  $B$  are two vectors, tangent to  $\eta$  at the point  $a \in \eta$ . Let  $\omega$  be some point in  $p^{-1}(a)$  and  $\tau_A$

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(respectively  $\tau_B$ ) the horizontal lifts of  $A$  (respectively  $B$ ) at  $\omega$ . Define  $g(A, B) = (\tau_A, \tau_B)$ . It is true that the right-hand side in this definition is independent of the choice of  $\omega$  in  $p^{-1}(a)$ , as a consequence of the gauge invariance of  $(\cdot, \cdot)$  and the properties of horizontal lifts.

**2.2. Expression of the metric in the covariant background gauge.** The connection form  $\chi_\omega$ , together with the riemannian structure on  $\mathcal{C}$  defines a splitting of  $T_\omega(\mathcal{C})$  into two orthogonal subspaces:

$$T_\omega(\mathcal{C}) = H_\omega \oplus V_\omega,$$

$H_\omega$  = horizontal subspace = kernel of  $\chi_\omega$ ,  $V_\omega$  = vertical subspace = image of  $\nabla_\omega$  in  $A^1$ . There exists a projection operator  $\Pi_\omega : T_\omega(\mathcal{C}) = A^1 \rightarrow H_\omega$ ,

$$\Pi_\omega = 1 - \nabla_\omega G_\omega \nabla_\omega^*.$$

$\Pi_\omega$  satisfies  $\Pi_\omega^2 = \Pi_\omega = \Pi_\omega^*$  (adjoint).

Let  $a_0$  be a point of  $\eta$  and  $\omega_0$  be same point in  $p^{-1}(a_0)$ . Let  $\mathcal{C}_0$  be the set of points  $\omega$  in  $\mathcal{C}$  which satisfy the background gauge condition, i.e.  $\nabla_{\omega_0}^*(\omega - \omega_0) = 0$ .  $\mathcal{C}_0$  provides a local section of the fibration  $\mathcal{C} \rightarrow^p \eta$  over a neighbourhood  $U_0$  of  $a_0$  and yields coordinates for this neighbourhood.

Let  $a$  be a point in  $U_0$  and  $\omega = p^{-1}(a) \cap \mathcal{C}_0$ . If  $A$  and  $B$  are vectors in  $T_a(\eta)$ , then the components of  $A$  and  $B$  are two vectors  $u_A$  and  $u_B$  (in  $T_\omega(\mathcal{C})$ ), which belong to  $\mathcal{C}_0$  (i.e.  $\nabla_{\omega_0}^* u_A = \nabla_{\omega_0}^* u_B = 0$ ) and such that  $p_*(u_A) = A$  and  $p_*(u_B) = B$ . Clearly

$$p_*(\Pi_\omega u_A) = p_*(u_A), \quad p_*(\Pi_\omega u_B) = p_*(u_B).$$

Consequently,  $\Pi_\omega(u_A)$  is the horizontal lift of  $A$  at  $\omega$  (respectively  $\Pi_\omega(u_B)$  is the lift of  $B$ ). Hence

$$g(A, B) = (\Pi_\omega u_A, \Pi_\omega u_B),$$

or

$$g(A, B) = (u_A, \Pi_\omega u_B) = (u_A, \Pi_{\omega_0} \Pi_\omega \Pi_{\omega_0} u_B).$$

Notice that  $g$  vanishes when  $\Pi_{\omega_0} \Pi_\omega = 0$ , i.e. when the orbit through  $\omega$  is tangent to  $\mathcal{C}_0$ . *This is precisely the point where the Gribov ambiguity appears, and where the coordinate system becomes singular.*

**3. Computation of det  $g$**

Let  $\Delta^{\text{FP}}$  be the usual Faddeev–Popov determinant,

i.e.  $\Delta^{\text{FP}} = \det(\nabla_{\omega_0}^* \nabla_\omega)$ , where  $\nabla_{\omega_0}^* \nabla_\omega$  acts on  $A^0$ .

*Proposition.* We have

$$(\det g)^{1/2} = \Delta^{\text{FP}} / (\det \square_{\omega_0})^{1/2} (\det \square_\omega)^{1/2},$$

or equivalently

$$(\det \square_{\omega_0})^{1/2} (\det g)^{1/2} = \Delta^{\text{FP}} (\det \square_\omega)^{-1/2}. \quad (1)$$

*Remark.* In computing  $\det g$ ,  $g$  is considered as a mapping from  $\mathcal{C}_0$  to  $\mathcal{C}_0$  and this determinant will be denoted by  $\det_{\mathcal{C}_0} g$ .

*Proof.* On  $\mathcal{C}_0$ , the operator  $g$  can be written  $g = 1 - \Pi_{\omega_0} \nabla_\omega G_\omega \nabla_\omega^* \Pi_{\omega_0}$ . Let us introduce the mapping  $Q : A^1 \rightarrow A^1$ , defined by

$$Q = \Pi_{\omega_0} \nabla_\omega G_\omega \nabla_\omega^* = \Pi_{\omega_0} (1 - \Pi_\omega).$$

We have

$$Q^* = \nabla_\omega G_\omega \nabla_\omega^* \Pi_{\omega_0} = (1 - \Pi_\omega) \Pi_{\omega_0}.$$

Then  $g = 1 - QQ^*$ .

Define the operator  $\gamma : A^0 \rightarrow A^0$  by  $\gamma = G_\omega \nabla_\omega^* \nabla_{\omega_0} G_{\omega_0} \nabla_{\omega_0}^* \nabla_\omega$ . From the definition:

$$\det \gamma = \frac{\det(\nabla_\omega^* \nabla_{\omega_0}) \det(\nabla_{\omega_0}^* \nabla_\omega)}{\det \square_{\omega_0} \det \square_\omega}.$$

But also

$$\det \gamma = \frac{(\Delta^{\text{FP}})^2}{\det \square_{\omega_0} \det \square_\omega},$$

if we use the gauge condition on  $\omega$ .

Since there exists an isomorphism between  $A^0$  and  $V_\omega$ :

$$\nabla_\omega : A^0 \rightarrow V_\omega, \quad (\nabla_\omega)^{-1} = G_\omega \nabla_\omega^* = \chi_\omega,$$

the operator  $\gamma$  can be written as an operator  $\gamma'$  on  $V_\omega$  with values in  $V_\omega$ :

$$\gamma' = \nabla_\omega \gamma \chi_\omega.$$

We have  $\gamma' = 1 - Q^*Q$ .

*Claim.*  $\det_{\mathcal{C}_0} g = \det_{V_\omega} \gamma' (= \det_{A^0} \gamma)$ .

*Proof.* Since  $Q^*Q$  is zero on  $H_\omega$ , we have:  $\det_{V_\omega} \gamma' = \det_{A^1} \gamma'$ . Since  $QQ^*$  is zero on  $V_{\omega_0}$ , we have:  $\det_{\mathcal{C}_0} g = \det_{A^1} g$ .

It is sufficient to show that  $QQ^*$  and  $Q^*Q$  have a non-zero spectrum on  $A^1$  to ensure that  $\det_{A^1} g =$

$\det_{A1} \gamma'$ . Suppose then that  $\phi$  is an eigenvector of  $Q^*Q$  associated to the nonzero eigenvalue  $\lambda$ . We have  $QQ^*(Q\phi) = \lambda Q\phi$ , and  $Q\phi \neq 0$ . Consequently,  $Q\phi$  is an eigenvector of  $QQ^*$  associated to the same eigenvalue  $\lambda$ , and  $Q$  defines an isomorphism between the eigenspaces of  $Q^*Q$  and the eigenspaces of  $QQ^*$  (for nonzero eigenvalues). Q.E.D.

*Remark.* The above computation is valid whatever base space  $M$  we use for  $P$  ( $M$  compact without boundary).

4. Path integral on the configuration space ( $M = S^3$ )

Our motivation in studying the metric on the space  $\eta$  is our aim to define, at least formally, a volume element which could be used in the Feynman path integral. What follows is purely formal, since we are dealing with infinite dimensional spaces but is a step towards a global definition of this integral.

The integral over the configuration space is contained in ref. [5, eq. (23)] when the gaussian integrations over  $E_k$  and  $A_0$  have been performed:

$$I = \int_{\text{path in } \mathcal{C}} \prod_{\text{time}} \left[ d\omega \det(\nabla_{\omega_0}^* \nabla_{\omega}) (\det \square_{\omega})^{-1/2} \right. \\ \left. \times \delta(\nabla_{\omega_0}^* [\omega - \omega_0]) \right] \exp \left\{ - \int \frac{1}{2} (\dot{\omega}, \Pi_{\omega} \dot{\omega}) + \frac{1}{2} F^2 \right\}$$

where  $\dot{\omega} = d\omega/dt$  is a tangent vector to  $\mathcal{C}$  at  $\omega$  and  $F^2 = \text{Tr}(F_{ij} F^{ij})$ .

Notice that  $(\dot{\omega}, \Pi_{\omega} \dot{\omega})$  can be considered as the square of a vector tangent to the orbit space, with respect to the riemannian structure  $g$ . It is the kinetic energy term, and  $\frac{1}{2}(\dot{\omega}, \Pi_{\omega} \dot{\omega}) + \frac{1}{2}F^2$  is the effective action  $\Sigma$  on  $\eta$ <sup>†1</sup>. We see here that  $g$  is the physically relevant metric on  $\eta$ .

Let  $I'$  be the path integral over  $\eta$ , defined with the volume element which  $g$  provides on  $\eta$ :

$$I' = \int_{\text{path on } \eta} \prod_{\text{time}} \left[ d\sigma (\det g(\sigma))^{1/2} \right] e^{-\Sigma(\sigma)},$$

where  $\sigma$  denotes the coordinates on  $\eta$ . We will show that  $I = I'$ .

In order to compare  $I$  and  $I'$ , let us rewrite  $I'$  as an integral over  $\mathcal{C}$ , using as usual a gauge fixing term

<sup>†1</sup>  $\Sigma$  is the euclidian action.

$\prod_{\text{time}} \delta(\nabla_{\omega_0}^* [\omega - \omega_0])$ , and the appropriate Jacobian  $\prod_{\text{time}} (\det \square_{\omega_0})^{1/2}$ :

$$I' = \int_{\text{path in } \mathcal{C}} \prod_{\text{time}} \left[ d\omega (\det g)^{1/2} (\det \square_{\omega_0})^{1/2} \right. \\ \left. \times \delta(\nabla_{\omega_0}^* [\omega - \omega_0]) \right] e^{-\Sigma(\omega)}.$$

We see directly from eq. (1) that  $I = I'$ .

5. Functional integral in the covariant case ( $M = S^4$ ) (see ref. [6])

The usual functional integral one uses is

$$I = \int d\omega e^{-S(\omega)} = N \int d\omega \delta(\nabla_{\omega_0}^* [\omega - \omega_0]) \Delta^{\text{FP}} e^{-S(\omega)},$$

where  $S = \frac{1}{4} \text{Tr} F_{\mu\nu} F^{\mu\nu}$  and  $N$  is an infinite factor standing for the volume of the group of gauge transformations.

We have seen that  $\Delta^{\text{FP}} = (\det g)^{1/2} (\det \square_{\omega})^{1/2} (\det \square_{\omega_0})^{1/2}$ .  $I$  can be written on  $\eta$  as:

$$I = N \int d\sigma (\det g)^{1/2} (\det \square_{\sigma})^{1/2} e^{-S(\sigma)}.$$

The volume element is then given by the natural volume element on  $\eta$  multiplied by a factor  $N(\det \square_{\sigma})^{1/2}$  which is nothing but the volume of the orbit over  $\sigma$ , measured in  $\mathcal{C}$  (see the appendix of ref. [7]). The presence of this factor comes from the integration over the group of gauge transformations. This integration over unphysical degrees of freedom is actually necessary to compensate the presence of the longitudinal part of the gauge potentials [8,9].

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