New results for the critical fermion flavour number of three-dimensional QED

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Based on works with Anatoly V. Kotikov and Vadim I. Shilin


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Outline

1. Introduction
2. Overview of results
3. Schwinger-Dyson gap equation ($1/N$-expansion at LO)
4. Schwinger-Dyson gap equation ($1/N$-expansion at NLO)
5. Mapping between large-$N$ QED$_3$ and reduced QED$_{4,3}$
6. Conclusion
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(2 + 1)-dimensional quantum electrodynamics (QED$_3$)

Extensive studies for more than three decades now:

- **Original interest:** [Pisarski ’84; Appelquist et al. ’84]
similarities to (3 + 1)-dimensional QCD and toy model to study systematically dynamical chiral symmetry breaking (D$\chi$SB)

- **Later:** [Semenoff ’84, Marston & Affleck ’89, Ioffe & Larkin ’89]
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Critical fermion flavour number in three-dimensional QED
“Chiral” (flavour) symmetry in QED₃

Massless QED₃ with N flavours of 4-component fermions

\[ L = \bar{\Psi}_\sigma (i\hat{\nabla} - e\hat{A})\Psi^\sigma - \frac{1}{4} F_{\mu\nu}^2 \quad (\sigma = 1, \cdots, N) \]

**Global \( U(2N) \) “chiral” (flavour) symmetry**

Only possible with 4-component spinors because in this case it is possible to add \( \gamma^3 \) and \( \gamma^5 \) anticommuting with \( \gamma^0, \gamma^1 \) and \( \gamma^2 \)

Parity-even mass term breaks \( U(2N) \rightarrow U(N) \times U(N) \)

\[ \bar{\Psi}_\sigma \Psi^\sigma = \bar{\chi}_\sigma \chi^\sigma - \bar{\chi}_{N+\sigma} \chi^{N+\sigma} \]

where \( \chi_i \) is a 2-component spinor \((i = 1, \cdots, 2N)\)

Question (parity-odd mass term neglected): \[\text{[Pisarski '84]}\]

Is it possible that “chiral” symmetry is dynamically broken in QED₃?

(with dynamical generation of a parity-even mass)
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(with dynamical generation of a parity-even mass)
Some properties of the model

QED$_3$ is super-renormalizable

- Dimensionful coupling constant $a = Ne^2/8$
- Loop-expansion plagued by IR singularities (starting from two-loop)
  [Jackiw & Templeton ’81] [Guendelman & Radulovic ’83, ’84]

Large-$N$ limit of QED$_3$ ($N \to \infty$ and $a$ fixed): IR softening

[Appelquist & Pisarski ’81, Appelquist & Heinz ’81]

$$D_{\mu\nu}(p) = \frac{g_{\mu\nu}}{p^2 [1 + \Pi(p)]} = \frac{g_{\mu\nu}}{p^2 [1 + a/|p|]} \xrightarrow{p \ll a} \frac{g_{\mu\nu}}{a|p|}$$

- Gauge propagator $\sim 1/p$ and dimensionless coupling $\sim 1/\sqrt{N}$
  ($\sim 1/p$ similar to reduced QED [ST ’12, Kotikov & ST ’13])
- Power counting in large-$N$ QED$_3$ similar to 4-dimensional theories
  - Model becomes IR finite
  - But also UV finite (no renormalization of the gauge field)
    scale (conformal) invariance!
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scale (conformal) invariance!
Dynamical chiral symmetry breaking in $\text{QED}_3$

- should take place at momentum scales $p \ll a$
  (breaks scale invariance)
- cannot take place at any finite order in $1/N$
  (requires a non-perturbative approach)
- may take place below some critical fermion flavour number $N_c$

Challenge: determine the value of $N_c$

Value of $N_c$ crucial to understand the phase structure of $\text{QED}_3$
(important consequences for particle/condensed matter physics)
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Some history

- [Pisarski '84] $D\chi_{SB}$ for all values of $N$ ($N_c \to \infty$)
  - Method: solves Schwinger-Dyson (SD) gap equation at leading order (LO) of $1/N$-expansion
  - Support: RG study [Pisarski '91], further SD studies [Pennington et al. '91, '92], lattice simulation [Azcoiti '93, '96]

- [Appelquist et al. '88] $D\chi_{SB}$ for $N < N_c = 32/\pi^2 = 3.24$ (Landau gauge)
  - Method: refined study of SD gap equation at LO of $1/N$-expansion
  - Support: most results in the literature are in favour of $0 < N_c < \infty$ including early lattice simulations [Dagotto et al. '89, '90]
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- **[Atkinson et al. '90]** no $D\chi_{SB}$ for any $N$ ($N_c \to 0$)
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In 30 years: very different results obtained for $N_c$!

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<th>Year</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\infty$</td>
<td>SD (LO in $1/N$)</td>
<td>1984</td>
</tr>
<tr>
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<td>1990, 1992</td>
</tr>
<tr>
<td>$\infty$</td>
<td>RG study</td>
<td>1991</td>
</tr>
<tr>
<td>$\infty$</td>
<td>lattice simulations</td>
<td>1993, 1996</td>
</tr>
<tr>
<td>$&lt; 4.4$</td>
<td>F-theorem</td>
<td>2015</td>
</tr>
<tr>
<td>$3.5 \pm 0.5$</td>
<td>lattice simulations</td>
<td>1988, 1989</td>
</tr>
<tr>
<td>$32/\pi^2 \approx 3.24$</td>
<td>SD (LO, Landau gauge)</td>
<td>1988</td>
</tr>
<tr>
<td>$2.89$</td>
<td>RG study (one-loop)</td>
<td>2016</td>
</tr>
<tr>
<td>$1 + \sqrt{2} = 2.41$</td>
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</tr>
<tr>
<td>$&lt; 3/2$</td>
<td>Free energy constraint</td>
<td>1999</td>
</tr>
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Table: Values of $N_c$ obtained over the years with different methods (at LO).
Beyond leading order

All these very different results reflect a poor understanding of the problem.

[Pisarski '91]: “difficult to presume we understand $\chi_{SB}$ in QCD$_4$ when we do not fully understand flavour-symmetry breaking in QED$_3$”

**Important question: stability of the critical point**

Consider the approach of [Appelquist et al. '88] (LO in $1/N$-expansion):

$$N_c = \frac{32}{\pi^2} = 3.24$$

is not large

Contribution of higher order corrections may be essential!
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Note: many works address the stability of the critical point but very few
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- non-local $\xi$-gauge
- additional resummation (of wave function renormalization)

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\text{LO + resummation: } N_c = (4/3)(32/\pi^2) = 4.32
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- fully gauge-invariant

- attempt to compute NLO corrections
  - approximate calculation of diagrams
  - different gauges used for different parts of the calculation
  - typo in the final result as noticed by [Gusynin & Pyatkovskiy ’16]

approximate NLO + resummation (corrected): $N_c = 3.52$
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  - for the most complicated master integrals: use of series representations via the Gegenbauer polynomial technique [Chetyrkin, Kataev & Tkachov ’80] [Kotikov ’96]
  - new functional relations for the most complicated master integrals
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- all computations extended to an arbitrary non-local gauge
- Nash’s resummation implemented
  
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<td>1993, 1996</td>
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<tr>
<td>$&lt; 4.4$</td>
<td>F-theorem</td>
<td>2015</td>
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<td>$(4/3)(32/\pi^2) = 4.32$</td>
<td>SD (LO, resummation)</td>
<td>1989</td>
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<td>$3.5 \pm 0.5$</td>
<td>lattice simulations</td>
<td>1988, 1989</td>
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<td>$3.31$</td>
<td>SD (NLO, Landau gauge)</td>
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<td>$3.29$</td>
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<td>$32/\pi^2 \approx 3.24$</td>
<td>SD (LO, Landau gauge)</td>
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<td>SD (NLO, resummation, $\forall \xi$)</td>
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<td>$2.89$</td>
<td>RG study (one-loop)</td>
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<td>$2.85$</td>
<td>SD (NLO, resummation)</td>
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<tr>
<td>$1 + \sqrt{2} = 2.41$</td>
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<td>$&lt; 9/4 = 2.25$</td>
<td>RG study (one-loop)</td>
<td>2015</td>
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<td>$&lt; 3/2$</td>
<td>Free energy constraint</td>
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<tr>
<td>$0$</td>
<td>lattice simulations</td>
<td>2015, 2016</td>
</tr>
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2. Overview of results

3. Schwinger-Dyson gap equation (1/N-expansion at LO)

4. Schwinger-Dyson gap equation (1/N-expansion at NLO)

5. Mapping between large-N QED$_3$ and reduced QED$_{4,3}$

6. Conclusion
Fermion propagator \((\Sigma(p))\): dynamically generated (parity-conserving) mass, \(A(p)\): fermion wave function:

\[
S^{-1}(p) = [1 + A(p)] (i\hat{\gamma} + \Sigma(p))
\]

Photon propagator (non-local \(\xi\)-gauge, \(\xi = 0\): Landau gauge):

\[
D_{\mu\nu}(p) = \frac{P^\xi_{\mu\nu}(p)}{p^2 [1 + \Pi(p)]}, \quad P^\xi_{\mu\nu}(p) = g_{\mu\nu} - (1 - \xi) \frac{p_\mu p_\nu}{p^2},
\]

SD equation for the fermion propagator \((\tilde{\Sigma}(p) = \Sigma(p)[1 + A(p)])\):

\[
\tilde{\Sigma}(p) = \frac{2a}{N} \text{Tr} \int \frac{d^3 k}{(2\pi)^3} \frac{\gamma^\mu D_{\mu\nu}(p - k) \Sigma(k) \Gamma^\nu(p, k)}{[1 + A(k)] (k^2 + \Sigma^2(k))} \quad A(p)p^2 = -\frac{2a}{N} \text{Tr} \int \frac{d^3 k}{(2\pi)^3} \frac{D_{\mu\nu}(p - k) \hat{\gamma}^\mu \hat{k} \Gamma^\nu(p, k)}{[1 + A(k)] (k^2 + \Sigma^2(k))}
\]

\(\Gamma^\nu(p, k)\): vertex function.
At leading order \( (a = Ne^2/8) \): \( A(p) = 0, \quad \Pi(p) = \frac{a}{|p|}, \quad \Gamma^\nu(p, k) = \gamma^\nu \)

A single diagram contributes to the gap equation (cross = mass insertion):

\[
\Sigma(p) = \includegraphics[width=0.2\textwidth]{diagram.png} = \frac{8(2 + \xi)a}{N} \int \frac{[d^3k] \Sigma(k)}{(k^2 + \Sigma^2(k)) \left[ (p - k)^2 + a|p - k| \right]}
\]

Following [Appelquist et al. ’88] and [Kotikov ’93]:

- focus on \( p \ll a \) and linearize

\[
\Sigma(p) = \frac{8(2 + \xi)}{N} \int \frac{d^3k}{(2\pi)^3} \frac{\Sigma(k)}{k^2 |p - k|}
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Half-integer index due to photon propagator \( |p - k| = [(p - k)^2]^{1/2} \)
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and use rules for integrating massless diagrams [Kazakov ’83]

\[
\Sigma^{(\text{LO})}(p) = \frac{4(2 + \xi) B}{N} \frac{(p^2)^{-\alpha}}{(4\pi)^{3/2}} \frac{2\beta}{\pi^{1/2}}
\]
At leading order \( a = Ne^2/8 \): \( A(p) = 0 \), \( \Pi(p) = \frac{a}{|p|} \), \( \Gamma^\nu(p, k) = \gamma^\nu \)

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The LO in $1/N$-expansion gap equation then reads:

$$1 = \frac{(2 + \xi) \beta}{L} \quad \text{where} \quad \beta = \frac{1}{\alpha (1/2 - \alpha)} \quad \text{and} \quad L \equiv \pi^2 N$$

Solving the gap equation, yields (in agreement with [Appelquist et al. '88]):

$$\alpha_\pm = \frac{1}{4} \left( 1 \pm \sqrt{1 - \frac{16(2 + \xi)}{L}} \right)$$
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$$N_c \equiv N_c(\xi) = \frac{16(2 + \xi)}{\pi^2} \quad (N_c(0) = 32/\pi^2)$$

such that for $N > N_c$: $\Sigma(p) = 0$, while for $N < N_c$ (Miransky scaling):

$$\Sigma(0) \simeq a \exp\left[ -\frac{2\pi}{\sqrt{N_c/N - 1}} \right]$$
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From the rules to integrate massless Feynman diagrams [Kazakov '83] it is also straightforward to compute the LO wave-function renormalization constant.

At leading order, we have (using dimensional regularization in $D = 3 - 2\varepsilon$):

$$A(p)p^2 = -\frac{2a}{N} \mu^{2\varepsilon} \text{Tr} \int \frac{d^D k}{(2\pi)^D} \frac{P_{\mu\nu}^\xi(p - k) \hat{p} \gamma^\mu \hat{k} \gamma^\nu}{k^2 |p - k|}$$

In $\overline{MS}$-scheme ($\overline{\mu}^2 = 4\pi e^{-\gamma_E} \mu^2$):

$$A(p) = \frac{\Gamma(1 + \varepsilon)(4\pi)^\varepsilon \mu^{2\varepsilon}}{p^{2\varepsilon}} C_1(\xi) = \frac{\overline{\mu}^{2\varepsilon}}{p^{2\varepsilon}} C_1(\xi) + O(\varepsilon)$$

where

$$C_1(\xi) = +\frac{2}{3\pi^2 N} \left( (2 - 3\xi) \left[ \frac{1}{\varepsilon} - 2 \ln 2 \right] + \frac{14}{3} - 6\xi \right)$$
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$$\lambda^{(1)}_A = \mu \frac{dA(p)}{d\mu} = \frac{4(2 - 3\xi)}{3\pi^2 N}$$
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\[
\Sigma^{(\text{NLO})}(p) = \left( \frac{8}{N} \right)^2 B \frac{(p^2)^{-\alpha}}{(4\pi)^3} (\Sigma_A + \Sigma_1 + 2 \Sigma_2 + \Sigma_3),
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and the gap equation has the form (\Sigma_i = \pi \Sigma_i, (i = 1, 2, 3, A)):

\[
1 = \frac{(2 + \xi)\beta}{L} + \frac{\Sigma_A(\xi) + \Sigma_1(\xi) + 2 \Sigma_2(\xi) + \Sigma_3(\xi)}{L^2}
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We need to compute the following diagrams (massless propagator-type):

\[
\Sigma_A = \text{Diagram}_A + \text{Diagram}_B
\]

\[
\Sigma_1 = \text{Diagram}_1,
\]

\[
\Sigma_2 = \text{Diagram}_2,
\]

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Massless propagator type 2-loop diagram

Basic building block of multi-loop calculations ([d^D k] = d^D k/(2\pi)^D):

\[ J(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = \int \int \frac{[d^D k_1] [d^D k_2]}{k_1^{2\alpha_1} k_2^{2\alpha_2} (k_2 - p)^{2\alpha_3} (k_1 - p)^{2\alpha_4} (k_2 - k_1)^{2\alpha_5}} \]

Arbitrary indices \( \alpha_i \) and external momentum \( p \) in Euclidean space \( (D) \)

Coefficient function (dimensionless as our \( \Sigma_i \)):

\[ G(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = \frac{(4\pi)^D}{(p^2)^D - \sum_{i=1}^{5} \alpha_i} J(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \]

Goal of multi-loop computation:

in \( D = n - 2\varepsilon \ (n = 3) \), compute \( G(\{\alpha_i\}) \) as a Laurent series in \( \varepsilon \to 0 \)
Long history: exact computations are crucial

- all indices integers: well-known and easy to compute, e.g. IBP
  [Vasil’ev, Pismak & Khonkonen '81] [Tkachov '81]
  [Chetyrkin & Tkachov '81]

- \( \alpha_i = n_i + a_i \varepsilon \) (\( \forall i \)): already non-trivial
  [Kazakov '83, '84, '85] [Broadhurst '86]

Automated in [Bierenbaum & Weinzierl '03] (\( D = 4 \): MZV)
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In our case, several peculiarities (difficulties):
- odd-dimensional QFT: \( \Gamma \)-functions with half-integer arguments
- photon propagator: some lines with 1/2-integer indices
- from mass insertion: some lines with arbitrary \( \alpha \)-index
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Technicalities (1)

Perturbation theory rules for massless Feynman diagrams [Kazakov '83]

In momentum space:

- Plain line with an arbitrary index $\alpha$:
  \[
  \alpha \quad \iff \quad \frac{1}{k^{2\alpha}}
  \]

- Chains reduce to the product of propagators:
  \[
  \alpha \beta \quad = \quad \alpha + \beta
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- Simple loops involve an integration (used in previous LO calculation):
  \[ \int \frac{[d^D k]}{k^{2\alpha}(p - k)^{2\beta}} = \frac{(p^2)^{D/2-\alpha-\beta}}{(4\pi)^{D/2}} G(\alpha, \beta) \]

where

\[ G(\alpha, \beta) = \frac{a(\alpha)a(\beta)}{a(\alpha + \beta - D/2)}, \quad a(\alpha) = \frac{\Gamma(D/2 - \alpha)}{\Gamma(\alpha)} \]
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  \]

  where

  $G(\alpha, \beta) = \frac{\Gamma(D/2 - \alpha)\Gamma(\beta)}{\Gamma(D/2 - \alpha - \beta)}$
### Uniqueness relation ($\tilde{\alpha} = D/2 - \alpha$):

- Polyakov '70
- D'Eramo, Parisi & Peliti '71
- Vasil’ev, Pismak & Khonkonen '81
- Usyukina '83
- Kazakov '83

\[
\alpha_3 \quad \alpha_1 \quad \alpha_2
\]

\[
\frac{\sum_{i} \alpha_i = D}{4\pi} \frac{G(\alpha_1, \alpha_2)}{D/2}
\]

(Note: unique triangle has index $\sum_i \alpha_i = D$)

### IBP relation:

- Vasil’ev, Pismak & Khonkonen '81
- Tkachov '81
- Chetyrkin & Tkachov '81

\[
(D - \alpha_2 - \alpha_3 - 2\alpha_5) = \alpha_2
\]

\[
\alpha_1 \quad \alpha_2 \quad \alpha_5
\]

\[
\alpha_4 \quad \alpha_3
\]

(Note: $\pm$ corresponds to add or subtract 1 to index $\alpha_i$)
Uniqueness relation ($\tilde{\alpha} = D/2 - \alpha$): [Polyakov '70] [D'Eramo, Parisi & Peliti '71] [Vasil’ev, Pismak & Khonkonen '81] [Usyukina '83] [Kazakov '83]

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(Note: unique triangle has index $\sum_i \alpha_i = D$)

IBP relation: [Vasil’ev, Pismak & Khonkonen '81] [Tkachov '81] [Chetyrkin & Tkachov '81]

\[ (D - \alpha_2 - \alpha_3 - 2\alpha_5) \]

(Note: ± corresponds to add or subtract 1 to index $\alpha_i$)
Example: this allows to compute [Vasil’ev et al. ’81] [Vasil’ev et al. ’93] [Kivel et al. ’94] [Kotikov & ST ’13] \((\lambda = D/2 - 1, \lambda = 1/2 \text{ in } D = 3)\)

\[ G(1, 1, 1, 1, \lambda) = \begin{array}{c}
1 \\
\lambda \\
1 \\
1
\end{array} = 3 \frac{\Gamma(\lambda)\Gamma(1 - \lambda)}{\Gamma(2\lambda)} \left[ \psi'(\lambda) - \psi'(1) \right] \]

But the following diagrams are beyond IBP and uniqueness:

\begin{align*}
&\begin{array}{c}
1 \\
\alpha \\
1
\end{array} &
\begin{array}{c}
\alpha \\
1 \\
\beta
\end{array} &
\begin{array}{c}
\alpha \\
\gamma \\
\beta
\end{array}
\end{align*}
Example: this allows to compute [Vasil’ev et al. ’81] [Vasil’ev et al. ’93] [Kivel et al. ’94] [Kotikov & ST ’13] (\(\lambda = D/2 - 1, \lambda = 1/2\) in \(D = 3\))

\[
G(1, 1, 1, 1, \lambda) = \frac{\Gamma(\lambda)\Gamma(1 - \lambda)}{\Gamma(2\lambda)} \left[ \psi'(\lambda) - \psi'(1) \right]
\]

But the following diagrams are beyond IBP and uniqueness:

\[
\tilde{I}_1(\alpha) = \frac{1}{2} \quad \tilde{I}_2(\alpha) = \frac{1}{2}
\]

... and our calculations require the evaluations of masters such as:
Example: this allows to compute [Vasil’ev et al. ’81] [Vasil’ev et al. ’93] [Kivel et al. ’94] [Kotikov & ST ’13] ($\lambda = D/2 - 1$, $\lambda = 1/2$ in $D = 3$)

$$G(1, 1, 1, 1, \lambda) = \frac{\Gamma(\lambda)\Gamma(1 - \lambda)}{\Gamma(2\lambda)} \left[ \psi'(\lambda) - \psi'(1) \right]$$

But the following diagrams are beyond IBP and uniqueness:

... and our calculations require the evaluations of masters such as:

$$\tilde{l}_1(\alpha) = \frac{1}{2}$$

$$\tilde{l}_2(\alpha) = \frac{1}{2}$$

... for which no exact solution is still available.
**Example:** this allows to compute [Vasil’ev et al. ’81] [Vasil’ev et al. ’93] [Kivel et al. ’94] [Kotikov & ST ’13] ($\lambda = D/2 - 1$, $\lambda = 1/2$ in $D = 3$)

\[
G(1, 1, 1, 1, \lambda) = \begin{array}{c}
1 \\
1 \\
\lambda \\
1 \\
1
\end{array} = 3 \frac{\Gamma(\lambda)\Gamma(1-\lambda)}{\Gamma(2\lambda)} \left[ \psi'(\lambda) - \psi'(1) \right]
\]

But the following diagrams are beyond IBP and uniqueness:

... and our calculations require the evaluations of masters such as:

\[
\tilde{I}_1(\alpha) = \begin{array}{c}
\frac{1}{2} \\
\alpha \\
\frac{1}{2} \\
1
\end{array} \quad \text{and} \quad \tilde{I}_2(\alpha) = \begin{array}{c}
\frac{1}{2} \\
1 \\
\alpha \\
\frac{1}{2}
\end{array}
\]

... for which no exact solution is still available.
Technicalities (2)

The Gegenbauer polynomial $x$-space technique

[Chetyrkin, Kataev & Tkachov '80] [Kotikov '96]

Gegenbauer polynomial $C_n^\beta$ of degree $n$ and index $\beta$ defined as:

$$\frac{1}{(1 - 2xw + w^2)^\beta} = \sum_{k=0}^{\infty} C_k^\beta(x) w^k$$

$$C_n^\beta(1) = \frac{\Gamma(n + 2\beta)}{\Gamma(2\beta) n!}$$

Orthogonality relation on the unit $D$-dimensional sphere ($\hat{x} = x/\sqrt{x^2}$):

$$\frac{1}{\Omega_D} \int d_D \hat{x} C_n^\lambda(\hat{z} \cdot \hat{x}) C_m^\lambda(\hat{x} \cdot \hat{z}) = \delta_{n,m} \frac{\lambda \Gamma(n + 2\lambda)}{\Gamma(2\lambda) (n + \lambda) n!}, \quad \lambda = \frac{D}{2} - 1,$$

$$d^D x = \frac{1}{2} x^{2\lambda} dx^2 d_D \hat{x} \quad \Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}$$

They allow to generalize the multi-pole expansion to arbitrary dimension $D$
For a propagator with arbitrary power $\beta$ ($\Theta(x) \equiv$ Heaviside (step) function):

$$\frac{1}{(x_1 - x_2)^{2\beta}} = \sum_{n=0}^{\infty} C_n^{\beta}(\hat{x}_1 \cdot \hat{x}_2) \left[ \frac{(x_1^2)^{n/2}}{(x_2^2)^{n/2+\beta}} \Theta(x_2^2 - x_1^2) + (x_1^2 \leftrightarrow x_2^2) \right],$$

where:

$$C_n^\delta(x) = \sum_{k=0}^{[n/2]} C_{n-2k}^\lambda(x) \left( \frac{n - 2k + \lambda}{k! \Gamma(\delta)} \frac{\Gamma(n + \delta - k)\Gamma(k + \delta - \lambda)}{\Gamma(n - k + \lambda + 1)\Gamma(\delta - \lambda)} \right)$$

Rules for integrating diagrams with Heaviside functions [Kotikov '96]

$$\int \frac{d^Dx}{x^{2\alpha}(x-y)^{2\beta}} \Theta(x^2 - y^2) = \frac{\pi^{D/2}}{(y^2)^{\alpha+\beta-\lambda-1}} \sum_{m=0}^{\infty} \frac{B(m, n|\beta, \lambda)}{m + \alpha + \beta - 1 - \lambda}$$

$$(\beta=\lambda) \quad \frac{\pi^{D/2}}{(y^2)^{\alpha-1}} \frac{1}{\Gamma(\lambda)} \frac{1}{(\alpha - 1)(n + \lambda)} \quad \text{(as one example)}$$

$$B(m, n|\beta, \lambda) = \frac{\Gamma(m + n + \beta)}{m!\Gamma(m + n + \lambda + 1)\Gamma(\beta)} \frac{\Gamma(m + \beta - \lambda)}{\Gamma(\beta - \lambda)}.$$

Critical fermion flavour number in three-dimensional QED
With these rules, one-fold series \(_3F_2\)-hypergeometric function of argument 1) obtained [Kotikov '96]:

\[
G(1, 1, 1, 1, \alpha) = -2 \frac{\Gamma(\lambda)\Gamma(\lambda - \alpha)\Gamma(1 - 2\lambda + \alpha)}{\Gamma(2\lambda)\Gamma(3\lambda - \alpha - 1)} \sum_{n=0}^{\infty} \frac{\Gamma(n + 2\lambda)\Gamma(n + 1)}{n!\Gamma(n + 1 + \alpha)} \frac{1}{n + 1 - \lambda + \alpha} + \pi \cot \pi(2\lambda - \alpha) \frac{\Gamma(2\lambda)}{\Gamma(2\lambda)}
\]

One-fold series (two \(_3F_2\)-hypergeometric functions of argument \(-1\)) obtained earlier by [Kazakov '84] (using functional relations):

\[
G(1, 1, 1, 1, \alpha) = -2 \frac{\Gamma^2(1 - \varepsilon)\Gamma(\varepsilon)\Gamma(-\varepsilon - \alpha)\Gamma(\alpha + 2\varepsilon)}{\Gamma(2 - 2\varepsilon)} \left[ \frac{1}{\Gamma(1 + \alpha)\Gamma(1 - 3\varepsilon - \alpha)} \right]
\times \sum_{n=1}^{\infty} (-1)^n \frac{\Gamma(n + 1 - 2\varepsilon)}{\Gamma(n + \varepsilon)} \left( \frac{1}{n + \alpha + \varepsilon} + \frac{1}{n - \alpha - 2\varepsilon} \right) + \cos[\pi\varepsilon]
\]
With these rules, one-fold series \((3F2\text{-hypergeometric function of argument 1})\) obtained \([\text{Kotikov '96}]\):

\[
G(1, 1, 1, 1, \alpha) = -2 \frac{\Gamma(\lambda)\Gamma(\lambda - \alpha)\Gamma(1 - 2\lambda + \alpha)}{\Gamma(2\lambda)\Gamma(3\lambda - \alpha - 1)} \sum_{n=0}^{\infty} \frac{\Gamma(n + 2\lambda)\Gamma(n + 1)}{n!\Gamma(n + 1 + \alpha)} \frac{1}{n + 1 - \lambda + \alpha} + \frac{\pi \cot \pi(2\lambda - \alpha)}{\Gamma(2\lambda)}
\]

One-fold series (two \(3F2\text{-hypergeometric functions of argument -1})\) obtained earlier by \([\text{Kazakov '84}]\) (using functional relations):

\[
G(1, 1, 1, 1, \alpha) = -2 \frac{\Gamma^2(1 - \varepsilon)\Gamma(\varepsilon)\Gamma(-\varepsilon - \alpha)\Gamma(\alpha + 2\varepsilon)}{\Gamma(2 - 2\varepsilon)} \left[ \frac{1}{\Gamma(1 + \alpha)\Gamma(1 - 3\varepsilon - \alpha)} \right]
\]

\[
\times \sum_{n=1}^{\infty} (-1)^n \frac{\Gamma(n + 1 - 2\varepsilon)}{\Gamma(n + \varepsilon)} \left( \frac{1}{n + \alpha + \varepsilon} + \frac{1}{n - \alpha - 2\varepsilon} \right) + \cos[\pi\varepsilon]
\]

Recently, the two results were proved to be equal \([\text{Kotikov & ST '16}]\)
With these rules, one-fold series (\(_3\!F_2\)-hypergeometric function of argument 1) obtained [Kotikov '96]:

\[
G(1, 1, 1, 1, \alpha) = \frac{\Gamma(\lambda)}{\Gamma(2\lambda)\Gamma(3\lambda - \alpha - 1)} \sum_{n=0}^{\infty} \frac{\Gamma(n + 2\lambda)\Gamma(n + 1)}{n! \Gamma(n + 1 + \alpha)} \frac{1}{n + 1 - \lambda + \alpha} + \frac{\pi \cot \pi(2\lambda - \alpha)}{\Gamma(2\lambda)}
\]

One-fold series (two \(_3\!F_2\)-hypergeometric functions of argument \(-1\)) obtained earlier by [Kazakov '84] (using functional relations):

\[
G(1, 1, 1, 1, \alpha) = -2 \frac{\Gamma^2(1 - \varepsilon)\Gamma(\varepsilon)\Gamma(-\varepsilon - \alpha)\Gamma(\alpha + 2\varepsilon)}{\Gamma(2 - 2\varepsilon)} \left[ \frac{1}{\Gamma(1 + \alpha)\Gamma(1 - 3\varepsilon - \alpha)} \right] \sum_{n=1}^{\infty} (-1)^n \frac{\Gamma(n + 1 - 2\varepsilon)}{\Gamma(n + \varepsilon)} \left( \frac{1}{n + \alpha + \varepsilon} + \frac{1}{n - \alpha - 2\varepsilon} \right) + \cos[\pi\varepsilon]
\]

Recently, the two results were proved to be equal [Kotikov & ST '16]
In a more complicated case, one-fold series (as a combination of two \( _3 F_2 \)-hypergeometric functions of argument 1) were also obtained in [Kotikov & ST ’14] from the rules of [Kotikov ’96]:

\[
I(\alpha, 1, \beta, 1, 1) = \frac{1}{\pi^D} \frac{1}{\bar{\alpha} - 1} \frac{1}{1 - \bar{\beta}} \times \\
\times \frac{\Gamma(\bar{\alpha})\Gamma(\bar{\beta})\Gamma(3 - \bar{\alpha} - \bar{\beta})}{\Gamma(\alpha)\Gamma(\lambda - 2 + \bar{\alpha} + \bar{\beta})} \frac{\Gamma(\lambda)}{\Gamma(2\lambda)} I(\bar{\alpha}, \bar{\beta})
\]

where:

\[
I(\bar{\alpha}, \bar{\beta}) = \frac{\Gamma(1 + \lambda - \bar{\alpha})}{\Gamma(3 - \bar{\alpha} - \bar{\beta})} \pi \sin[\pi \bar{\alpha}] \sin[\pi(\lambda - 1 + \bar{\beta})] \sin[\pi(\bar{\alpha} + \bar{\beta} + \lambda - 1)] \\
+ \sum_{n=0}^{\infty} \frac{\Gamma(n + 2\lambda)}{n!} \left( \frac{1}{n + \lambda + \bar{\alpha} - 1} \frac{\Gamma(n + 1)}{\Gamma(n + 2 + \lambda - \bar{\beta})} + \frac{1}{n + \lambda + 1 - \bar{\alpha}} \times \\
\times \frac{\Gamma(n + 2 - \bar{\alpha})\Gamma(2 - \bar{\beta})\Gamma(\lambda)}{\Gamma(n + 3 + \lambda - \bar{\alpha} - \bar{\beta})\Gamma(3 - \bar{\alpha} - \bar{\beta})\Gamma(\lambda + \bar{\alpha} - 1)} \sin[\pi(\bar{\beta} + \lambda - 1)] \right)
\]
For $\Sigma_2$, related master integral represented in terms of a two-fold series

$$\Sigma_2 = \ldots$$

$$\tilde{l}_1(\alpha) = \ldots = \frac{(4\pi)^3}{(p^2)^{-\alpha}} l_1(\alpha)$$
**Back to QED:** for the most complicated integrals, the rules [Kotikov ’96] yield multiple-series representations [Kotikov, Shilin & ST ’16]

For $\Sigma_2$, related master integral represented in terms of a two-fold series

\[
\Sigma_2 = \quad \tilde{I}_1(\alpha) = \quad \frac{1}{2} \quad 1 \quad \frac{1}{2} = \quad \frac{(4\pi)^3}{(p^2)^{-\alpha}} l_1(\alpha)
\]

\[
\tilde{I}_1(\alpha) = \frac{(4\pi)^3}{(p^2)^{-\alpha}} \int \frac{[d^3k_1][d^3k_2]}{|p-k_1| k_1^{2\alpha} (k_1-k_2)^2 (p-k_2)^2 |k_2|} = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{B(l, n, 1, 1/2)}{(n+1/2) \Gamma(1/2)}
\]

\[
\times \left[ \frac{2}{n+1/2} \left( \frac{1}{l+n+\alpha} + \frac{1}{l+n+3/2-\alpha} \right) + \frac{1}{(l+n+\alpha)^2} + \frac{1}{(l+n+3/2-\alpha)^2} \right]
\]
For the most complicated integrals, the rules [Kotikov '96] yield multiple-series representations [Kotikov, Shilin & ST '16]

For $\Sigma_2$, related master integral represented in terms of a two-fold series

$$
\Sigma_2 = \quad \tilde{I}_1(\alpha) = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1/2
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
= \frac{(4\pi)^3}{(p^2)^{-\alpha}} I_1(\alpha)
$$

Moreover, it obeys the following functional relation:

$$
\tilde{I}_1(\alpha + 1) = \frac{(\alpha - 1/2)^2}{\alpha^2} \tilde{I}_1(\alpha) - \frac{1}{\pi \alpha^2} \left[ \psi'(\alpha) - \psi'(1/2 - \alpha) \right].
$$

(Obtained by analogy with the ones in [Kazakov '84])
Back to QED: for the most complicated integrals, the rules [Kotikov '96] yield multiple-series representations [Kotikov, Shilin & ST '16]

For $\Sigma_2$, related master integral represented in terms of a two-fold series

$$\Sigma_2 = \ldots$$

$$\tilde{I}_1(\alpha) = \ldots$$

Moreover, it obeys the following functional relation:

$$\tilde{I}_1(\alpha + 1) = \frac{(\alpha - 1/2)^2}{\alpha^2} \tilde{I}_1(\alpha) - \frac{1}{\pi \alpha^2} \left[ \Psi'(\alpha) - \Psi'(1/2 - \alpha) \right].$$

(obtained by analogy with the ones in [Kazakov '84])
In the case of $\Sigma_3$, two master integrals contribute:

$$\Sigma_3 = \begin{tikzpicture} 
\node at (0,0) [circle,draw,inner sep=0pt,minimum size=0.5cm] {}; 
\node at (2,0) [circle,draw,inner sep=0pt,minimum size=0.5cm] {}; 
\node at (4,0) [circle,draw,inner sep=0pt,minimum size=0.5cm] {}; 
\node at (6,0) [circle,draw,inner sep=0pt,minimum size=0.5cm] {}; 
\node at (8,0) [circle,draw,inner sep=0pt,minimum size=0.5cm] {}; 
\node at (10,0) [circle,draw,inner sep=0pt,minimum size=0.5cm] {}; 
\draw[thick] (0,0) -- (4,0); 
\draw[thick] (4,0) -- (8,0); 
\draw[thick] (8,0) -- (10,0); 
\draw[thick] (0,0) -- (2,2); 
\draw[thick] (0,0) -- (2,-2); 
\draw[thick] (2,-2) -- (4,-2); 
\draw[thick] (2,2) -- (4,2); 
\draw[thick] (8,0) -- (10,2); 
\draw[thick] (8,0) -- (10,-2); 
\end{tikzpicture}$$

$$\tilde{I}(\alpha, \gamma) = \frac{(4\pi)^3}{(p^2)^{\alpha-\gamma+1/2}} \int \frac{[d^3 k_1][d^3 k_2]}{(p-k_1)^2 \gamma k_1^{2\alpha} (k_1 - k_2)^2 (p-k_2)^2 |k_2|}$$

$$\tilde{I}_2(\alpha) = \begin{tikzpicture} 
\node at (0,0) [circle,draw,inner sep=0pt,minimum size=0.5cm] {}; 
\node at (2,0) [circle,draw,inner sep=0pt,minimum size=0.5cm] {}; 
\node at (4,0) [circle,draw,inner sep=0pt,minimum size=0.5cm] {}; 
\node at (6,0) [circle,draw,inner sep=0pt,minimum size=0.5cm] {}; 
\node at (8,0) [circle,draw,inner sep=0pt,minimum size=0.5cm] {}; 
\node at (10,0) [circle,draw,inner sep=0pt,minimum size=0.5cm] {}; 
\draw[thick] (0,0) -- (4,0); 
\draw[thick] (4,0) -- (8,0); 
\draw[thick] (8,0) -- (10,0); 
\draw[thick] (0,0) -- (2,2); 
\draw[thick] (0,0) -- (2,-2); 
\draw[thick] (2,-2) -- (4,-2); 
\draw[thick] (2,2) -- (4,2); 
\draw[thick] (8,0) -- (10,2); 
\draw[thick] (8,0) -- (10,-2); 
\end{tikzpicture}$$

$$\tilde{I}_3(\alpha) = \begin{tikzpicture} 
\node at (0,0) [circle,draw,inner sep=0pt,minimum size=0.5cm] {}; 
\node at (2,0) [circle,draw,inner sep=0pt,minimum size=0.5cm] {}; 
\node at (4,0) [circle,draw,inner sep=0pt,minimum size=0.5cm] {}; 
\node at (6,0) [circle,draw,inner sep=0pt,minimum size=0.5cm] {}; 
\node at (8,0) [circle,draw,inner sep=0pt,minimum size=0.5cm] {}; 
\node at (10,0) [circle,draw,inner sep=0pt,minimum size=0.5cm] {}; 
\draw[thick] (0,0) -- (4,0); 
\draw[thick] (4,0) -- (8,0); 
\draw[thick] (8,0) -- (10,0); 
\draw[thick] (0,0) -- (2,2); 
\draw[thick] (0,0) -- (2,-2); 
\draw[thick] (2,-2) -- (4,-2); 
\draw[thick] (2,2) -- (4,2); 
\draw[thick] (8,0) -- (10,2); 
\draw[thick] (8,0) -- (10,-2); 
\end{tikzpicture}$$

Only one is independent. Functional relations:

$$\tilde{I}_2(\alpha) = \tilde{I}_2(3/2 - \alpha), \quad \tilde{I}_3(\alpha) = \frac{2}{4\alpha - 1} \left( \alpha \tilde{I}_2(1 + \alpha) - (1/2 - \alpha)\tilde{I}_2(\alpha) \right) - \frac{\beta^2}{\pi}$$

Critical fermion flavour number in three-dimensional QED
In the case of $\Sigma_3$, two master integrals contribute:

\[
\Sigma_3 = \quad \tilde{I}(\alpha, \gamma) = \quad \frac{(4\pi)^3}{(p^2)^{-\alpha-\gamma+1/2}} \int \frac{[d^3 k_1][d^3 k_2]}{(p - k_1)^{2\gamma} k_1^{2\alpha} (k_1 - k_2)^2 (p - k_2)^2 |k_2|}
\]

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\]
Functional relations:

\[ \tilde{I}_2(\alpha) = \tilde{I}_2(3/2 - \alpha), \quad \tilde{I}_3(\alpha) = \frac{2}{4\alpha - 1} \left( \alpha \tilde{I}_2(1 + \alpha) - (1/2 - \alpha)\tilde{I}_2(\alpha) \right) - \frac{\beta^2}{\pi} \]

Representation of \( \tilde{I}_2(\alpha) \) in terms of a three-fold series

\[
\tilde{I}_2(\alpha) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} B(m, n, \beta, 1/2) \sum_{l=0}^{\infty} B(l, n, 1, 1/2) \times C(n, m, l, \alpha),
\]

\[
C(n, m, l, \alpha) = \frac{1}{(m + n + \alpha)(l + n + \alpha)} + \frac{1}{(m + n + \alpha)(l + m + n + 1)}
\]

\[
+ \frac{1}{(m + n + 1/2)(l + m + n + \alpha)} + \frac{1}{(m + n + 1/2)(l + n + 3/2 - \alpha)}
\]

\[
+ \frac{1}{(n + l + \alpha)(l + m + n + \alpha)} + \frac{1}{(l + n + 3/2 - \alpha)(l + n + m + \alpha)}.
\]
Back to the gap equation at NLO

\[ 1 = \frac{(2 + \xi) \beta}{L} + \frac{\Sigma_A(\xi) + \Sigma_1(\xi) + 2 \Sigma_2(\xi) + \Sigma_3(\xi)}{L^2} \]

All diagrams can be computed exactly

Contribution of $\Sigma_A$ originates from LO $A(p)$ (singular):

\[ \Sigma_A(\xi) = 4 \frac{\mu^{2\varepsilon}}{p^{2\varepsilon}} \beta \left[ \left( \frac{4}{3} (1 - \xi) - \xi^2 \right) \left[ \frac{1}{\varepsilon} + \psi_1 - \frac{\beta}{4} \right] + \left( \frac{16}{9} - \frac{4}{9} \xi - 2\xi^2 \right) \right] \]

where $\psi_1 = \psi(\alpha) + \psi(1/2 - \alpha) - 2\psi(1) + \frac{3}{1/2 - \alpha} - 2 \ln 2$
Back to the gap equation at NLO

\[ 1 = \frac{(2 + \xi) \beta}{L} + \frac{\Sigma_A(\xi) + \Sigma_1(\xi) + 2 \Sigma_2(\xi) + \Sigma_3(\xi)}{L^2} \]

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where \( \psi_1 = \psi(\alpha) + \psi(1/2 - \alpha) - 2\psi(1) + \frac{3}{1/2 - \alpha} - 2 \ln 2 \)

Contribution of \( \Sigma_1 \) originates from 2-loop polarization operator in \( D = 3 \) [Gracey '93] [Gusynin, Hams & Reenders '01] [ST '12] [Kotikov & ST '13] (finite):

\[ \Sigma_1(\xi) = -2(2 + \xi) \beta \hat{\Pi}, \quad \hat{\Pi} = \frac{92}{9} - \pi^2, \]

Notice: \( \xi \)-dependence comes from the fact that we work in a non-local gauge
Back to the gap equation at NLO

\[ 1 = \frac{(2 + \xi)\beta}{L} + \frac{\Sigma_A(\xi) + \Sigma_1(\xi) + 2\Sigma_2(\xi) + \Sigma_3(\xi)}{L^2} \]

All diagrams can be computed exactly

Contribution of \(\Sigma_A\) originates from LO \(A(p)\) (singular):

\[ \Sigma_A(\xi) = 4 \frac{\mu^{2\varepsilon}}{p^{2\varepsilon}} \beta \left[ \left( \frac{4}{3}(1 - \xi) - \xi^2 \right) \left( \frac{1}{\varepsilon} + \Psi_1 - \frac{\beta}{4} \right) + \left( \frac{16}{9} - \frac{4}{9}\xi - 2\xi^2 \right) \right] \]

where \(\Psi_1 = \psi(\alpha) + \psi(1/2 - \alpha) - 2\psi(1) + \frac{3}{1/2 - \alpha} - 2 \ln 2\)

Contribution of \(\Sigma_1\) originates from 2-loop polarization operator in \(D = 3\) [Gracey ’93] [Gusynin, Hams & Reenders ’01] [ST ’12] [Kotikov & ST ’13] (finite):

\[ \Sigma_1(\xi) = -2(2 + \xi)\beta \hat{\Pi}, \quad \hat{\Pi} = \frac{92}{9} - \pi^2, \]

Notice: \(\xi\)-dependence comes from the fact that we work in a non-local gauge
The contribution $\Sigma_2$ is singular:

$$\Sigma_2(\xi) = \frac{-2\mu^2 \epsilon}{p^{2\epsilon}} \beta \left[ \frac{(2 + \xi)(2 - 3\xi)}{3} \left( \frac{1}{\epsilon} + \psi_1 - \frac{\beta}{4} \right) + \frac{\beta}{4} \left( \frac{14}{3} (1 - \xi) + \xi^2 \right) 
+ \frac{28}{9} + \frac{8}{9} \xi - 4\xi^2 \right] + (1 - \xi) \hat{\Sigma}_2,$$

where $\hat{\Sigma}_2$ is the "complicated" part (depending on $\tilde{l}_1(\alpha)$):

$$\hat{\Sigma}_2(\alpha) = (4\alpha - 1)\beta \left[ \psi'(\alpha) - \psi'(1/2 - \alpha) \right] + \frac{\pi \tilde{l}_1(\alpha)}{2\alpha} + \frac{\pi \tilde{l}_1(\alpha + 1)}{2(1/2 - \alpha)}.$$

Singularities in $\Sigma_A(\xi)$ and $\Sigma_2(\xi)$ cancel each other and the sum is finite:

$$\Sigma_{2A}(\xi) = \Sigma_A(\xi) + 2\Sigma_2(\xi),$$

$$\Sigma_{2A}(\xi) = 2(1 - \xi)\hat{\Sigma}_2(\alpha) - \left( \frac{14}{3} (1 - \xi) + \xi^2 \right) \beta^2 - 8\beta \left( \frac{2}{3} (1 + \xi) - \xi^2 \right).$$
The contribution $\bar{\Sigma}_2$ is singular:

$$\bar{\Sigma}_2(\xi) = \frac{-2\mu^{2\varepsilon}}{p^{2\varepsilon}} \beta \left[ \frac{(2 + \xi)(2 - 3\xi)}{3} \left( \frac{1}{\varepsilon} + \psi_1 - \frac{\beta}{4} \right) + \frac{\beta}{4} \left( \frac{14}{3} (1 - \xi) + \xi^2 \right) 
+ \frac{28}{9} + \frac{8}{9} \xi - 4\xi^2 \right] + (1 - \xi) \hat{\Sigma}_2,$$

where $\hat{\Sigma}_2$ is the “complicated” part (depending on $\tilde{l}_1(\alpha)$):

$$\hat{\Sigma}_2(\alpha) = (4\alpha - 1)\beta \left[ \psi'(\alpha) - \psi'(1/2 - \alpha) \right] + \frac{\pi \tilde{l}_1(\alpha)}{2\alpha} + \frac{\pi \tilde{l}_1(\alpha + 1)}{2(1/2 - \alpha)}.$$

Singularities in $\bar{\Sigma}_A(\xi)$ and $\bar{\Sigma}_2(\xi)$ cancel each other and the sum is finite:

$$\bar{\Sigma}_{2A}(\xi) = \bar{\Sigma}_A(\xi) + 2\bar{\Sigma}_2(\xi),$$

$$\bar{\Sigma}_{2A}(\xi) = 2(1 - \xi)\hat{\Sigma}_2(\alpha) - \left( \frac{14}{3} (1 - \xi) + \xi^2 \right) \beta^2 - 8\beta \left( \frac{2}{3} (1 + \xi) - \xi^2 \right).$$
Finally, the contribution of $\Sigma_3$ is finite too:

$$\Sigma_3(\xi) = \hat{\Sigma}_3(\alpha, \xi) + \left(3 + 4\xi - 2\xi^2\right)\beta^2,$$

where $\hat{\Sigma}_3$ is the “complicated” part (depending on $\tilde{l}_2(\alpha)$ and $\tilde{l}_3(\alpha)$):

$$\hat{\Sigma}_3(\alpha, \xi) = \frac{1}{4} \left(1 + 8\xi + \xi^2 + 2\alpha(1 - \xi^2)\right)\pi \tilde{l}_2(\alpha) + \frac{1}{2} \left(1 + 4\xi - \alpha(1 - \xi^2)\right)\pi \tilde{l}_2(1 + \alpha) + \frac{1}{4} \left(-7 - 16\xi + 3\xi^2\right)\pi \tilde{l}_3(\alpha).$$

**Notice:** while $\hat{\Sigma}_2$ is gauge-invariant, $\hat{\Sigma}_3$ depends on $\xi$. 
Finally, the contribution of $\Sigma_3$ is finite too:

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$$\hat{\Sigma}_3(\alpha, \xi) = \frac{1}{4} \left(1 + 8\xi + \xi^2 + 2\alpha(1 - \xi^2)\right)\pi \tilde{l}_2(\alpha)$$

$$+ \frac{1}{2} \left(1 + 4\xi - \alpha(1 - \xi^2)\right)\pi \tilde{l}_2(1 + \alpha)$$

$$+ \frac{1}{4} \left(-7 - 16\xi + 3\xi^2\right)\pi \tilde{l}_3(\alpha).$$

Notice: while $\hat{\Sigma}_2$ is gauge-invariant, $\hat{\Sigma}_3$ depends on $\xi$. 
Gap equation (1)

Combing all previous results yields an explicit form for the gap equation:

\[
1 = \frac{(2 + \xi)\beta}{L} + \frac{1}{L^2} \left[ 8S(\alpha, \xi) - 2(2 + \xi)\hat{\Pi}\beta - \left( \frac{5}{3} - \frac{26}{3}\xi + 3\xi^2 \right)\beta^2 - 8\beta \left( \frac{2}{3}(1 - \xi) - \xi^2 \right) \right],
\]

where \( S(\alpha, \xi) \) contains all the “complicated” parts:

\[
S(\alpha, \xi) = \left( \hat{\Sigma}_3(\alpha, \xi) + 2(1 - \xi)\hat{\Sigma}_2(\alpha) \right)/8.
\]

At the critical point \( \alpha = 1/4 \) (\( \beta = 16 \)): \( L = \pi^2 N \)

\[
L_c^2 - 16(2 + \xi)L - 8 \left[ S(\xi) - 4(2 + \xi)\hat{\Pi} - 16 \left( 4 - 50\xi/3 + 5\xi^2 \right) \right] = 0
\]

\[
S(\xi) = S(\alpha = 1/4, \xi) = \left( \hat{\Sigma}_3(\xi) + 2(1 - \xi)\hat{\Sigma}_2 \right)/8,
\hat{\Sigma}_2 = \hat{\Sigma}_2(\alpha = 1/4), \quad \hat{\Sigma}_3(\xi) = \hat{\Sigma}_3(\alpha = 1/4, \xi)
\]
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\[
S(\xi) = S(\alpha = 1/4, \xi) = (\hat{\Sigma}_3(\xi) + 2(1 - \xi)\hat{\Sigma}_2)/8,
\]

\[
\hat{\Sigma}_2 = \hat{\Sigma}_2(\alpha = 1/4), \quad \hat{\Sigma}_3(\xi) = \hat{\Sigma}_3(\alpha = 1/4, \xi)
\]
Solving the gap equation yields two standard solutions:

\[ L_{c, \pm} = 8(2 + \xi) \pm \sqrt{d_1(\xi)}, \]

\[ d_1(\xi) = 8 \left[ S(\xi) - 8 \left( 4 - \frac{112}{3} \xi + 9 \xi^2 + \frac{2 + \xi}{2} \hat{\Pi} \right) \right] . \]

To get a numerical estimate for \( N_c \), we use the series representations to evaluate the integrals:

\[ \pi \tilde{I}_1(\alpha = 1/4) \equiv R_1, \quad \pi \tilde{I}_2(\alpha = 1/4 + i\delta) \equiv R_2 - iP_2\delta + O(\delta^2) \]

where \( \delta \rightarrow 0 \) regulates an artificial singularity in \( \pi \tilde{I}_3(1/4) = R_2 + P_2/4 \).

With 10000 iterations for each series, the following numerical estimates are obtained:

\[ R_1 = 163.7428, \quad R_2 = 209.175, \quad P_2 = 1260.720 , \]

from which the complicated part of the self-energies can be evaluated:

\[ \hat{\Sigma}_2 = 4R_1, \quad \hat{\Sigma}_3(\xi) = (\xi^2 - 1)R_2 - (7 + 16\xi - 3\xi^2) P_2/16 . \]
Results: exact NLO (without resummation)

Combining these values with the one of $\hat{\Pi}$, yields:

\[
L_c(\xi = 2/3) = 30.51, \quad L_c(\xi = 0) = 32.45
\]
\[
N_c(\xi = 2/3) = 3.09, \quad N_c(\xi = 0) = 3.29
\]

“−” solutions are unphysical
and there is no solution in the Feynman gauge ($\xi = 1$)

The range of $\xi$-values for which there is a solution corresponds to:

\[-2.36 \leq \xi \leq 0.88\]
Gap equation (2)

Previous results show that LO $\sim \beta$ while NLO has $\sim \beta$ and $\sim \beta^2$ contribution:

$$1 = \frac{(2 + \xi)\beta}{L} + \frac{1}{L^2} \left[ 8S(\alpha, \xi) - 2(2 + \xi)\hat{\Pi} \beta - \left( \frac{5}{3} - \frac{26}{3} \xi + 3\xi^2 \right) \beta^2 
- 8\beta \left( \frac{2}{3} (1 - \xi) - \xi^2 \right) \right]$$

Extracting terms $\sim \beta$ and $\sim \beta^2$ from the “complicated” part:

$$S(\alpha, \xi) = \left( \hat{\Sigma}_3(\alpha, \xi) + 2(1 - \xi)\hat{\Sigma}_2(\alpha) \right)/8.$$  

yields another, equivalent, gap equation:

$$1 = \frac{(2 + \xi)\beta}{L} + \frac{1}{L^2} \left[ 8\tilde{S}(\alpha, \xi) - 2(2 + \xi)\hat{\Pi} \beta + \left( \frac{2}{3} - \xi \right) (2 + \xi) \beta^2 
+ 4\beta \left( \xi^2 - \frac{4}{3} \xi - \frac{16}{3} \right) \right],$$

where \( \tilde{S}(\alpha, \xi) = \left( \tilde{\Sigma}_3(\alpha, \xi) + 2(1 - \xi)\tilde{\Sigma}_2(\alpha) \right)/8 \) is “the rest”. 

Critical fermion flavour number in three-dimensional QED

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How “the rest” looks like:

\[ \hat{\Sigma}_2(\alpha) = \beta (3\beta - 8) + \tilde{\Sigma}_2(\alpha), \]
\[ \tilde{\Sigma}_2 = \tilde{\Sigma}_2(\alpha = 1/4) = 4\tilde{R}_1, \]
\[ \tilde{R}_1 = 3.7428. \]

\[ \tilde{\Sigma}_3(\xi) = -4\xi(4 + \xi)\beta + \tilde{\Sigma}_3(\alpha, \xi), \]
\[ \tilde{\Sigma}_3(\alpha, \xi) = \frac{1}{4} (1 + 8\xi + \xi^2 + 2\alpha(1 - \xi^2)) \pi J_2(\alpha) \]
\[ + \frac{1}{2} (1 + 4\xi - \alpha(1 - \xi^2)) \pi J_2(1 + \alpha) - \frac{1}{4} (-7 - 16\xi + 3\xi^2) \pi J_3(\alpha), \]
\[ \tilde{\Sigma}_3(\xi) = \tilde{\Sigma}_3(\alpha = 1/4, \xi) = (\xi^2 - 1) \tilde{R}_2 - (7 + 16\xi - 3\xi^2) \frac{\tilde{P}_2}{16}, \]
\[ \pi J_2(\alpha = 1/4) = \pi J_2(\alpha = 5/4) = \tilde{R}_2^N = 17.175, \quad \tilde{R}_2 = \tilde{R}_2^N - 16 = 1.175, \]
\[ \pi J_3(\alpha = 1/4) = \tilde{R}_2^N + \frac{\tilde{P}_2^N}{4}, \quad \tilde{P}_2 = \tilde{P}_2^N = -19.28 . \]
Nash’s resummation

The last form of the gap equation

\[
1 = \frac{(2 + \xi)\beta}{L} + \frac{1}{L^2} \left[ 8\tilde{S}(\alpha, \xi) - 2(2 + \xi)\hat{\Pi}\beta + \left(\frac{2}{3} - \xi\right)(2 + \xi)\beta^2 \\
+ 4\beta \left(\xi^2 - \frac{4}{3}\xi - \frac{16}{3}\right) \right],
\]

is a convenient starting point to implement a resummation of the wave-function renormalization constant [Nash ’89]

Recall that \(\lambda^{(1)}\) at LO and \(\lambda^{(2)}\) at NLO from [Gracey ’93]:

\[
\lambda_A = \frac{\lambda^{(1)}}{L} + \frac{\lambda^{(2)}}{L^2} + \cdots, \quad \lambda^{(1)} = 4 \left(\frac{2}{3} - \xi\right), \quad \lambda^{(2)} = -8 \left(\frac{8}{27} + \left(\frac{2}{3} - \xi\right)\hat{\Pi}\right)
\]

Crucial observation: the NLO term \(\sim \beta^2\) is proportional to \(\lambda^{(1)}\) (in the gap equation the LO and NLO contain, respectively, the zeroth and first-order terms in \(\lambda_A\))

\[\Rightarrow\text{resum the full expansion of }\lambda_A\text{ at the level of the gap equation!}\]

\((\lambda^{(2)}\text{ required to achieve NLO accuracy})\)
Nash’s resummation

The last form of the gap equation

\[ 1 = \frac{(2 + \xi)\beta}{L} + \frac{1}{L^2} \left[ 8\tilde{S}(\alpha, \xi) - 2(2 + \xi)\hat{\Pi}\beta + \left(\frac{2}{3} - \xi\right)(2 + \xi) \beta^2 \\
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is a convenient starting point to implement a resummation of the wave-function renormalization constant \[\text{[Nash '89]}\]

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**Crucial observation:** the NLO term \(\sim \beta^2\) is proportional to \(\lambda^{(1)}\) (in the gap equation the LO and NLO contain, respectively, the zeroth and first-order terms in \(\lambda_A\))

\[\Rightarrow\] resum the full expansion of \(\lambda_A\) at the level of the gap equation! (\(\lambda^{(2)}\) required to achieve NLO accuracy)
For more details, the beautiful observation of [Nash '89] is that the gap equation
\[
1 = \frac{(2 + \xi)\beta}{L} + \frac{1}{L^2} \left[ 8\tilde{S}(\alpha, \xi) - 2(2 + \xi)\hat{\Pi}\beta + \left(\frac{2}{3} - \xi\right)(2 + \xi)\beta^2 \right. \\
\left. + 4\beta \left(\xi^2 - \frac{4}{3}\xi - \frac{16}{3}\right)\right],
\]
can be re-written (in Appelquist form [Appelquist et al. '88]):
\[
1 = \frac{4(2 + \xi)}{L\Sigma(p)} \int_0^a \frac{d|k|\Sigma(|k|)}{\max(|k|, |p|)} \left\{ 1 + \frac{4(2 - 3\xi)}{3L} \ln \left[ \frac{\max(|k|, |p|)}{\min(|k|, |p|)} \right] \right\} + \frac{\Delta(\alpha, \xi)}{L^2},
\]
where \(\Delta(\alpha, \xi) = 8\tilde{S}(\alpha, \xi) - 4\beta \left(\xi^2 + 4\xi + \frac{8}{3} + \frac{2 + \xi}{2}\hat{\Pi}\right)\).

For resummation:
\[
\int_0^a \frac{d|k|\Sigma(|k|)}{\max(|k|, |p|)} \left\{ 1 + \frac{\lambda^{(1)}}{L} \ln \left[ \frac{\max(|k|, |p|)}{\min(|k|, |p|)} \right] \right\} \rightarrow \int_0^a \frac{d|k|\Sigma(|k|)}{\max(|k|, |p|)} \left[ \frac{\max(|k|, |p|)}{\min(|k|, |p|)} \right]^{\lambda_A}
\]
After resummation, the gap equation reads:

\[ 1 = \frac{8\beta}{3L} + \frac{\beta}{4L^2} \left( \lambda^{(2)} - 4\lambda^{(1)} \left( \frac{14}{3} + \xi \right) \right) + \frac{\Delta(\alpha, \xi)}{L^2}, \]

where the LO term is now gauge independent [Nash '89].

More explicitly, our careful analysis shows that:

\[ 1 = \frac{8\beta}{3L} + \frac{1}{L^2} \left[ 8\tilde{S}(\alpha, \xi) - \frac{16}{3} \beta \left( \frac{40}{9} + \hat{\Pi} \right) \right], \]

there is a strong suppression of the gauge dependence even at NLO

\( \xi \)-dependent terms do exist but they enter the gap equation only through the rest, \( \tilde{S} \), which is very small numerically

At the critical point (\( \alpha = 1/4 \) or \( \beta = 16 \)): (\( L = \pi^2 N \))

\[ L_c^2 - \frac{128}{3} L_c - \left[ 8\tilde{S}(\xi) - \frac{256}{3} \left( \frac{40}{9} + \hat{\Pi} \right) \right] = 0. \]
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\]
We have again two standard solutions:

\[ L_{c, \pm} = \frac{64}{3} \pm \sqrt{d_2(\xi)}, \quad d_2(\xi) = \left( \frac{64}{3} \right)^2 + \left[ 8\bar{S}(\xi) - \frac{256}{3} \left( \frac{40}{9} + \hat{\Pi} \right) \right]. \]

The numerical estimate for \( N_c \), is obtained by using the values of \( \bar{R}_1, \bar{R}_2 \) and \( \bar{P}_2 \) obtained from the series representation together with:

\begin{align*}
\bar{S}(\xi = 0) &= \bar{R}_1 - \frac{\bar{R}_2}{8} - \frac{7\bar{P}_2}{128}, \\
\bar{S}(\xi = 1) &= -\frac{5\bar{P}_2}{32}, \\
\bar{S}(\xi = 2/3) &= \frac{\bar{R}_1}{3} - \frac{5\bar{R}_2}{72} - \frac{49\bar{P}_2}{384},
\end{align*}

together with the value of \( \hat{\Pi} \).
Results: exact NLO with Nash’s resummation

\[ L_c(1) = 29.69, \quad L_c(2/3) = 29.98, \quad L_c(0) = 30.44, \]
\[ N_c(1) = 3.0084, \quad N_c(2/3) = 3.0377, \quad N_c(0) = 3.0844. \]

Remarks:

- “−” solutions unphysical but there is a solution in the Feynman gauge
- solutions exist for a broad range of values of \( \xi \): \(-8.412 \leq \xi \leq 4.042\) (consistent with the weak \( \xi \)-dependence of the gap equation)
- for \( \xi = 2/3 \) the value of \( N_c \) is very stable as it decreases only by 1-2% during resummation
  \( (\xi = 2/3 \) could be the “right(est)” gauge in agreement with [Gorbar, Gusynin & Miransky ’01] \)
- if we neglect the rest (\( \tilde{S}(\xi) = 0 \)) the gap equation becomes \( \xi \)-independent and we have: \( \bar{L}_c = 28.0981, \quad \bar{N}_c = 2.85 \)
  (in full agreement with recent results of [Gusynin & Pyatkovskiy ’16])
Remarkable crossing in the vicinity of the Landau gauge $N_c(0) = 3.0844$

NLO and NLO+resummation cross at:

$$N_c(\xi_1 = -0.4367) = 3.1074, \quad N_c(\xi_2 = 0.7092) = 3.0342.$$ 

NLO and NLO+resummation have a maximum at ($dN_c(\xi_0)/d\xi = 0$):

$$N_c^{(NLO)}(\xi_0 = 0.1903) = 3.3095, \quad N_c^{(NLO+R)}(\xi_0 = -2.1849) = 3.1471.$$
Reduced QED_{4,3}

Fermion field in 2 + 1-dimensions and photon field in 3 + 1-dimensions:

\[ S = \int d^3x \bar{\psi}_\sigma i \gamma_\nu \psi^\nu + \int d^4x \left[ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 \right] \]

Boundary effective Lagrangian (in 3 dimensions): non-local

\[ L = \bar{\psi}_\sigma i \left( \gamma_\nu + ie \tilde{A}^{\nu} \right) \psi^\nu - \frac{1}{4} \tilde{F}^{\mu\nu} \frac{2}{[-\Box]^{1/2}} \tilde{F}_{\mu\nu} + \frac{1}{2\tilde{\xi}} \tilde{A}^{\mu} \frac{2 \partial_\mu \partial_\nu}{[-\Box]^{1/2}} \tilde{A}^{\nu} \]

\[ \tilde{\xi} = (1 + \xi)/2: \text{ gf parameter associated to reduced gauge field } \tilde{A}^{\mu} \]
Reduced QED$_{4,3}$

Fermion field in $2 + 1$-dimensions and photon field in $3 + 1$-dimensions:

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Model known in some form or the other for a long time (under different names): [Gusynin et al. ’01] [Marino ’93] [Dorey & Mavromatos ’92] [Kovner & Rosenstein ’92] [Kaplan et al. ’09]
Reduced QED\(_{4,3}\)

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\]

Boundary effective Lagrangian (in 3 dimensions): non-local

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L = \bar{\psi}_\sigma i \left( \partial + ie\tilde{A} \right) \psi^\sigma - \frac{1}{4} \tilde{F}_{\mu\nu} \frac{2}{[-\Box]^{1/2}} \tilde{F}_{\mu\nu} + \frac{1}{2\tilde{\xi}} \tilde{A}_\mu \frac{2 \partial_\mu \partial_\nu}{[-\Box]^{1/2}} \tilde{A}_\nu
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On the other hand planar Dirac-Weyl liquids, e.g., graphene, have an IR Lorentz-invariant fixed point as proved long ago by Spanish scientists [Gonzalez, Guinea & Vozmediano '94]
**Reduced QED\(_{4,3}\)**

Fermion field in \(2 + 1\)-dimensions and photon field in \(3 + 1\)-dimensions:

\[
S = \int d^3 x \, \bar{\psi}_\sigma i D \psi^\sigma + \int d^4 x \left[ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 \right]
\]

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L = \bar{\psi}_\sigma i \left( \partial + i e \tilde{A} \right) \psi^\sigma - \frac{1}{4} \tilde{F}^{\mu\nu} \frac{2}{[-\Box]^{1/2}} \tilde{F}_{\mu\nu} + \frac{1}{2\xi} \tilde{A}_\mu \frac{2 \partial_\mu \partial_\nu}{[-\Box]^{1/2}} \tilde{A}^\nu
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Reduced $\text{QED}_{4,3}$ is the relativistic effective field theory describing planar Dirac-Weyl liquids at the IR Lorentz-invariant fixed point [ST '12]

Many recent works: quantum Hall physics [Marino et al. '14, '15], optical properties [Raya et al. '15, '16], 1/2-filled FQHE systems [Son '15], LKF [Ahmad et al. '16], duality [Hsiao & Son '17], bCFT [Herzog & Huang '17]...
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A kind of bulk-boundary correspondence [Kotikov & ST ’13, ’14, ’16]

Mapping between
large-$N$ $\text{QED}_3$ (coupling $\sim 1/N$) and $\text{QED}_{4,3}$ (coupling $\alpha = e^2/(4\pi)$)

Origin of the mapping: photon propagators have the same form

$$D_{\text{RQED}}^{\mu\nu}(p) = \frac{d^{\mu\nu}(\eta/2)}{2|p|}, \quad D_{\text{QED}_3}^{\mu\nu}(p) = \frac{d^{\mu\nu}(\tilde{\eta})}{a|p|}, \quad d^{\mu\nu}(\eta) = g^{\mu\nu} - \eta \frac{p^\mu p^\nu}{p^2}$$

Both theories have power counting similar to four-dimensional ones and are scale (conformal) invariant
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Many recent works: quantum Hall physics [Marino et al. '14, '15], optical properties [Raya et al. '15, '16], 1/2-filled FQHE systems [Son '15], LKF [Ahmad et al. '16], duality [Hsiao & Son '17], bCFT [Herzog & Huang '17]...

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Origin of the mapping: photon propagators have the same form

$$D^{\mu\nu}_{\text{RQED}}(p) = \frac{d^{\mu\nu}(\eta/2)}{2|p|}, \quad D^{\mu\nu}_{\text{QED3}}(p) = \frac{d^{\mu\nu}(\tilde{\eta})}{a|p|} \quad d^{\mu\nu}(\eta) = g^{\mu\nu} - \eta \frac{p^\mu p^\nu}{p^2}$$

Both theories have power counting similar to four-dimensional ones and are scale (conformal) invariant
Transformations (from large-$N$ QED to reduced QED$_{4,3}$)

\[
\frac{1}{L} \equiv \frac{1}{\pi^2 N} \to \frac{\alpha}{4\pi} \equiv \frac{e^2}{(4\pi)^2}, \quad \tilde{\eta} \to \eta \left(\xi \to \frac{1 + \xi}{2}\right)
\]

\[
\hat{\Pi}_2 = \frac{92}{9} - \pi^2 \quad \to \quad \hat{\Pi}_1 = \frac{N\pi^2}{2} \quad \text{and} \quad \tilde{\xi}\hat{\Pi}_1 = 0
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<table>
<thead>
<tr>
<th>$\alpha_c$ ($N_c$)</th>
<th>Method</th>
<th>Year</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.65</td>
<td>SD (LO, dynamic RPA, running $v$)</td>
<td>2013</td>
</tr>
<tr>
<td>3.7</td>
<td>FRG, Bethe-Salpeter</td>
<td>2016</td>
</tr>
<tr>
<td>$3.2 &lt; \alpha_c &lt; 3.3$</td>
<td>SD (LO, dynamic RPA, running $v$)</td>
<td>2012</td>
</tr>
<tr>
<td>3.1</td>
<td>SD (LO, bare vertex approximation)</td>
<td>2015</td>
</tr>
<tr>
<td>1.62</td>
<td>SD (LO, static RPA)</td>
<td>2002</td>
</tr>
<tr>
<td>(3.52)</td>
<td>SD (LO)</td>
<td>2009</td>
</tr>
<tr>
<td>1.13 (3.6)</td>
<td>SD (LO, static RPA, running $v$)</td>
<td>2008</td>
</tr>
<tr>
<td>1.11 ± 0.06</td>
<td>Lattice simulations</td>
<td>2008</td>
</tr>
<tr>
<td>$1.03 &lt; \alpha_c &lt; 1.08$</td>
<td>SD (NLO, RPA, resummation, $v/c \rightarrow 1$)</td>
<td>2016</td>
</tr>
<tr>
<td>$3.17 &lt; N_c &lt; 3.24$</td>
<td>SD (NLO, RPA, $v/c \rightarrow 1$)</td>
<td>2016</td>
</tr>
<tr>
<td>$0.94 &lt; \alpha_c &lt; 1.02$</td>
<td>SD (NLO, RPA, $v/c \rightarrow 1$)</td>
<td>2016</td>
</tr>
<tr>
<td>$3.24 &lt; N_c &lt; 3.36$</td>
<td>SD (NLO, RPA, $v/c \rightarrow 1$)</td>
<td>2016</td>
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<tr>
<td>0.99</td>
<td>RG study</td>
<td>2012</td>
</tr>
<tr>
<td>0.92</td>
<td>SD (LO, dynamic RPA)</td>
<td>2009</td>
</tr>
<tr>
<td>0.9 ± 0.2</td>
<td>Lattice simulations</td>
<td>2012</td>
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<tr>
<td>0.833</td>
<td>RG study</td>
<td>2008</td>
</tr>
</tbody>
</table>

Graphene: $N = 2$ and $\alpha_g = 2.2$. But $\alpha = 1/137$ ($v/c \rightarrow 1$).
Outline

1 Introduction

2 Overview of results

3 Schwinger-Dyson gap equation (1/N-expansion at LO)

4 Schwinger-Dyson gap equation (1/N-expansion at NLO)

5 Mapping between large-N QED$_3$ and reduced QED$_{4,3}$

6 Conclusion
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