

APPENDIX B

Bessel functions

| z | $J_0(z)$ | $J_1(z)$ | $J_2(z)$ | $J_3(z)$ | $J_4(z)$ |
|------|----------------------|----------------|----------------|-------------|-------------|
| 0.00 | 1.00000 00000 00000 | 0.00000 00000 | 0.00000 00000 | 0.00000 000 | 0.00000 000 |
| 0.02 | 0.99990 00024 99972 | 0.00999 95000 | 0.00004 99983 | 0.00000 000 | 0.00000 000 |
| 0.05 | 0.99937 50976 49468 | 0.02499 21883 | 0.00031 24349 | 0.00000 260 | 0.00000 002 |
| 0.1 | 0.99750 15620 66040 | 0.04993 75260 | 0.00124 89587 | 0.00002 082 | 0.00000 026 |
| 0.2 | 0.99002 49722 39576 | 0.09950 08326 | 0.00498 33542 | 0.00016 625 | 0.00000 416 |
| 0.4 | 0.96039 82266 59563 | 0.19602 65780 | 0.01973 46631 | 0.00132 005 | 0.00006 614 |
| 0.6 | 0.91200 48634 97211 | 0.28670 09881 | 0.04366 50967 | 0.00439 966 | 0.00033 147 |
| 0.8 | 0.84628 73527 50480 | 0.36884 20461 | 0.07581 77625 | 0.01024 678 | 0.00103 299 |
| 1.0 | 0.76519 76865 57967 | 0.44005 05857 | 0.11490 34849 | 0.01956 335 | 0.00247 664 |
| 1.2 | 0.67113 27442 64363 | 0.49828 90576 | 0.15934 90183 | 0.03287 434 | 0.00502 267 |
| 1.4 | 0.56695 51203 74289 | 0.54194 77139 | 0.20735 58995 | 0.05049 771 | 0.00906 287 |
| 1.6 | 0.45540 21676 39381 | 0.56989 59353 | 0.25696 77514 | 0.07252 344 | 0.01499 516 |
| 1.8 | 0.33998 64110 42558 | 0.58151 69517 | 0.30614 35353 | 0.09880 202 | 0.02319 652 |
| 2.0 | 0.22389 07791 41236 | 0.57672 48078 | 0.35283 40286 | 0.12894 325 | 0.03399 572 |
| 2.2 | 0.11036 22669 22174 | 0.55596 30498 | 0.39505 86875 | 0.16232 547 | 0.04764 715 |
| 2.4 | 0.00250 76832 97244 | 0.52018 52682 | 0.43098 00402 | 0.19811 480 | 0.06430 696 |
| 2.6 | -0.09680 49543 97038 | 0.47081 82665 | 0.45897 28517 | 0.23529 381 | 0.08401 287 |
| 2.8 | -0.18503 60333 64387 | 0.40970 92469 | 0.47768 54954 | 0.27269 860 | 0.10666 866 |
| 3.0 | -0.26005 19549 01933 | 0.33905 89585 | 0.48609 12606 | 0.30906 272 | 0.13203 418 |
| 3.5 | -0.38012 77399 87263 | 0.13737 75274 | 0.45862 91842 | 0.38677 011 | 0.20440 529 |
| 4.0 | -0.39714 98098 63847 | -0.06604 33280 | 0.36412 81459 | 0.43017 147 | 0.28112 9 |
| 4.5 | -0.32054 25089 85121 | -0.23106 04319 | 0.21784 89837 | 0.42470 397 | 0.34842 3 |
| 5.0 | -0.17759 67713 14338 | -0.32757 91376 | 0.04656 51163 | 0.36483 123 | 0.39123 2 |
| 5.5 | -0.00684 38694 17819 | -0.34143 82154 | -0.11731 54816 | 0.25611 8 | 0.39671 7 |
| 6.0 | 0.15064 52572 50997 | -0.27668 38581 | -0.24287 32100 | 0.11476 8 | 0.35764 2 |
| 6.5 | 0.26009 46055 81606 | -0.15384 13014 | -0.30743 03906 | -0.03534 7 | 0.27480 3 |
| 7.0 | 0.30007 92705 19556 | -0.00468 28235 | -0.30141 72201 | -0.16755 6 | 0.15779 8 |
| 8.0 | 0.17165 08071 37554 | 0.23463 63469 | -0.11299 17204 | -0.29113 2 | -0.10535 7 |
| 9.0 | -0.09033 36111 82876 | 0.24531 17866 | 0.14484 73415 | -0.18093 5 | -0.26547 1 |
| 10.0 | -0.24593 57644 51348 | 0.04347 27462 | 0.25463 03137 | 0.05837 9 | -0.21960 3 |
| 11.0 | -0.17119 03004 07196 | -0.17678 52990 | 0.13904 75188 | 0.22734 8 | -0.01504 0 |
| 12.0 | 0.04768 93107 96834 | -0.22344 71045 | -0.08493 04949 | 0.19513 7 | 0.18249 9 |
| 13.0 | 0.20692 61023 77068 | -0.07031 80521 | -0.21774 42642 | 0.00332 0 | 0.21927 6 |
| 14.0 | 0.17107 34761 10459 | 0.13337 51547 | -0.15201 98826 | -0.17680 9 | 0.07624 4 |
| 15.0 | -0.01422 44728 26781 | 0.20510 40386 | 0.04157 16780 | -0.19401 8 | -0.11917 9 |
| 16.0 | -0.17489 90739 83629 | 0.09039 71757 | 0.18619 87209 | -0.04384 7 | -0.20264 2 |

| z | $J_5(z)$ | $J_6(z)$ | $J_7(z)$ | $J_8(z)$ | $J_9(z)$ | $J_{10}(z)$ | $J_{11}(z)$ | $J_{12}(z)$ |
|------|-----------|-----------|-----------|-----------|-----------|-------------|-------------|-------------|
| 0.0 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 |
| 0.5 | 0.000008 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 |
| 1.0 | 0.000250 | 0.000021 | 0.000002 | 0.000000 | 0.000000 | 0.000000 | 0.000000 | 0.000000 |
| 1.5 | 0.001799 | 0.000228 | 0.000025 | 0.000002 | 0.000000 | 0.000000 | 0.000000 | 0.000000 |
| 2.0 | 0.007040 | 0.001202 | 0.000175 | 0.000022 | 0.000002 | 0.000000 | 0.000000 | 0.000000 |
| 2.5 | 0.019502 | 0.004225 | 0.000777 | 0.000124 | 0.000018 | 0.000002 | 0.000000 | 0.000000 |
| 3.0 | 0.043028 | 0.011394 | 0.002547 | 0.000493 | 0.000084 | 0.000013 | 0.000002 | 0.000000 |
| 3.5 | 0.080442 | 0.025429 | 0.006743 | 0.001543 | 0.000311 | 0.000056 | 0.000009 | 0.000001 |
| 4.0 | 0.132087 | 0.049088 | 0.015176 | 0.004029 | 0.000939 | 0.000195 | 0.000037 | 0.000006 |
| 4.5 | 0.194715 | 0.084276 | 0.030022 | 0.009126 | 0.002425 | 0.000573 | 0.000122 | 0.000024 |
| 5.0 | 0.261141 | 0.131049 | 0.053376 | 0.018405 | 0.005520 | 0.001468 | 0.000351 | 0.000076 |
| 5.5 | 0.320925 | 0.186783 | 0.086601 | 0.033657 | 0.011309 | 0.003356 | 0.000893 | 0.000216 |
| 6.0 | 0.362088 | 0.245837 | 0.129587 | 0.056532 | 0.021165 | 0.006964 | 0.002048 | 0.000545 |
| 6.5 | 0.373565 | 0.299913 | 0.180121 | 0.088039 | 0.036590 | 0.013288 | 0.004297 | 0.001254 |
| 7.0 | 0.347896 | 0.339197 | 0.233584 | 0.127971 | 0.058921 | 0.023539 | 0.008335 | 0.002656 |
| 8.0 | 0.185775 | 0.337576 | 0.320589 | 0.223455 | 0.126321 | 0.060767 | 0.025597 | 0.009624 |
| 9.0 | -0.055039 | 0.204317 | 0.327461 | 0.305067 | 0.214881 | 0.124694 | 0.062217 | 0.027393 |
| 10.0 | -0.234062 | -0.014459 | 0.216711 | 0.317854 | 0.291856 | 0.207486 | 0.123117 | 0.063370 |
| 11.0 | -0.238286 | -0.201584 | 0.018376 | 0.224972 | 0.308856 | 0.280428 | 0.201014 | 0.121600 |
| 12.0 | -0.073471 | -0.243725 | -0.170254 | 0.045095 | 0.230381 | 0.300476 | 0.270412 | 0.195280 |
| 13.0 | 0.131620 | -0.118031 | -0.240571 | -0.141046 | 0.066976 | 0.233782 | 0.292688 | 0.261537 |
| 14.0 | 0.220378 | 0.081168 | -0.150805 | -0.231973 | -0.114307 | 0.085007 | 0.235745 | 0.285450 |
| 15.0 | 0.130456 | 0.206150 | 0.034464 | -0.173984 | -0.220046 | -0.090072 | 0.099950 | 0.236666 |

Table of zeros of Bessel functions:

| k | J_0 | J_1 | J_2 | J_3 | J_4 | J_5 | J_6 | J_7 |
|-----|------------|----------|----------|----------|----------|----------|----------|----------|
| 1 | 2.4048256 | 3.83171 | 5.13562 | 6.38016 | 7.58834 | 8.77148 | 9.93611 | 11.08637 |
| 2 | 5.5200781 | 7.01559 | 8.41724 | 9.76102 | 11.06471 | 12.33860 | 13.58929 | 14.82127 |
| 3 | 8.6537279 | 10.17347 | 11.61984 | 13.01520 | 14.37254 | 15.70017 | 17.00382 | 18.28758 |
| 4 | 11.7915344 | 13.32369 | 14.79595 | 16.22347 | 17.61597 | 18.98013 | 20.32079 | 21.64154 |
| 5 | 14.9309177 | 16.47063 | 17.95982 | 19.40942 | 20.82693 | 22.21780 | 23.58608 | 24.93493 |
| 6 | 18.0710640 | 19.61586 | 21.11700 | 22.58273 | 24.01902 | 25.43034 | 26.82015 | 28.19119 |
| 7 | 21.2116366 | 22.76008 | 24.27011 | 25.74817 | 27.19909 | 28.62662 | 30.03372 | 31.42279 |
| 8 | 24.3524715 | 25.90367 | 27.42057 | 28.90835 | 30.37101 | 31.81172 | 33.23304 | 34.63709 |
| 9 | 27.4934791 | 29.04683 | 30.56920 | 32.06485 | 33.53714 | 34.98878 | 36.42202 | 37.83872 |
| 10 | 30.6346065 | 32.18968 | 33.71652 | 35.21867 | 36.69900 | 38.15987 | 39.60324 | 41.03077 |
| 11 | 33.7758202 | 35.33231 | 36.86286 | 38.37047 | 39.85763 | 41.32638 | 42.77848 | 44.21541 |
| 12 | 36.9170984 | 38.47477 | 40.00845 | 41.52072 | 43.01374 | 44.48932 | 45.94902 | 47.39417 |

The k th zero of J_n is denoted $j_{n,k}$.

Fourier series

$$\sin(z \sin \theta) = 2 \sum_{n=0}^{\infty} J_{2n+1}(z) \sin(2n+1)\theta$$

$$\cos(z \sin \theta) = J_0(z) + 2 \sum_{n=1}^{\infty} J_{2n}(z) \cos 2n\theta$$

$$J_n(z) = \frac{1}{\pi} \int_0^{\pi} \cos(n\theta - z \sin \theta) d\theta.$$

Differential equation

$$J_n''(z) + \frac{1}{z}J_n'(z) + \left(1 - \frac{n^2}{z^2}\right)J_n(z) = 0$$

Power series

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z}{2}\right)^{n+2k}}{k!(n+k)!}$$

Generating function

$$e^{\frac{1}{2}z(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} J_n(z)t^n$$

Limiting values

If n is constant, z is real and $|z| \rightarrow \infty$,

$$J_n(z) = \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{1}{2}\left(n + \frac{1}{2}\right)\pi\right) + O(|z|^{-3/2}).$$

[Here, $O(|z|^{-3/2})$ represents an error term which is bounded by some constant multiple of $|z|^{-3/2}$]

If z is constant and $n \rightarrow \infty$, $J_n(z) \sim \frac{1}{\sqrt{2\pi n}} \left(\frac{ez}{2n}\right)^n$.

[The \sim notation means that the ratio of these two quantities tends to one as $n \rightarrow \infty$]

For n fixed, as $k \rightarrow \infty$, $j_{n,k} \sim (k + \frac{1}{2}n - \frac{1}{4})\pi$.

Other formulas

$$J_{-n}(z) = (-1)^n J_n(z)$$

$$J_n'(z) = \frac{1}{2}(J_{n-1}(z) - J_{n+1}(z))$$

$$J_n(z) = \frac{z}{2n}(J_{n-1}(z) + J_{n+1}(z))$$

$$\frac{d}{dz}(z^n J_n(z)) = z^n J_{n-1}(z)$$

$$\sum_{n=-\infty}^{\infty} J_n(z)^2 = 1$$

In particular, $|J_n(z)| \leq 1$ for all n and z , and if $n \neq 0$ then $|J_n(z)| \leq \frac{1}{\sqrt{2}}$.

FM Synthesis

$$\sin(\phi + z \sin \theta) = \sum_{n=-\infty}^{\infty} J_n(z) \sin(\phi + n\theta)$$

The following table shows how index of modulation (I) varies as a function of operator output level (an integer in the range 0–99) on the Yamaha six operator synthesizers DX7, DX7IID, DX7IIFD, DX7S, DX5, DX1, TX7, TX816, TX216, TX802 and TF1:

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|----|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 0 | 0.0002 | 0.0003 | 0.0005 | 0.0007 | 0.0010 | 0.0012 | 0.0016 | 0.0019 | 0.0023 | 0.0027 |
| 10 | 0.0032 | 0.0038 | 0.0045 | 0.0054 | 0.0064 | 0.0076 | 0.0083 | 0.0091 | 0.0108 | 0.0118 |
| 20 | 0.0140 | 0.0152 | 0.0166 | 0.0181 | 0.0198 | 0.0216 | 0.0235 | 0.0256 | 0.0280 | 0.0305 |
| 30 | 0.0332 | 0.0362 | 0.0395 | 0.0431 | 0.0470 | 0.0513 | 0.0559 | 0.0610 | 0.0665 | 0.0725 |
| 40 | 0.0791 | 0.0862 | 0.0940 | 0.1025 | 0.1118 | 0.1219 | 0.1330 | 0.1450 | 0.1581 | 0.1724 |
| 50 | 0.1880 | 0.2050 | 0.2236 | 0.2438 | 0.2659 | 0.2900 | 0.3162 | 0.3448 | 0.3760 | 0.4101 |
| 60 | 0.4472 | 0.4877 | 0.5318 | 0.5799 | 0.6324 | 0.6897 | 0.7521 | 0.8202 | 0.8944 | 0.9754 |
| 70 | 1.0636 | 1.1599 | 1.2649 | 1.3794 | 1.5042 | 1.6403 | 1.7888 | 1.9507 | 2.1273 | 2.3198 |
| 80 | 2.5298 | 2.7587 | 3.0084 | 3.2807 | 3.5776 | 3.9014 | 4.2545 | 4.6396 | 5.0595 | 5.5174 |
| 90 | 6.0168 | 6.5614 | 7.1552 | 7.8028 | 8.5090 | 9.2792 | 10.119 | 11.035 | 12.034 | 13.123 |

The following table shows how index of modulation (I) varies as a function of operator output level (an integer in the range 0–99) on the Yamaha four operator synthesizers DX11, DX21, DX27, DX27S, DX100 and TX81Z:

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|----|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 0 | 0.0004 | 0.0006 | 0.0009 | 0.0013 | 0.0018 | 0.0024 | 0.0031 | 0.0036 | 0.0043 | 0.0052 |
| 10 | 0.0061 | 0.0073 | 0.0087 | 0.0103 | 0.0123 | 0.0146 | 0.0159 | 0.0174 | 0.0206 | 0.0225 |
| 20 | 0.0268 | 0.0292 | 0.0318 | 0.0347 | 0.0379 | 0.0413 | 0.0450 | 0.0491 | 0.0535 | 0.0584 |
| 30 | 0.0637 | 0.0694 | 0.0757 | 0.0826 | 0.0900 | 0.0982 | 0.1071 | 0.1168 | 0.1273 | 0.1388 |
| 40 | 0.1514 | 0.1651 | 0.1801 | 0.1963 | 0.2141 | 0.2335 | 0.2546 | 0.2777 | 0.3028 | 0.3302 |
| 50 | 0.3601 | 0.3927 | 0.4282 | 0.4670 | 0.5093 | 0.5554 | 0.6056 | 0.6604 | 0.7202 | 0.7854 |
| 60 | 0.8565 | 0.9340 | 1.0185 | 1.1107 | 1.2112 | 1.3209 | 1.4404 | 1.5708 | 1.7130 | 1.8680 |
| 70 | 2.0371 | 2.2214 | 2.4225 | 2.6418 | 2.8809 | 3.1416 | 3.4259 | 3.7360 | 4.0741 | 4.4429 |
| 80 | 4.8450 | 5.2835 | 5.7617 | 6.2832 | 6.8519 | 7.4720 | 8.1483 | 8.8858 | 9.6900 | 10.567 |
| 90 | 11.523 | 12.566 | 13.704 | 14.944 | 16.297 | 17.772 | 19.380 | 21.134 | 23.047 | 25.133 |

APPENDIX C

Complex numbers

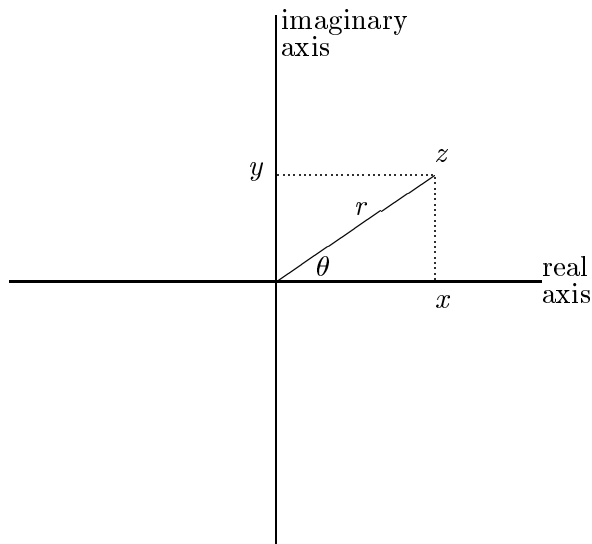
We use i to denote $\sqrt{-1}$, and the general complex number is of the form $a + ib$ where a and b are real numbers. Addition and multiplication are given by

$$(a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$$

$$(a_1 + ib_1)(a_2 + ib_2) = (a_1a_2 - b_1b_2) + i(a_1b_2 + b_1a_2).$$

The real numbers a and b can be thought of as the Cartesian coordinates of the complex number $a + ib$, so that complex numbers correspond to points on the plane. In this language, the real numbers are contained in the complex numbers as the x axis, and the points on the y axis are called pure imaginary numbers.

For the purpose of multiplication, it is easier to work in polar coordinates. If $z = x + iy$ is a complex number, we define the *absolute value* of z to be $|z| = \sqrt{x^2 + y^2}$. The *argument* of z is the angle θ formed by the line from zero to z . Angle is measured counterclockwise from the x axis.



The *complex conjugate* of $z = x + iy$ is defined to be $\bar{z} = x - iy$, so that

$$z\bar{z} = |z|^2 = x^2 + y^2.$$

So division by a nonzero complex number z is achieved by multiplying by

$$\frac{\bar{z}}{|z|^2} = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2},$$

which is the *multiplicative inverse* of z .

The *exponential function* is defined for a complex argument $z = x + iy$ by

$$e^z = e^x(\cos y + i \sin y).$$

This means that conversion from Cartesian coordinates to polar coordinates is given by

$$z = x + iy = re^{i\theta},$$

where $r = \sqrt{x^2 + y^2}$ and $\tan \theta = y/x$. Translation in the other direction is given by $x = r \cos \theta$ and $y = r \sin \theta$. The trigonometric identities

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B.$$

are equivalent to the statement that if z_1 and z_2 are complex numbers then

$$e^{z_1} e^{z_2} = e^{z_1 + z_2}.$$

So we have Euler's formula

$$e^{i\theta} = \cos \theta + i \sin \theta \tag{C.1}$$

and

$$\cos \theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \tag{C.2}$$

$$\sin \theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}). \tag{C.3}$$

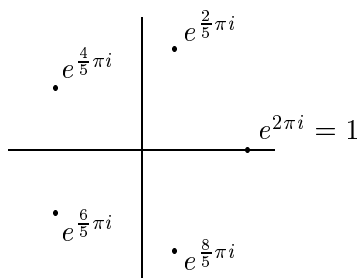
Using (C.1), the relation $(e^{i\theta})^n = e^{in\theta}$ translates as de Moivre's Theorem

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

The complex n th roots of unity (i.e., of the number one) are the numbers

$$e^{2\pi im/n} = \cos 2\pi m/n + i \sin 2\pi m/n$$

for $0 \leq m \leq n-1$. These are equally spaced around the unit circle in the complex plane. For example, here is a picture of the complex fifth roots of unity.



Remark. Engineers use the letter j instead of i .

Hyperbolic functions: In Section 3.7 the analysis of the xylophone involves the *hyperbolic functions* $\cosh x$ and $\sinh x$. These are defined by analogy with equations (C.2) and (C.3) via

$$\cosh x = \frac{1}{2}(e^x + e^{-x}) \quad (\text{C.4})$$

$$\sinh x = \frac{1}{2}(e^x - e^{-x}). \quad (\text{C.5})$$

The standard identities for these functions are

$$\cosh^2 x - \sinh^2 x = 1,$$

and

$$\sinh(A + B) = \sinh A \cosh B + \cosh A \sinh B$$

$$\cosh(A + B) = \cosh A \cosh B + \sinh A \sinh B.$$

The values at zero are given by

$$\sinh(0) = 0, \quad \cosh(0) = 1.$$

The derivatives are given by

$$\frac{d}{dx} \sinh x = \cosh x, \quad \frac{d}{dx} \cosh x = \sinh x.$$

Note the changes in sign from the corresponding trigonometric formulas.

APPENDIX D

Dictionary

As an aide to reading the literature on the subject in French, German, Italian, Latin and Spanish, as well as the literature on ancient Greek music, here is a dictionary of common terms.

- Abklingen (G.), *decay*
 Abgeleiteter Akkord (G.), *inversion of a chord*
 Absatz (G.), *cadence*
 Abstimmung (G.), *tuning*
 accord (Fr.), *chord*
 accordage (Fr.), *accordatura* (It.), *tuning, intonation*
 accordo (It.), *acorde* (Sp.), *chord*
 afinación (Sp.), *tuning*
 affaiblissement (Fr.), *decay*
 aigu (Fr.), *acute, high*
 Akkord (G.), *chord*
 allgemein (G.), *general*
 amplificateur (Fr.), *amplificatore* (It.), *amplificador* (Sp.), *amplifier*
 Anklang (G.), *tune, harmony, accord*
 armoneggiare (It.), *to harmonize*
 armonica (It.), *armónico* (Sp.), *harmonic*
 atenuamiento (Sp.), *attenuazione* (It.), *decay*
 audición (Sp.), *audition* (Fr.), *hearing*
 auferions (archaic Eng.), *wire strings*
 Aufhaltung (G.), *suspension (harmony)*
 aulos (Gk.), *ancient Greek reed instrument*
 Ausdruck (G.), *expression*
 battements (Fr.), *battimenti* (It.), *beats*
 bec (Fr.), *becco* (It.), *mouthpiece*
 bécarre (Fr.), *becuado* (Sp.), *natural* (‡)
 Bedingung (G.), *condition*
 Beispiel (G.), *example*
 beliebig (G.), *arbitrary*
 bémol (Fr.), *bemol* (Sp.), *bemolle* (It.), *flat* (b)
 bequadro (It.), *natural* (‡)
 beweisen (G.), *to prove*
 bruit (Fr.), *noise*
 Bund (G.), *fret*
 cadenza d'inganno (It.), *deceptive cadence*
 caisse (Fr.), *drum*
 Canonici, *followers of the Pythagorean system of music, where consonance is based on ratios, see also Musici*
 chevalet (Fr.), *bridge of stringed instrument*
 chiave (It.), *clave* (Sp.), *clavis* (L.), *clef, key*
 chiffage (Fr.), *time signature*
 clavecin (Fr.), *harpsichord*
 cloche (Fr.), *bell*
 concento (It.), *concentus* (L.), *harmony*
 controreazione (It.), *feedback*
 conversio (L.), *inversion*
 cor (Fr.), *horn*
 cuarta (Sp.), *fourth*
 Dach (G.), *sounding board*
 daher (G.), *hence*
 Darstellung (G.), *representation*
 demi-ton (Fr.), *semitone*
 denarius (L.), *numbers 1–10*
 diapason (Fr., It.), *diapasón* (Sp.), *pitch*
 diapason (Gk.), *octave*
 diapente (Gk.), *fifth*
 diastema (Gk.), *interval*
 diatessaron (Gk.), *fourth*
 diazeuxis (Gk.), *separation of two tetrachords by a tone*
 dièse (Fr.), *diesis* (It.), *sharp* (#)
 disdiapason (Gk.), *two octaves*
 dodécaphonique (Fr.), *twelve tone*

- Doppelbee (G.), *double flat* (bb)
 Doppelkreuz (G.), *double sharp* (x)
 Dreiklang (G.), *triad*
 Dur (G.), *major*
 durchgehend (G.), *transient*
 échantillonneur (Fr.), *sampler*
 échelle (Fr.), *scale*
 einfach (G.), *simple*
 Einheit (G.), *unity*
 Einklang (G.), *consonance*
 Einselement (G.), *identity element*
 emmeleia (Gk.), *consonance*
 enmascaramiento (Sp.), *masking*
 ensemble (Fr.), *set*
 entier (Fr.), *integer*
 entonación (Sp.), *intonation*
 entsprechen (G.), *to correspond to*
 erhöhen (G.), *to raise, increase*
 erweitern (G.), *to extend, augment*
 escala (Sp.), *scale*
 espectro (Sp.), *spectrum*
 estribo (Sp.), étrier (Fr.), *stapes*
 faux (Fr.), *out of tune*
 Folge (G.), *sequence, series*
 gama (Sp.), gamma (It.), gamme (Fr.),
 scale
 ganancia (Sp.), *gain*
 ganze Zahl (G.), *integer*
 ganzer Ton (G.), *whole tone*
 Gegenpunkt (G.), *counterpoint*
 gerade (G.), *even, just, exactly*
 Gesetz (G.), *law, rule*
 giusto (It.), *just*
 gleichschwebende (G.), *equal beating*
 gleichstufige (G.), *equal (temperament)*
 Gleichung (G.), *equation*
 gleichzeitig (G.), *simultaneous*
 Glied (G.), *term*
 Grundton (G.), *fundamental*
 guadagno (It.), *gain*
 Halbton (G.), *semitone*
 hautbois (Fr.), *oboe*
 hauteur (Fr.), Höhe (G.), *pitch*
 helicon (Gk.), *instrument used for*
 calculating ratios
 hemiolios (Gk.), *ratio 3:2*
 Hörbar (G.), *audible*
 Hören (G.), *hearing*
 impair (Fr.), *odd*
 imparfait (Fr.), *imperfect*
 Kettenbruch (G.), *continued fractions*
 Klang (G.), *timbre*
 Klangstufe (G.), *degree of scale*
 klein (G.), *small, minor*
 Kombinationston (G.), *combination tone*
 Komma (G.), *comma*
 Kraft (G.), *energy*
 Kreuz (G.), *sharp* (#)
 laud (Sp.), Laute (G.), *lute*
 Leistung (G.), *power*
 leiten (G.), *to derive, deduce*
 Leiter (G.), *scale*
 ley (Sp.), *law*
 limaçon (Fr.), *cochlea*
 Lösung (G.), *solution*
 maggiore (It.), majeur (Fr.), mayor (Sp.),
 major
 marche d'harmonie (Fr.), *harmonic*
 sequence
 Menge (G.), *set*
 menor (Sp.), *minor*
 mehrstimmig (G.), *polyphonic*
 mésotonique (Fr.), *meantone*
 minore (It.), *minor*
 mitteltönig (G.), *meantone*
 Moll (G.), *flat* (b), *minor*
 Musici, *followers of the Aristoxenian*
 system of music, in which the ear
 is the judge of consonance,
 see also Canonici
 Muster (G.), *pattern*
 Nachhall (G.), *reverberation*
 Nenner (G.), *denominator*
 neuvième (Fr.), *ninth*
 Notenschlüssel (G.), *clef*
 Oberwelle (G.), *harmonic*
 offen (G.), *open*
 Ohr (G.), *ear*
 Ohrmuschel (G.), *auricle*
 oído (Sp.), *ear*
 onda (It., Sp.), *wave*
 onda portante (It.), onda portadora
 (Sp.), *carrier*
 onde (Fr.), *wave*

- orecchio (It.), oreille (Fr.), *ear*
 ouïe (Fr.), *hearing; sound-hole*
 padigione (It.), *auricle*
 pair (Fr.), par (Sp.), *even*
 parfait (Fr.), *perfect*
 pavillon (Fr.), *auricle*
 portée (Fr.), *staff, stave*
 porteuse (Fr.), *carrier*
 potencia (Sp.), potenza (It.), *puissance*
 (Fr.), *power*
 profondeur (Fr.), *depth*
 pulsaciones (Sp.), *beats*
 Quadrat (G.), *natural* (♮)
 quadrivium (L.), *The four disciplines:*
arithmetic, geometry, astronomy
and music
 quarta (It., L.), quarte (Fr.), Quarte
 (G.), *fourth*
 quaternarius (L.), *numbers 1–4*
 quinta (It., L., Sp.), quinte (Fr.), Quinte
 (G.), *fifth*
 réaction (Fr.), *feedback*
 reine Stimmung (G.), *just intonation*
 renversement (Fr.), *inversion*
 retard (Fr.), *delay*
 retroalimentación (Sp.), *feedback*
 ronde (Fr.), *whole note* (USA),
semibreve (GB)
 Rückkopplung (G.), *feedback*
 Saite (G.), *string*
 Schall (G.), *sound*
 Schlag (G.), *beat*
 Schlüssel (G.), *clef*
 Schnecke (G.), *cochlea*
 Schwebungen (G.), *beats*
 Schwingungen (G.), *vibrations*
 senarius (L.), *numbers 1–6*
 sensible (Fr.), *leading note*
 septenarius (L.), *numbers 1–7*
 septime (L.), *seventh*
 Septimenakkord (G.), *chord of the*
seventh
 série de hauteurs (Fr.), *tone row*
 sesquialtera (L.), *ratio 3:2*
 sesquitertia (L.), *ratio 4:3*
 Sext (G.), sexta (L.), *sixth*
 sibilo (It.), sifflement (Fr.), silbo (Sp.),
hiss
 sillet (Fr.), *bridge*
 Skala (G.), *scale*
 son (Fr.), *sound*
 son combiné (Fr.), *combination tone*
 son différentiel (Fr.), *difference tone*
 sonido (Sp.), *sound*
 sonido de combinación (Sp.),
combination tone
 sonorità (It.), *harmony, resonance*
 sonus (L.), *sound*
 sostenido (Sp.), *sharp* (♯)
 spectre (Fr.), *spectrum*
 staffa (It.), *stapes*
 stark (G.), *loud*
 Stege (G.), *bridge*
 Steigbügel (G.), *stapes*
 Stufe (G.), *scale degree*
 subsemitonia (L.), *split keys*
 suono (It.), *sound*
 suono di combinazione (It.), *combination*
tone
 synaphe (Gk.), *conjunction, or*
overlapping of two tetrachords
 Takt (G.), *time, measure, bar*
 tambour (Fr.), tamburo (It.), tambor
 (Sp.), *drum*
 Tastame (It.), Tastatur, Tastenbrett,
 Tastenleiter (G.), Tastatura,
 Tastiera (It.), *keyboard of piano*
or organ
 tasto (It.), tecla (Sp.), *fret*
 teilbar (G.), *divisible*
 Teilmenge (G.), *subset*
 Teilung (G.), *division*
 Temperatur (G.), *temperament*
 temperiert (G.), *tempered*
 temps (Fr.), *beat, measure*
 tercera (Sp.), tertia (L.), Terz (G.), terza
 (It.), tierce (Fr.), *third*
 ton (Fr.), *pitch, tone, key*
 tonalité (Fr.), Tonart (G.), *key*
 Tonausweichung (G.), *modulation*
 Tonhöhe (G.), *pitch*
 tono medio (It., Sp.), *meantone*
 Tonschluss (G.), *cadence*
 Tonstufe (G.), *scale degree*
 touche (Fr.), *fret*

- Träger (G.), *carrier*
 tripla (L.), *ratio 3:1*
 Trommel (G.), *drum*
 tuyau à bouche (Fr.), *open pipe*
 tuyau d'orgue (Fr.), *organ pipe*
 tympan (Fr.), *eardrum*
 Übereinstimmung (G.), *consonance*,
 harmony
 Übermässig (G.), *augmented*
 udibile (It.), *audible*
 udito (It.), *hearing*
 uguale (It.), *equal*
 Umkehrung (G.), *inversion*
 Unterdominant (G.), *subdominant*
 Unterhalbton (G.), *leading note*
 Unterleitton (G.), *dominant seventh*
 Untergruppe (G.), *subgroup*
 valeur propre (Fr.), *eigenvalue*
 vent (Fr.), *wind*
 Ventil (G.), ventile (It.), *valve, on wind*
 instruments
 vents (Fr.), *wind instruments*
 Verbindung (G.), *combination, union*
 Verdeckung (G.), *masking*
 Verhältnis (G.), *ratio, proportion*
 Verknüpfung (G.), *operation*
 vermindert (G.), *diminished*
 versetzen (G.), *to transpose*
 Versetzungszeichen (G.), *accidentals*
 Verspätung (G.), *delay*
 Verstärker (G.), *amplifier*
 Verstärkung (G.), *gain*
 verstimmt (G.), *out of tune*
 verwandt (G.), *related*
 Verzerrung (G.), *distortion*
 Vollkommenheit (G.), *perfection*
 Welle (G.), *wave*
 Zahl (G.), *number*
 Zeichen (G.), *sign, note*
 Zischen (G.), *hiss*
 Zuklang (G.), *unison, consonance*

APPENDIX E

Equal tempered scales

| q | p_3 | e_3 | p_5 | e_5 | p_7 | e_7 | e_{35} | e_{357} | $e_{5 \cdot q^2}$ | $e_{35 \cdot q^{\frac{3}{2}}}$ | $e_{357 \cdot q^{\frac{4}{3}}}$ |
|-----|-------|----------|-------|----------|-------|----------|----------|-----------|-------------------|--------------------------------|---------------------------------|
| 2 | 1 | +213.686 | 1 | -101.955 | 2 | +231.174 | 166.245 | 190.365 | 392 | 470 | 480 |
| 3 | 1 | +13.686 | 2 | +98.045 | 2 | -168.826 | 70.000 | 112.993 | 882 | 364 | 489 |
| 4 | 1 | -86.314 | 2 | -101.955 | 3 | -68.826 | 94.459 | 86.760 | 1631 | 756 | 551 |
| 5 | 2 | +93.686 | 3 | +18.045 | 4 | -8.826 | 67.464 | 55.319 | 451 | 754 | 473 |
| 6 | 2 | +13.686 | 4 | +98.045 | 5 | +31.174 | 70.000 | 59.922 | 3530 | 1029 | 653 |
| 7 | 2 | -43.457 | 4 | -16.241 | 6 | +59.746 | 32.804 | 43.672 | 796 | 608 | 585 |
| 8 | 3 | +63.686 | 5 | +48.045 | 6 | -68.826 | 56.410 | 60.831 | 3075 | 1276 | 973 |
| 9 | 3 | +13.686 | 5 | -35.288 | 7 | -35.493 | 26.764 | 23.104 | 2858 | 723 | 433 |
| 10 | 3 | -26.314 | 6 | +18.045 | 8 | -8.826 | 22.561 | 19.113 | 1804 | 713 | 412 |
| 11 | 4 | +50.050 | 6 | -47.410 | 9 | +12.992 | 48.748 | 40.503 | 5737 | 1778 | 991 |
| 12 | 4 | +13.686 | 7 | -1.955 | 10 | +31.174 | 9.776 | 19.689 | 282 | 406 | 541 |
| 13 | 4 | -17.083 | 8 | +36.507 | 10 | -45.749 | 28.500 | 35.202 | 6170 | 1336 | 1076 |
| 14 | 5 | +42.258 | 8 | -16.241 | 11 | -25.969 | 32.012 | 30.132 | 3183 | 1677 | 1017 |
| 15 | 5 | +13.686 | 9 | +18.045 | 12 | -8.826 | 16.015 | 14.034 | 4060 | 930 | 519 |
| 16 | 5 | -11.314 | 9 | -26.955 | 13 | +6.174 | 20.671 | 17.250 | 6900 | 1323 | 695 |
| 17 | 5 | -33.373 | 10 | +3.927 | 14 | +19.409 | 23.761 | 22.404 | 1135 | 1665 | 979 |
| 18 | 6 | +13.686 | 11 | +31.378 | 15 | +31.174 | 24.207 | 26.732 | 10167 | 1849 | 1261 |
| 19 | 6 | -7.366 | 11 | -7.218 | 15 | -23.457 | 7.293 | 13.745 | 2606 | 604 | 697 |
| 20 | 6 | -26.314 | 12 | +18.045 | 16 | -8.826 | 22.561 | 19.113 | 7218 | 2018 | 1038 |
| 21 | 7 | +13.686 | 12 | -16.241 | 17 | +2.603 | 15.018 | 12.354 | 7162 | 1445 | 716 |
| 22 | 7 | -4.496 | 13 | +7.136 | 18 | +12.992 | 5.964 | 8.943 | 3454 | 615 | 551 |
| 31 | 10 | +0.783 | 18 | -5.181 | 25 | -1.084 | 3.705 | 3.089 | 4979 | 639 | 301 |
| 41 | 13 | -5.826 | 24 | +0.484 | 33 | -2.972 | 4.134 | 3.786 | 814 | 1085 | 535 |
| 53 | 17 | -1.408 | 31 | -0.068 | 43 | +4.759 | 0.997 | 2.866 | 192 | 385 | 570 |
| 65 | 21 | +1.379 | 38 | -0.417 | 52 | -8.826 | 1.018 | 5.163 | 1760 | 534 | 1349 |
| 68 | 22 | +1.922 | 40 | +3.927 | 55 | +1.762 | 3.092 | 2.722 | 18160 | 1734 | 755 |
| 72 | 23 | -2.980 | 42 | -1.955 | 58 | -2.159 | 2.520 | 2.406 | 10135 | 1540 | 721 |
| 84 | 27 | -0.599 | 49 | -1.955 | 68 | +2.603 | 1.446 | 1.911 | 13794 | 1113 | 703 |
| 99 | 32 | +1.565 | 58 | +1.075 | 80 | -0.871 | 1.343 | 1.206 | 10539 | 1323 | 552 |
| 118 | 38 | +0.127 | 69 | -0.260 | 95 | -2.724 | 0.205 | 1.582 | 3621 | 262 | 915 |
| 130 | 42 | +1.379 | 76 | -0.417 | 105 | +0.405 | 1.018 | 0.864 | 7040 | 1509 | 569 |
| 140 | 45 | -0.599 | 82 | +0.902 | 113 | -0.254 | 0.766 | 0.642 | 17682 | 1269 | 467 |
| 171 | 55 | -0.349 | 100 | -0.201 | 138 | -0.405 | 0.285 | 0.330 | 5866 | 636 | 313 |
| 441 | 142 | +0.081 | 258 | +0.086 | 356 | -0.118 | 0.083 | 0.096 | 16689 | 772 | 324 |
| 494 | 159 | -0.079 | 289 | +0.069 | 399 | +0.405 | 0.074 | 0.241 | 16909 | 815 | 943 |
| 612 | 197 | -0.039 | 358 | +0.006 | 494 | -0.198 | 0.028 | 0.117 | 2166 | 424 | 607 |
| 665 | 214 | -0.148 | 389 | -0.0001 | 537 | +0.197 | 0.105 | 0.142 | 50 | 1798 | 825 |

This table shows how well the scales based around equal divisions of the octave approximate the 5:4 major third, the 3:2 perfect fifth and the 7:4 seventh harmonic. The first column (q) gives the number of divisions to the octave. The second column (p_3) shows the scale degree closest to the 5:4 major third (counting from zero for the tonic), and the next column (e_3) shows the error in cents:

$$e_3 = 1200 \left(\frac{p_3}{q} - \log_2 \left(\frac{5}{4} \right) \right).$$

Similarly, the next two columns (p_5 and e_5) show the scale degree closest to the 3:2 perfect fifth and the error in cents:

$$e_5 = 1200 \left(\frac{p_5}{q} - \log_2 \left(\frac{3}{2} \right) \right).$$

The two columns after that (p_7 and e_7) show the scale degree closest to the 7:4 seventh harmonic and the error in cents:

$$e_7 = 1200 \left(\frac{p_7}{q} - \log_2 \left(\frac{7}{4} \right) \right).$$

We write e_{35} for the root mean square (RMS) error of the major third and perfect fifth:

$$e_{35} = \sqrt{(e_3^2 + e_5^2)/2}$$

and e_{357} for the RMS error for the major third, perfect fifth and seventh harmonic:

$$e_{357} = \sqrt{(e_3^2 + e_5^2 + e_7^2)/3}.$$

Theorem 6.2.3 shows that the quantity $e_5 \cdot q^2$ is a good measure of how well the perfect fifth is approximated by p_5/q of an octave, with respect to the number of notes in the scale. This theorem shows that there are infinitely many values of q for which $e_5 \cdot q^2 < 1200$, while on average we should expect this quantity to grow linearly with q .

Similarly, Theorem 6.2.5 with $k = 2$ shows that the quantity $e_{35} \cdot q^{\frac{3}{2}}$ is a good measure of how well the major third and perfect fifth are simultaneously approximated, and shows that there are infinitely many values of q for which $e_{35} \cdot q^{\frac{3}{2}} < 1200$, while on average we should expect this quantity to grow like the square root of q . Theorem 6.2.5 with $k = 3$ shows that the quantity $e_{357} \cdot q^{\frac{4}{3}}$ is a good measure of how well all three intervals: major third, perfect fifth and seventh harmonic are simultaneously approximated, and shows that there are infinitely many values of q for which $e_{357} \cdot q^{\frac{4}{3}} < 1200$, while on average we should expect this quantity to grow like the cube root of q .

Particularly good values of $e_5 \cdot q^2$, $e_{35} \cdot q^{\frac{3}{2}}$ and $e_{357} \cdot q^{\frac{4}{3}}$ are indicated in bold face in the last three columns of the table.

APPENDIX I

Intervals

This is a table of intervals not exceeding one octave (or a tritave in the case of the Bohlen–Pierce, or BP scale). A much more extensive table may be found in Appendix XX to Helmholtz [43] (page 453), which was added by the translator, Alexander Ellis. Names of notes in the BP scale are denoted with a subscript BP, to save confusion with notes which may have the same name in the octave based scale.

The first column is equal to 1200 times the logarithm to base two of the ratio given in the second column. Logarithms to base two can be calculated by taking the natural logarithm and dividing by $\ln 2$. So the first column is equal to

$$\frac{1200}{\ln 2} \approx 1731.234$$

times the natural logarithm of the second column.

We have given all intervals to three decimal places for theoretical purposes. While intervals of less than a few cents are imperceptible to the human ear in a melodic context, in harmony very small changes can cause large changes in beats and roughness of chords. Three decimal places gives great enough accuracy that errors accumulated over several calculations should not give rise to perceptible discrepancies.

| Cents | Interval ratio | Eitz | Name, etc. | Ref |
|---------|-------------------------|-----------------------------------|---|-------|
| 0.000 | 1:1 | C^0, C_{BP}^0 | Fundamental | §4.1 |
| 1.000 | $2^{\frac{1}{1200}}:1$ | | Cent | §5.4 |
| 1.805 | $2^{\frac{1}{665}}:1$ | | Degree of 665 tone scale | §6.4 |
| 1.953 | 32805:32768 | $B\sharp^{-1}$ | Schisma | §5.6 |
| 3.986 | $10^{\frac{1}{1000}}:1$ | | Savart | §5.4 |
| 14.191 | 245:243 | C_{BP}^{+1} | BP-minor diesis | §6.7 |
| 19.553 | 2048:2025 | $D\flat\flat^{+2}$ | Diaschisma | §5.6 |
| 21.506 | 81:80 | C^{+1} | Syntonic, or ordinary comma | §5.5 |
| 22.642 | $2^{\frac{1}{53}}:1$ | | Degree of 53 tone scale | §6.3 |
| 23.460 | $3^{12}:2^{19}$ | $B\sharp^0$ | Pythagorean comma | §5.2 |
| 27.264 | 64:63 | | Septimal comma | §5.6 |
| 35.099 | | | Carlos' γ scale degree | §6.6 |
| 41.059 | 128:125 | $D\flat\flat^{+3}$ | Great diesis | §5.10 |
| 49.772 | $7^{13}:3^{23}$ | $D\flat\flat_{BP}^0$ | BP 7/3 comma | §6.7 |
| 63.833 | | | Carlos' β scale degree | §6.6 |
| 70.672 | 25:24 | $C\sharp^{-2}$ | Small (just) semitone | §5.5 |
| 77.965 | | | Carlos' α scale degree | §6.6 |
| 90.225 | 256:243 | $D\flat^0$ | Diesis or Limma | §5.2 |
| 100.000 | $2^{\frac{1}{12}}:1$ | $\approx C\sharp^{-\frac{7}{11}}$ | Equal semitone | §5.12 |
| 111.731 | 16:15 | $D\flat^{+1}$ | Just minor semitone (ti-do, mi-fa) | §5.5 |
| 113.685 | 2187:2048 | $C\sharp^0$ | Pythagorean apotomē | §5.2 |
| 133.238 | 27:25 | $D\flat_{BP}^{-2}$ | | §6.7 |
| 146.304 | $3^{\frac{1}{13}}:1$ | | BP-equal semitone | §6.7 |
| 182.404 | 10:9 | D^{-1} | Just minor tone (re-mi, so-la) | §5.5 |
| 193.157 | $\sqrt{5}:2$ | $D^{-\frac{1}{2}}$ | Meantone whole tone | §5.10 |
| 200.000 | $2^{\frac{1}{6}}:1$ | $\approx D^{-\frac{2}{11}}$ | Equal whole tone | §5.12 |
| 203.910 | 9:8 | D^0 | Just major tone (do-re, fa-so, la-ti); | §5.5 |
| | | | Pythagorean major tone; | §5.2 |
| | | | Nineth harmonic | §4.1 |
| 294.135 | 32:27 | $E\flat^0$ | Pythagorean minor third | §5.2 |
| 300.000 | $2^{\frac{1}{4}}:1$ | $\approx E\flat^{+\frac{3}{11}}$ | Equal minor third | §5.12 |
| 315.641 | 6:5 | $E\flat^{+1}$ | Just minor third (mi-so, la-do, ti-re) | §5.5 |
| 386.314 | 5:4 | E^{-1} | Just major third (do-mi, fa-la, so-ti); | §5.5 |
| | | | Meantone major third; | §5.10 |
| | | | Fifth harmonic | §4.1 |
| 400.000 | $2^{\frac{1}{3}}:1$ | $\approx E^{-\frac{4}{11}}$ | Equal major third | §5.12 |
| 407.820 | 81:64 | E^0 | Pythagorean major third | §5.2 |
| 498.045 | 4:3 | F^0 | Perfect fourth | §5.2 |

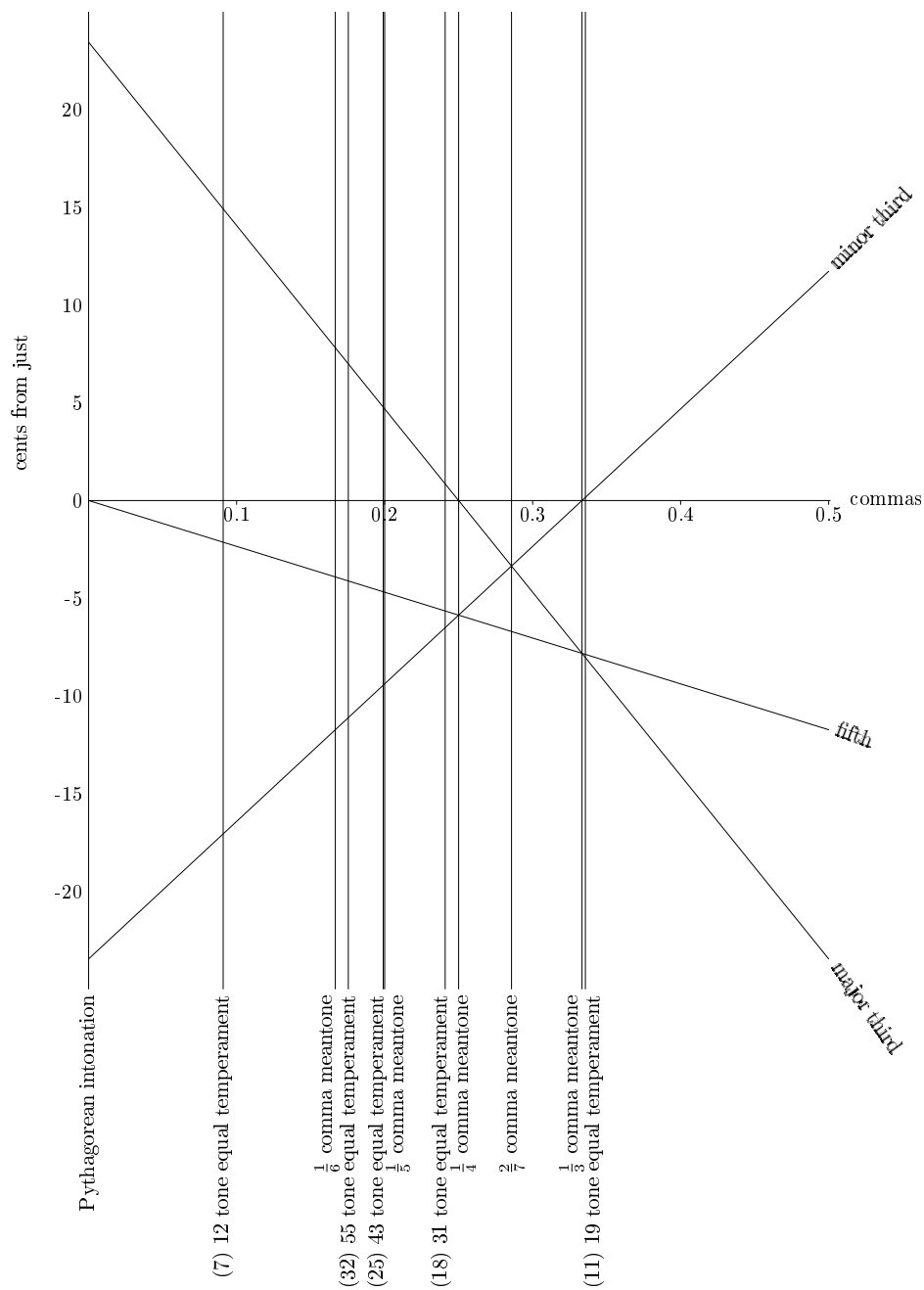
| Cents | Interval ratio | Eitz | Name, etc. | Ref |
|----------|-----------------------|------------------------------------|---------------------------------------|-------|
| 500.000 | $2^{\frac{5}{12}}:1$ | $\approx F^{+\frac{1}{11}}$ | Equal fourth | §5.12 |
| 503.422 | $2:5^{\frac{1}{4}}$ | $F^{+\frac{1}{4}}$ | Meantone fourth | §5.10 |
| 551.318 | 11:8 | | Eleventh harmonic | §4.1 |
| 600.000 | $\sqrt{2}:1$ | $\approx F^{\sharp -\frac{6}{11}}$ | Equal tritone | §5.12 |
| 611.731 | 729:512 | $F^{\sharp 0}$ | Pythagorean tritone | §5.2 |
| 696.579 | $5^{\frac{1}{4}}:1$ | $G^{-\frac{1}{4}}$ | Meantone fifth | §5.10 |
| 700.000 | $2^{\frac{7}{12}}:1$ | $\approx G^{-\frac{1}{11}}$ | Equal fifth | §5.12 |
| 701.955 | 3:2 | G^0 | Just and Pythagorean (perfect) fifth; | §5.2 |
| | | | Third harmonic | §4.1 |
| 792.180 | 128:81 | $A^b 0$ | Pythagorean minor sixth | §5.2 |
| 800.000 | $2^{\frac{2}{3}}:1$ | $\approx A^b^{+\frac{4}{11}}$ | Equal minor sixth | §5.12 |
| 813.687 | 8:5 | A^b^{+1} | Just minor sixth | §5.5 |
| 840.528 | 13:8 | | Thirteenth harmonic | §4.1 |
| 884.359 | 5:3 | A^{-1} | Just major sixth | §5.5 |
| 889.735 | $5^{\frac{3}{4}}:2$ | $A^{-\frac{3}{4}}$ | Meantone major sixth | §5.10 |
| 900.000 | $2^{\frac{3}{4}}:1$ | $\approx A^{-\frac{3}{11}}$ | Equal major sixth | §5.12 |
| 905.865 | 27:16 | A^0 | Pythagorean major sixth | §5.2 |
| 968.826 | 7:4 | | Seventh harmonic | §4.1 |
| 996.091 | 16:9 | $B^b 0$ | Pythagorean minor seventh | §5.2 |
| 1000.000 | $2^{\frac{5}{8}}:1$ | $\approx B^b^{+\frac{2}{11}}$ | Equal minor seventh | §5.12 |
| 1082.892 | $5^{\frac{5}{4}}:4$ | $B^{-\frac{5}{4}}$ | Meantone major seventh | §5.10 |
| 1088.269 | 15:8 | B^{-1} | Just major seventh; | §5.5 |
| | | | Fifteenth harmonic | §4.1 |
| 1100.000 | $2^{\frac{11}{12}}:1$ | $\approx B^{-\frac{5}{11}}$ | Equal major seventh | §5.12 |
| 1109.775 | 243:128 | B^0 | Pythagorean major seventh | §5.2 |
| 1200.000 | 2:1 | C^0 | Octave; Second harmonic | §4.1 |
| 1466.871 | 7:3 | A_{BP}^0 | BP-tenth | §6.7 |
| 1901.955 | 3:1 | C_{BP}^0 | BP-Tritave | §6.7 |

APPENDIX J

Just, equal and meantone scales compared

The figure on the next page has its horizontal axis measured in multiples of the (syntonic) comma, and the vertical axis measured in cents. Each vertical line represents a regular scale, generated by its fifth. The size of the fifth in the scale is equal to the Pythagorean fifth (ratio of 3:2, or 701.955 cents) minus the multiple of the comma given by the position along the horizontal axis. The three sloping lines show how far from the just values the fifth, major third and minor third are in these scales. This figure is relevant to Exercise **2** in §6.4.

It is worth noting that if $\frac{1}{11}$ comma meantone were drawn on this diagram, it would be indistinguishable from 12 tone equal temperament; see §5.12.



Regular scales and their deviations from just intonation

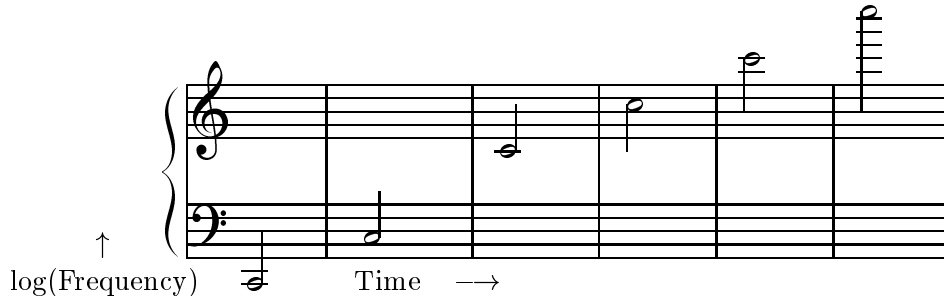
Music theory

The diagram shows a 12-tone scale with 12 vertical bars. Each bar contains two stacked boxes. The notes are arranged in a sequence of 12 bars, with some bars containing two notes (e.g., C and D) and others containing one (e.g., E). The notes are color-coded: C (blue), D (orange), E (green), F (red), G (purple), A (brown), B (pink).

On a modern keyboard, each of the twelve intervals making up an octave represents the same frequency ratio, called a *semitone*. The name comes from the fact that two semitones make a *tone*. The twelfth power of the semitone's frequency ratio is a factor of 2:1, so a semitone represents a frequency ratio of $2^{\frac{1}{12}}$:1. The arrangement where all the semitones are equal in this way is called *equal temperament*. Frequency is an exponential function of position on the keyboard, and so the keyboard is really a *logarithmic* representation of frequency.

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Staff notation works in a similar way, except that the logarithmic frequency is represented vertically, and the horizontal direction represents time. So music notation paper can be regarded as graph paper with a linear horizontal time axis and a logarithmic vertical frequency axis.



In the above diagram, each note is twice the frequency of the previous one, so they are equally spaced on the logarithmic frequency scale (except for the break between the bass and treble clefs). The gap between adjacent notes is one octave, so the gap between the lowest and highest note is described *additively* as five octaves, representing a *multiplicative* frequency ratio of $2^5:1$.

There are two clefs on this diagram. The upper one is called the *treble clef*, with lines representing the notes E, G, B, D, F, beginning with the E two white notes above middle C and working up the lines. The spaces between them represent the notes F, A, C, E between them, so that this takes care of all the white notes between the E above middle C and the F an octave and a semitone above that. The black notes are represented in by using the line or space with the likewise lettered white note with a sharp (#) or flat (b) sign in front.

The lower clef is called the *bass clef*, with lines representing the notes G, B, D, F, A, with the last note representing the A two white notes below middle C and the first note representing the G an octave and a tone below that.

Middle C itself is represented using a *leger line*, either below the treble clef or above the bass clef.



The frequency ratio represented by seven semitones, for example the interval from C to the G above it, is called a *perfect fifth*. Well, actually, this isn't quite true. A perfect fifth is supposed to be a frequency ratio of $3:2$, or $1.5:1$, whereas seven semitones on our modern equal tempered scale produce a frequency ratio of $2^{7/12}:1$ or roughly $1.4983:1$. The perfect fifth is a consonant interval, just as the octave is, for reasons described in Chapter 4. So seven semitones is very close to a consonant interval. It is very difficult to

discern the difference between a perfect fifth and an equal tempered fifth except by listening for beats; the difference is about one fiftieth of a semitone.

The perfect fourth represents the interval of 4:3, which is also consonant. The difference between a perfect fourth and the equal tempered fourth of five semitones is exactly the same as the difference between the perfect fifth and the equal tempered fifth, because they are obtained from the corresponding versions of a fifth by subtracting from an octave.

The frequency ratio represented by four semitones, for example the interval from C to the E above it, is called a *major third*. This represents a frequency ratio of $2^{4/12}:1$ or $\sqrt[3]{2}:1$, or roughly 1.25992:1. The *just major third* is defined to be the frequency ratio of 5:4 or 1.25:1. Again it is the just major third which represents the consonant interval, and the major third on our modern equal tempered scale is an approximation to it. The approximation is quite a bit worse than it was for the perfect fifth. The difference between a just major third and an equal tempered major third is quite audible; the difference is about one seventh of a semitone.

The frequency ratio represented by three semitones, for example the interval from E to the G above it, is called a *minor third*. This represents a frequency ratio of $2^{3/12}:1$ or $\sqrt[4]{2}:1$, or roughly 1.1892:1. The consonant *just minor third* is defined to be the frequency ratio of 6:5 or 1.2:1. The equal tempered minor third again differs from it by about a seventh of a semitone.

A major third plus a minor third makes up a fifth, either in the just/perfect versions or the equal tempered versions. So the intervals C to E (major third) plus E to G (minor third) make C to G (fifth). In the just/perfect versions, this gives ratios 4:5:6 for a *just major chord* C—E—G. We refer to C as the *root* of this chord. The chord is named after its root, so that this is a C major chord.



4:5:6

If we used the frequency ratios 3:4:5, it would just give an *inversion* of this chord, which is regarded as a variant form of the C major chord, because of the principle of octave equivalence.



3:4:5

while the frequency ratios 2:3:4 give a much simpler chord with a fifth and an octave.



2:3:4

So the just major chord 4:5:6 is the chord that is basic to the western system of musical harmony. On an equal tempered keyboard, this is approximated with the chord $2^{7/12}:2^{4/12}:1$, which is a good approximation except for the somewhat sharp major third.

The *major scale* is formed by taking three major chords on three notes separated by intervals of a fifth. So for example the scale of C major is formed from the notes of the chords F major, C major and G major. Between them, these account for the white notes on the keyboard, which make up the scale of C major. So in just intonation, the C major scale would have the following frequency ratios.

| C | D | E | F | G | A | B | C | D |
|---------------|---------------|---------------|---------------|---------------|---------------|----------------|---------------|---------------|
| $\frac{1}{1}$ | $\frac{9}{8}$ | $\frac{5}{4}$ | $\frac{4}{3}$ | $\frac{3}{2}$ | $\frac{5}{3}$ | $\frac{15}{8}$ | $\frac{2}{1}$ | $\frac{9}{4}$ |
| 4 | : | 5 | : | 6 | : | (8) | | |
| <hr/> | | | | | | | | |
| | | 4 | : | 5 | : | 6 | | |
| <hr/> | | | | | | | | |
| (3) | : | 4 | : | 5 | : | 6 | | |

Here, we have made use of 2:1 octaves to transfer ratios between the right and left end of the diagram.

The basic problem with this scale is that the interval from D to A is almost, but not quite equal to a perfect fifth. It is just close enough that it sounds like a nasty, out of tune fifth. It is short of a perfect fifth by a ratio of 81:80. This interval is called a *syntonic comma*. In this text, when we use the word comma without further qualification, it will always mean the syntonic comma. This and other commas are investigated in Section 5.6.

The *meantone* scale addresses this problem by distributing the syntonic comma equally between the four fifths C–G–D–A–E. So in the meantone scale, the fifths are one quarter of a comma smaller than the perfect fifth, and the major thirds are just. In the meantone scale, a number of different keys work well, but the more remote keys do not. For further details, see Section 5.10.

To make all keys work well, the meantone scale must be bent to meet around the back. A number of different versions of this compromise have been used historically, the first ones being due to Werckmeister. Some of these well tempered scales are described in Section 5.11. Meantone and well tempered scales were in common use for about four centuries before equal temperament became widespread in the late nineteenth and early twentieth century.

APPENDIX O

Online papers

Several journals have good selections of papers available online. Access usually requires you to be logged on from an academic establishment which subscribes to the journal in question. Here is a selection of what is available from a typical academic institution.

From <http://www.jstor.org> you can obtain online copies of papers from the American Mathematics Monthly, a publication which concentrates on undergraduate level mathematics. Papers include the following, in chronological order.

- J. M. Barbour, *Synthetic musical scales*, Amer. Math. Monthly 36 (3) (1929), 155–160.
- J. M. Barbour, *A sixteenth century approximation for π* , Amer. Math. Monthly 40 (2) (1933), 69–73.
- J. M. Barbour, *Music and ternary continued fractions*, Amer. Math. Monthly 55 (9) (1948), 545–555.
- J. B. Rosser, *Generalized ternary continued fractions*, Amer. Math. Monthly 57 (8) (1950), 528–535.
- J. M. Barbour, *A geometrical approximation to the roots of numbers*, Amer. Math. Monthly 64 (1) (1957), 1–9.
- Mark Kac, *Can one hear the shape of a drum?* Amer. Math. Monthly 73 (4) (1966), 1–23.
- John Rogers and Bary Mitchell, *A problem in mathematics and music*, Amer. Math. Monthly 75 (8) (1968), 871–873.
- A. L. Leigh Silver, *Musimatics, or the nun's fiddle*, Amer. Math. Monthly 78 (4) (1971), 351–357.
- G. D. Hasley and Edwin Hewitt, *More on the superparticular ratios in music*, Amer. Math. Monthly 79 (10) (1972), 1096–1100.
- I. J. Schoenberg, *On the location of the frets on the guitar*, Amer. Math. Monthly 83 (7) (1976), 550–552.
- David Gale, *Tone perception and decomposition of periodic function*, Amer. Math. Monthly 86 (1) (1979), 36–42.
- Murray Schechter, *Tempered scales and continued fractions*, Amer. Math. Monthly 87 (1) (1980), 40–42.
- David L. Reiner, *Enumeration in music theory*, Amer. Math. Monthly 92 (1) (1985), 51–54.
- John Clough and Gerald Myerson, *Musical scales and the generalized circle of fifths*, Amer.

Math. Monthly 93 (9) (1986), 695–701.

S. J. Chapman, *Drums that sound the same*, Amer. Math. Monthly 102 (2) (1995), 124–138.

Rachel W. Hall and Krešimir Josić, *The mathematics of musical instruments*, Amer. Math. Monthly 108 (4) (2001), 347–357.

There are occasionally relevant articles in the SIAM¹ journals, also available from <http://www.jstor.org>. Examples include the following.

A. A. Goldstein, *Optimal temperament*, SIAM Review 19 (3) (1977), 554–562.

A. Inselberg, *Cochlear dynamics: the evolution of a mathematical model*, SIAM Review 20 (2) (1978), 301–351.

Robert Burridge, Jay Kappraff and Christine Mordeshi, *The Sitar string, a vibrating string with a one-sided inelastic constraint*, SIAM J. Appl. Math. 42 (6) (1982), 1231–1251.

M. H. Protter, *Can one hear the shape of a drum? Revisited*, SIAM Review 29 (2) (1987), 185–197.

Tobin A. Driscoll, *Eigenmodes of isospectral drums*, SIAM Review 39 (1) (1997), 1–17.

From <http://ojps.aip.org/jasa/> (then hit “browse html” or “search”) you can obtain online copies of articles from the Journal of the Acoustical Society of America (JASA) from 1997 to the current issue. Here is a selection of some relevant articles that can be downloaded.

Donald L. Sullivan, *Accurate frequency tracking of timpani spectral lines*, JASA 101 (1) (1997), 530–538.

Antoine Chaigne and Vincent Doutaut, *Numerical simulations of xylophones. I. Time-domain modeling of the vibrating bars*, JASA 101 (1) (1997), 539–557.

Hugh J. McDermott and Colette M. McKay, *Musical pitch perception with electrical stimulation of the cochlea*, JASA 101 (3) (1997), 1622–1631.

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Knut Guettler and Anders Askenfelt, *Acceptance limits for the duration of pre-Helmholtz transients in bowed string attacks*, JASA 101 (5) (1997), 2903–2913.

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David M. Mills, *Interpretation of distortion product otoacoustic emission measurements. I. Two stimulus tones*, JASA 102 (1) (1997), 413–429.

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- Guy Vandegrift and Eccles Wall, *The spatial inhomogeneity of pressure inside a violin at main air resonance*, JASA 102 (1) (1997), 622–627.
- Harold A. Conklin, Jr., *Piano strings and “phantom” partials*, JASA 102 (1) (1997), 659.
- I. Winkler, M. Tervaniemi and R. Näätänen, *Two separate codes for missing-fundamental pitch in the human auditory cortex*, JASA 102 (2) (1997), 1072–1082.
- Alain de Cheveigné, *Harmonic fusion and pitch shifts of mistuned partials*, JASA 102 (2) (1997), 1083–1087.
- Robert P. Carlyon, *The effects of two temporal cues on pitch judgments*, JASA 102 (2) (1997), 1097–1105.
- N. Giordano, *Simple model of a piano soundboard*, JASA 102 (2) (1997), 1159–1168.
- Ray Meddis and Lowel O’Mard, *A unitary model of pitch perception*, JASA 102 (3) (1997), 1811–1820.
- Bruno H. Repp, *Acoustics, perception, and production of legato articulation on a computer-controlled grand piano*, JASA 102 (3) (1997), 1878–1890.
- William A. Sethares, *Specifying spectra for musical scales*, JASA 102 (4) (1997), 2422–2431.
- Eric D. Scheirer, *Tempo and beat analysis of acoustic musical signals*, JASA 103 (1) (1998), 588–601.
- Myeong-Hwa Lee, Jeong-No Lee and Kwang-Sup Soh, *Chaos in segments from Korean traditional singing and Western singing*, JASA 103 (2) (1998), 1175–1182.
- Alain de Cheveigné, *Cancellation model of pitch perception*, JASA 103 (3) (1998), 1261–1271.
- Louise J. White and Christopher J. Plack, *Temporal processing of the pitch of complex tones*, JASA 103 (4) (1998), 2051–2063.
- N. Giordano, *Mechanical impedance of a piano soundboard*, JASA 103 (4) (1998), 2128–2133.
- Henry T. Bahnson, James F. Antaki and Quinter C. Beery, *Acoustical and physical dynamics of the diatonic harmonica*, JASA 103 (4) (1998), 2134–2144.
- Jian-Yu Lin and William M. Hartmann, *The pitch of a mistuned harmonic: evidence for a template model*, JASA 103 (5) (1998), 2608–2617.
- Shigeru Yoshikawa, *Jet-wave amplification in organ pipes*, JASA 103 (5) (1998), 2706–2717.
- Teresa D. Wilson and Douglas H. Keefe, *Characterizing the clarinet tone: measurements of Lyapunov exponents, correlation dimension, and unsteadiness*, JASA 104 (1) (1998), 550–561.
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N. H. Fletcher, *A class of chaotic bird calls?*, JASA 108 (2) (2000), 821–826.

Gabriel Weinreich, Colin Holmes and Maureen Melody, *Air-wood coupling and the Swiss-cheese violin*, JASA 108 (5) (2000), 2389–2402.

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R. Ranvaud, W. F. Thompson, L. Silveira-Moriyama and L.-L. Balkwill, *The speed of pitch resolution in a musical context*, JASA 109 (6) (2001), 3021–3030.

Jeffrey M. Brunstrom and Brian Roberts, *Effects of asynchrony and ear of presentation on the pitch of mistuned partials in harmonic and frequency-shifted complex tones*, JASA 110 (1) (2001), 391–401.

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Werner Goebel, *Melody lead in piano performance: Expressive device or artifact?*, JASA 110 (1) (2001), 563–572.

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Diana S. Dabby, *Musical variations from a chaotic mapping*, Chaos 6 (2) (1996), 95–107.

From <http://www.elsevier.com> you can download the following papers.

R. C. Read, *Combinatorial problems in the theory of music*, Discrete Mathematics 167/168 (1997), 543–551.

Ján Haluška, *Equal temperament and Pythagorean tuning: a geometrical interpretation in the plane*, Fuzzy Sets and Systems 114 (2000), 261–269.

From <http://www.idealibrary.com>, you can obtain online copies of papers from a number of journals; for example, the following papers come from the Journal of Sound and Vibration.

F. Gautier and N. Tahani, *Vibroacoustic behaviour of a simplified musical wind instrument*, Journal of Sound and Vibration 213 (1) (1998), 107–125.

S. Gaudet, C. Gauthier and V. G. LeBlanc, *On the vibrations of an N-string*, Journal of Sound and Vibration 238 (1) (2000), 147–169.

APPENDIX P

Partial derivatives

Partial derivatives are what happens when we differentiate a function of more than one variable. For example, a geographical map which indicates height above sea level, by some device such as coloration or contours, can be regarded as describing a function $z = f(x, y)$. Here, x and y represent the two coordinates of the map, and z denotes height above sea level. If we move due east, which we take to be the direction of the x axis, then we are keeping y constant and changing x . So the slope in this direction would be the derivative of $z = f(x, y)$ with respect to x , regarding y as a constant. This derivative is denoted $\frac{\partial z}{\partial x}$. More formally,

$$\frac{\partial z}{\partial x} = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}.$$

Similarly, $\frac{\partial z}{\partial y}$ is the derivative of z with respect to y , regarding x as a constant. As an example, let $z = x^4 + x^2y - 2y^2$. Then we have $\frac{\partial z}{\partial x} = 4x^3 + 2xy$, because x^2y is being regarded as a constant multiple of x^2 , and $-2y^2$ is just a constant. Similarly, $\frac{\partial z}{\partial y} = x^2 - 4y$, because x^4 is a constant and x^2y is a constant multiple of y .

Second partial derivatives are defined similarly, but we now find that we can mix the variables. As well as $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$, we can now form $\frac{\partial^2 z}{\partial x \partial y}$ by taking the partial derivative of $\frac{\partial z}{\partial y}$ with respect to x , regarding y as constant, and we can also form $\frac{\partial^2 z}{\partial y \partial x}$ by taking partial derivatives in the opposite order. So in the above example, we have

$$\frac{\partial^2 z}{\partial x^2} = 12x^2 + 2y, \quad \frac{\partial^2 z}{\partial y^2} = -4, \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = 2x.$$

In fact, the two mixed partial derivatives agree under some fairly mild hypotheses.

THEOREM P.1. Suppose that the partial derivatives $\frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^2 z}{\partial y \partial x}$ both exist and are both continuous at some point (i.e., for some chosen values of x and y). Then they are equal at that point.

PROOF. See any book on elementary analysis; for example, J. C. Burkhill, *A first course in mathematical analysis*, CUP, 1962, theorem 8.3. \square

Partial derivatives work in exactly the same way for functions of more variables. So for example if $f(x, y, z) = xy^2 \sin z$ then we have $\frac{\partial f}{\partial x} = y^2 \sin z$, $\frac{\partial f}{\partial y} = 2xy \sin z$, and $\frac{\partial f}{\partial z} = xy^2 \cos z$. For each pair of variables, the two mixed partial derivatives with respect to those variables agree provided they are both continuous.

The chain rule for partial derivatives needs some care. Suppose, by way of example, that z is a function of u , v and w , and that each of u , v and w is a function of x and y . Then z can also be regarded as a function of x and y . A change in the value of x , keeping y constant, will result in a change of all of u , v and w , and each of these changes will result in a change in the value of z . These changes have to be added as follows:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} + \frac{\partial z}{\partial w} \frac{\partial w}{\partial x}.$$

Similarly, we have

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y} + \frac{\partial z}{\partial w} \frac{\partial w}{\partial y}.$$

It is essential to keep track of which variables are independent, intermediate, and dependent. In this example, the independent variables are x and y , the intermediate ones are u , v and w , and the dependent variable is z .

A good illustration of the chain rule for partial derivatives is given by the conversion from Cartesian to polar coordinates. If z is a function of x and y then it can also be regarded as a function of r and θ . To convert from polar to Cartesian coordinates, we use $x = r \cos \theta$ and $y = r \sin \theta$, and to convert back we use $r = \sqrt{x^2 + y^2}$ and $\tan \theta = y/x$. Let us convert the quantity

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2},$$

into polar coordinates, assuming that all mixed second partial derivatives are continuous, so that the above theorem applies. This calculation will be needed in §3.5, where we investigate the vibrational modes of the drum. For this purpose, it is actually technically slightly easier to regard x and y as the intermediate variables and r and θ as the independent variables, although it would be quite permissible to interchange their roles. The dependent variable is z . We have

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r} = \cos \theta \frac{\partial z}{\partial x} + \sin \theta \frac{\partial z}{\partial y}. \quad (\text{P.1})$$

To take the second derivative, we do the same again.

$$\begin{aligned}
 \frac{\partial^2 z}{\partial r^2} &= \cos \theta \frac{\partial}{\partial r} \left(\frac{\partial z}{\partial x} \right) + \sin \theta \frac{\partial}{\partial r} \left(\frac{\partial z}{\partial y} \right) \\
 &= \cos \theta \left(\cos \theta \frac{\partial^2 z}{\partial x^2} + \sin \theta \frac{\partial^2 z}{\partial y \partial x} \right) + \sin \theta \left(\cos \theta \frac{\partial^2 z}{\partial x \partial y} + \sin \theta \frac{\partial^2 z}{\partial y^2} \right) \\
 &= \cos^2 \theta \frac{\partial^2 z}{\partial x^2} + 2 \sin \theta \cos \theta \frac{\partial^2 z}{\partial x \partial y} + \sin^2 \theta \frac{\partial^2 z}{\partial y^2}.
 \end{aligned} \tag{P.2}$$

Similarly, we have

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial \theta} = (-r \sin \theta) \frac{\partial z}{\partial x} + (r \cos \theta) \frac{\partial z}{\partial y},$$

and

$$\begin{aligned}
 \frac{\partial^2 z}{\partial \theta^2} &= (-r \sin \theta) \frac{\partial}{\partial \theta} \left(\frac{\partial z}{\partial x} \right) + (-r \cos \theta) \frac{\partial z}{\partial x} \\
 &\quad + (r \cos \theta) \frac{\partial}{\partial \theta} \left(\frac{\partial z}{\partial y} \right) + (-r \sin \theta) \frac{\partial z}{\partial y} \\
 &= (-r \sin \theta) \left((-r \sin \theta) \frac{\partial^2 z}{\partial x^2} + (r \cos \theta) \frac{\partial^2 z}{\partial y \partial x} \right) + (-r \cos \theta) \frac{\partial z}{\partial x} \\
 &\quad + (r \cos \theta) \left((-r \sin \theta) \frac{\partial^2 z}{\partial x \partial y} + (r \cos \theta) \frac{\partial^2 z}{\partial y^2} \right) + (-r \cos \theta) \frac{\partial z}{\partial y} \\
 &= r^2 \left(\sin^2 \theta \frac{\partial^2 z}{\partial x^2} - 2 \sin \theta \cos \theta \frac{\partial^2 z}{\partial x \partial y} + \cos^2 \theta \frac{\partial^2 z}{\partial y^2} \right) \\
 &\quad - r \left(\cos \theta \frac{\partial z}{\partial x} + \sin \theta \frac{\partial z}{\partial y} \right).
 \end{aligned} \tag{P.3}$$

Comparing the formula (P.2) for $\frac{\partial^2 z}{\partial r^2}$ with the formula (P.3) for $\frac{\partial^2 z}{\partial \theta^2}$, and using the fact that $\sin^2 \theta + \cos^2 \theta = 1$, we see that

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 z}{\partial \theta^2} = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} - \frac{1}{r} \left(\cos \theta \frac{\partial z}{\partial x} + \sin \theta \frac{\partial z}{\partial y} \right).$$

Finally, looking back at equation (P.1) for $\frac{\partial z}{\partial r}$, we obtain the formula we were looking for, namely

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r} \frac{\partial z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 z}{\partial \theta^2} = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}. \tag{P.4}$$

APPENDIX R

Recordings

Go to the entry “compact discs” in the index to find the points in the text which refer to these recordings.

Johann Sebastian Bach, *The Complete Organ Music*, recorded by Hans Fagius, Volumes 6 and 8, BIS-CD-397/398 (1989) and BIS-CD-443/444 (1989 & 1990). These recordings are played on the reconstructed 1764 Wahlberg organ, Fredrikskyrkan, Karlskrona, Sweden. This organ was reconstructed using the original temperament, which was Neidhardt’s Circulating Temperament No. 3 “für eine grosse Stadt” (for a large town).

Clarence Barlow’s “OTodeBLU” is in 17 tone equal temperament, played on two pianos. This piece was composed in celebration of John Pierce’s eightieth birthday, and appeared as track 15 on the Computer Music Journal’s Sound Anthology CD, 1995, to accompany volumes 15–19 of the journal. The CD can be obtained from MIT press for \$15.

Easley Blackwood has composed a set of microtonal compositions in each of the equally tempered scales from 13 tone to 24 tone, as part of a research project funded by the National Endowment for the Humanities to explore the tonal and modal behavior of these temperaments. He devised notations for each tuning, and his compositions were designed to illustrate chord progressions and practical application of his notations. The results are available on compact disc as Cedille Records CDR 90000 018, Easley Blackwood: *Microtonal Compositions* (1994). Copies of the scores of the works can be obtained from Blackwood Enterprises, 5300 South Shore Drive, Chicago, IL 60615, USA for a nominal cost.

Dietrich Buxtehude, *Orgelwerke*, Volumes 1–7, recorded by Harald Vogel, published by Dabringhaus and Grimm. These works are recorded on a variety of European organs in different temperaments. Extensive details are given in the liner notes.

CD1 Tracks 1–8: Norden – St. Jakobi/Kleine organ in Werckmeister III;

Tracks 9–15: Norden – St. Ludgeri organ in modified $\frac{1}{5}$ Pythagorean comma meantone with $C_{\sharp}^{-\frac{6}{5}p}$, $G_{\sharp}^{-\frac{6}{5}p}$, $Bb^{+\frac{1}{5}p}$ and Eb^0 ;

CD2 Tracks 1–6: Stade – St. Cosmae organ in modified quarter comma meantone with¹ $C_{\sharp}^{-\frac{3}{2}}$, $G_{\sharp}^{-\frac{3}{2}}$, F^0 , Bb^0 , $Eb^{-\frac{1}{5}}$;

Tracks 7–15: Weener – Georgskirche organ in Werckmeister III;

CD3 Tracks 1–10: Grasberg organ in Neidhardt No. 3;

Tracks 11–14: Damp – Herrenhaus organ in modified meantone with pitches taken from original pipe lengths;

¹The liner notes are written as though $G_{\sharp}^{-\frac{3}{2}}$ were equal to $Ab^{-\frac{2}{5}}$, which is not quite true. But the discrepancy is only about 0.2 cents.

CD4 Tracks 1–8: Noordbroeck organ in Werckmeister III;

Tracks 9–15: Groningen – Aa-Kerk organ in (almost) equal temperament;

CD5 Tracks 1–5: Pilsum organ in modified $\frac{1}{5}$ Pythagorean comma meantone (the same as the Norden – St. Ludgeri organ described above);

Tracks 6–7: Buttforde organ;

Tracks 8–10: Langwarden organ in modified quarter comma meantone with $G\sharp^{-\frac{7}{4}}$, $Bb^{-\frac{1}{4}}$, $Eb^{-\frac{1}{4}}$;

Tracks 11–13: Basedow organ in quarter comma meantone;

Tracks 14–15: Groß Eichsen organ in quarter comma meantone;

CD6 Tracks 1–10: Roskilde organ in Neidhardt (no. 3?);

Track 11: Helsingør organ (unspecified temperament);

Tracks 12–15: Torrlösa organ (unspecified temperament);

CD7 Tracks 1–10 modified $\frac{1}{5}$ comma meantone with² $C\sharp^{-\frac{6}{5}}$, $G\sharp^{-\frac{6}{5}}$, $Bb^{+\frac{1}{5}}$ and $Eb^{\frac{1}{5}-\frac{1}{10}P}$.

William Byrd, *Cantones Sacrae 1575, The Cardinall's Music*, conducted by David Skinner. Track 12, *Diliges Dominum*, exhibits temporal reflectional symmetry, so that it is a perfect palindrome.

Wendy Carlos, *Switched-On Bach 2000*, Audio CD, Telarc, 1992. CD-80323. Carlos' original "Switched-On Bach" recording was performed on a Moog analog synthesizer, back in the late 1960s. The Moog is only capable of playing in equal temperament. Improvements in technology inspired her to release this new recording, using a variety of temperaments and modern methods of digital synthesis. The temperaments used are $\frac{1}{5}$ and $\frac{1}{4}$ comma meantone, and various circular (irregular) temperaments.

Wendy Carlos, *Beauty in the Beast*, Audion, 1986, Passport Records, Inc., SYNCD 200. Tracks 4 and 5 make use of super just scales.

Charles Carpenter has two CDs, titled *Frog à la Pêche* (Caterwaul Records, CAT8221, 1994) and *Splat* (Caterwaul Records, CAT4969, 1996), composed using the Bohlen–Pierce scale, and played in a progressive rock/jazz style. These recordings can be ordered directly from <http://www.kspace.com/carpenter> for \$13.95 each. Although Carpenter does not restrict himself to sounds composed mainly of odd harmonics, his compositions are nonetheless compelling.

Perry Cook (ed.), *Music, cognition and computerized sound. An introduction to psychoacoustics* [15] comes with an accompanying CD full of sound examples.

Michael Harrison, *From Ancient Worlds*, for Harmonic Piano, New Albion Records, Inc., 1992. NA 042 CD. The pieces on this recording all make use of his 24 tone super just scale.

In Joseph Haydn's *Sonata 41* in A (Hob. XVI:26), the movement *Menuetto al rovescio* is a perfect palindrome. This piece can be found as track 16 on the Naxos CD number 8.553127, Haydn, *Piano sonatas, Vol. 4*, with Jenő Jandó at the piano.

A. J. M. Houtsma and T. D. Rossing and W. M. Wagenaars, *Auditory Demonstrations*, Audio CD and accompanying booklet, Philips, 1987. This classic collection of sound examples illustrates a number of acoustic and psychoacoustic phenomena. It can be obtained from the Acoustical Society of America at <http://asa.aip.org/discs.html> for \$26 + shipping.

²The liner notes identify $Ab^{-\frac{1}{10}P}$ with $G\sharp^{-\frac{6}{5}}$, in accordance with the approximation of Kirnberger and Farey described in §5.12.

Enid Katahn, *Beethoven in the Temperaments* (Gasparo GSCD-332, 1997). Katahn plays Beethoven's Sonatas Op. 13, *Pathétique* and Op. 14 Nr. 1 using the Prinz temperament, and Sonatas Op. 27 Nr. 2, *Moonlight* and Op. 53 *Waldstein* in Thomas Young's temperament. The instrument is a modern Steinway concert grand rather than a period instrument. The tuning and liner notes are by Edward Foote.

Enid Katahn and Edward Foote have also brought out a recording, *Six degrees of tonality* (Gasparo GSCD-344, 2000). This begins with Scarlatti's Sonata K. 96 in quarter comma meantone, followed by Mozart's *Fantasie* Kv. 397 in Prelleur temperament, a Haydn sonata in Kirnberger III, a Beethoven sonata in Young temperament, Chopin's *Fantaisie-Improvisation* in DeMorgan temperament, and Grieg's *Glochengeläute* in Coleman 11 temperament. Finally, and in many ways the most interesting part of this recording, the Mozart *Fantasie* is played in quarter comma meantone, Prelleur temperament and equal temperament in succession, which allows a very direct comparison to be made. Unfortunately, the tempi are slightly different, which makes this recording not very useful for a blind test.

Bernard Lagacé has recorded a CD of music of various composers on the C. B. Fisk organ at Wellesley College, Massachusetts, USA, tuned in quarter comma meantone temperament. This recording is available from Titanic Records Ti-207, 1991.

Guillaume de Machaut (1300–1377), *Messe de Notre Dame* and other works. The Hilliard Ensemble, Hyperion, 1989, CDA66358. This recording is sung in Pythagorean intonation throughout. The mass alternates polyphonic with monophonic sections. The double leading-note cadences at the end of each polyphonic section are particularly striking in Pythagorean intonation. Track 19 of this recording is *Ma fin est mon commencement* (My end is my beginning). This is an example of retrograde canon, meaning that it exhibits temporal reflectional symmetry.

Mathews and Pierce, *Current directions in computer music research* [66] comes with a companion CD containing numerous examples; note that track 76 is erroneous, cf. Pierce [84], page 257.

Edward Parmentier, *Seventeenth Century French Harpsichord Music*, Wildboar, 1985, WLBR 8502. This collection contains pieces by Johann Jakob Froberger, Louis Couperin, Jacques Champion de Chambonnières, and Jean-Henri d'Anglebert. The recording was made using a Keith Hill copy of a 1640 harpsichord by Joannes Couchet, tuned in $\frac{1}{3}$ comma meantone temperament.

Many of Harry Partch's compositions have been rereleased on CD by Composers Recordings Inc., 73 Spring Street, Suite 506, New York, NY 10012-5800. As a starting point, I would recommend *The Bewitched*, CRI CD 7001, originally released on Partch's own label, Gate 5. This piece makes extensive use of his 43 tone super just scale.

A number of Robert Rich's recordings are in some form of super just scale. His basic scale is mostly 5-limit with a 7:5 tritone:

$$1:1, 16:15, 9:8, 6:5, 5:4, 4:3, 7:5, 3:2, 8:5, 5:3, 9:5, 15:8.$$

This appears throughout the CDs *Numena*, *Geometry*, *Rainforest*, and others. One of the nicest examples of this tuning is *The Raining Room* on the CD *Rainforest*, Hearts of Space HS11014-2. He also uses the 7-limit scale

$$1:1, 15:14, 9:8, 7:6, 5:4, 4:3, 7:5, 3:2, 14:9, 5:3, 7:4, 15:8.$$

This appears on *Sagrada Familia* on the CD *Gaudi*, Hearts of Space HS11028-2. See <http://www.amoeba.com> for a more complete discography of Robert Rich's work.

William Sethares, *Xentonality*, Music in 10-, 17- and 19-tet. See Frog Peak Music <http://www.frogpeak.org> to get hold of this recording.

Sethares, *Tuning, timbre, spectrum, scale* [105] comes with a CD full of examples.

Isao Tomita, *Pictures at an Exhibition* (Mussorgsky), BMG 60576-2-RG. This recording was made on analog synthesizers in 1974, and is remarkably sophisticated for that era.

Johann Gottfried Walther, *Organ Works*, Volumes 1 and 2, played by Craig Cramer on the organ of St. Bonifacius, Tröchtelborn, Germany. Naxos CD numbers 8.554316 and 8.554317. This organ was restored in Kellner's reconstruction of Bach's temperament, see §5.11. For more information about the organ (details are not given in the CD liner notes), see <http://www.gdo.de/neurest/troehtelborn.html>.

Aldert Winkelman, *Works by Mattheson, Couperin, and others*. Clavigram VRS 1735-2. This recording is hard to obtain. The pieces by Johann Mattheson, François Couperin, Johann Jakob Froberger, Joannes de Gruytters and Jacques Duphy are played on a harpsichord tuned to Werckmeister III. The pieces by Louis Couperin and Gottlieb Muffat are played on a spinet tuned in quarter comma meantone.

APPENDIX W

The wave equation

This appendix is a supplement to Section 3.6. Its purpose is to justify the method of separation of variables for the wave equation, and to explain why a drum has “enough” eigenvalues. The account of the solution of the wave equation given here is deliberately much more compressed than the account usually given in books on partial differential equations, to emphasize the shape of the reasoning rather than the more computational aspects usually emphasized. The level of mathematical sophistication needed to follow this appendix is rather greater than for the rest of the book, but it should be accessible to someone who has taken standard undergraduate courses in vector calculus, analysis and linear algebra.

We discuss solutions z of the two dimensional wave equation

$$\frac{\partial^2 z}{\partial t^2} = c^2 \nabla^2 z, \quad (\text{W.1})$$

on a closed, bounded domain Ω . We assume that z is identically zero on the boundary S . Initial conditions are given by specifying the values of z and $\frac{\partial z}{\partial t}$ at $t = 0$.

Throughout this appendix, Ω is a closed, bounded, simply connected domain in \mathbb{R}^2 with piecewise twice continuously differentiable boundary S , such that the pieces of the boundary meet at nonzero interior angles. We write \mathbf{x} for the position vector (x, y) on Ω , and $d\mathbf{x}$ for the element $dx dy$ of area on Ω . We write \mathbf{n} for the outward normal vector to S , and $d\sigma$ denotes the element of length on S . With this notation, the divergence theorem states that if $f(\mathbf{x})$ is a continuously differentiable function on Ω then

$$\int_S f \cdot \mathbf{n} d\sigma = \int_{\Omega} \nabla f d\mathbf{x}. \quad (\text{W.2})$$

In order to solve the wave equation, we begin with a study of Laplace's equation

$$\nabla^2 \phi = 0$$

on Ω , with Dirichlet boundary conditions. In other words, the value of ϕ is given on the boundary S .

Green's Identities

Let Ω be a closed bounded region with boundary S . Suppose that $f(\mathbf{x})$ and $g(\mathbf{x})$ are functions on Ω . Then we have

$$\nabla \cdot (f \nabla g) = f \nabla^2 g + \nabla f \cdot \nabla g. \quad (\text{W.3})$$

If Ω is a closed bounded region with boundary S , then integrating over Ω and using the divergence theorem (W.2), we get Green's first identity.

THEOREM W.1 (Green's First Identity). *Let $f(\mathbf{x})$ be continuously differentiable, and $g(\mathbf{x})$ be twice continuously differentiable on Ω . Then*

$$\int_S (f \nabla g) \cdot \mathbf{n} \, d\sigma = \int_\Omega (f \nabla^2 g + \nabla f \cdot \nabla g) \, d\mathbf{x}. \quad (\text{W.4})$$

Reversing the roles of f and g and subtracting gives Green's second identity.

THEOREM W.2 (Green's Second Identity). *Let $f(\mathbf{x})$ and $g(\mathbf{x})$ be twice continuously differentiable on Ω . Then*

$$\int_S (f \nabla g - g \nabla f) \cdot \mathbf{n} \, d\sigma = \int_\Omega (f \nabla^2 g - g \nabla^2 f) \, d\mathbf{x}. \quad (\text{W.5})$$

Gauss' formula

We start with the function of two variables \mathbf{x} and \mathbf{x}' in Ω given by $z = \ln |\mathbf{x} - \mathbf{x}'|$. For functions of two variables, it makes sense to apply ∇ with respect to \mathbf{x} keeping \mathbf{x}' constant, or vice versa. These are analogs of partial differentiation. To distinguish between these two options, we write $\nabla_{\mathbf{x}}$ or $\nabla_{\mathbf{x}'}$.

An easy calculation in terms of coordinates shows that as long as $\mathbf{x} \neq \mathbf{x}'$, we have

$$\nabla_{\mathbf{x}'} \ln |\mathbf{x} - \mathbf{x}'| = - \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^2} \quad (\text{W.6})$$

and

$$\nabla_{\mathbf{x}'}^2 \ln |\mathbf{x} - \mathbf{x}'| = 0. \quad (\text{W.7})$$

For $\mathbf{x} = \mathbf{x}'$, the quantity $\nabla_{\mathbf{x}'}^2 \ln |\mathbf{x} - \mathbf{x}'|$ doesn't make sense, because the logarithm isn't defined. But if we pretend that it is continuously differentiable, and integrate using the divergence theorem (W.2) we get

$$\int_\Omega \nabla_{\mathbf{x}'}^2 \ln |\mathbf{x} - \mathbf{x}'| \, d\mathbf{x}' = \int_S \nabla_{\mathbf{x}'} \ln |\mathbf{x} - \mathbf{x}'| \cdot \mathbf{n}' \, d\sigma' = - \int_S \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^2} \cdot \mathbf{n}' \, d\sigma', \quad (\text{W.8})$$

where \mathbf{n}' and σ' are with respect to \mathbf{x}' . The shape of the region Ω doesn't matter in this calculation, as long as \mathbf{x}' is in the interior, because of equation (W.7). If we measure using \mathbf{x} as the origin and make the region a unit disk centered at the origin, then the calculation reduces to $\int_S \mathbf{x}' \cdot \mathbf{n}' \, d\sigma'$. But

in this case \mathbf{x}' and \mathbf{n}' are unit vectors in the same direction, so $\mathbf{x}' \cdot \mathbf{n}' = 1$. Since the circumference of the unit circle is 2π , the integral gives 2π ,

$$\int_S \nabla_{\mathbf{x}'} \ln |\mathbf{x} - \mathbf{x}'| \cdot \mathbf{n}' d\sigma' = 2\pi. \quad (\text{W.9})$$

The interpretation of this calculation is that although $\ln |\mathbf{x} - \mathbf{x}'|$ is not differentiable with respect to \mathbf{x}' at $\mathbf{x}' = \mathbf{x}$, we can think of $\nabla_{\mathbf{x}'}^2 \ln |\mathbf{x} - \mathbf{x}'|$ as a distribution, in the sense in which we introduced the term in Section 2.15. We have to replace $\int_{-\infty}^{\infty}$ with \int_{Ω} , so that the delta function $\delta(\mathbf{x})$ is defined to be zero for $\mathbf{x} \neq \mathbf{0}$, and $\int_{\Omega} \delta(\mathbf{x}) d\mathbf{x} = 1$. In terms of this delta function, the above calculation can be expressed as saying that

$$\nabla_{\mathbf{x}'}^2 \ln |\mathbf{x} - \mathbf{x}'| = 2\pi \delta(\mathbf{x} - \mathbf{x}'). \quad (\text{W.10})$$

So far, we have assumed that \mathbf{x}' is in the interior of Ω . For a point \mathbf{x}' outside Ω , the integrand in equation (W.8) is zero so the integral is zero. If \mathbf{x}' is on the boundary S , and it is a point where S is continuously differentiable, then instead of a circle, in the above calculation we have to integrate over a semicircle. So the integral is π instead of 2π . At a corner with angle θ , we are integrating over a sector of a circle with angle θ , so the integral is θ . So we define a function $p(\mathbf{x})$ on \mathbb{R}^2 by

$$p(\mathbf{x}) = \begin{cases} 2\pi & \text{if } \mathbf{x} \text{ is in the interior of } \Omega, \\ 0 & \text{if } \mathbf{x} \text{ is not in } \Omega, \\ \pi & \text{if } \mathbf{x} \text{ is a continuously differentiable point on } S, \\ \theta & \text{if } \mathbf{x} \text{ is a corner of } S \text{ with interior angle } \theta. \end{cases}$$

Then the extension of equation (W.9) to the plane is **Gauss' formula**

$$\int_S \nabla_{\mathbf{x}'} \ln |\mathbf{x} - \mathbf{x}'| \cdot \mathbf{n}' d\sigma' = p(\mathbf{x}). \quad (\text{W.11})$$

If $f(\mathbf{x})$ is any continuous function on Ω , then we have

$$\int_{\Omega} f(\mathbf{x}') \nabla_{\mathbf{x}'}^2 \ln |\mathbf{x} - \mathbf{x}'| d\mathbf{x}' = p(\mathbf{x}) f(\mathbf{x}). \quad (\text{W.12})$$

This is because the integrand is zero except near $\mathbf{x} = \mathbf{x}'$, so $f(\mathbf{x}')$ may as well be replaced by $f(\mathbf{x})$ and taken out of the integral before applying the divergence theorem.

Remark. The above calculation was performed in two dimensions. The corresponding calculation in three dimensions uses the function $1/|\mathbf{x} - \mathbf{x}'|$ instead of $\ln |\mathbf{x} - \mathbf{x}'|$. The unit circle is replaced by the unit sphere, of surface area 4π , and the analog of equation (W.9) is

$$\int_S \nabla_{\mathbf{x}'} \frac{1}{|\mathbf{x} - \mathbf{x}'|} \cdot \mathbf{n}' d\sigma' = 4\pi.$$

The definition of $h(\mathbf{x}, \mathbf{x}')$ and $G(\mathbf{x}, \mathbf{x}')$ below are adjusted accordingly.

Similarly, in n dimensions ($n \geq 3$), the corresponding formula is

$$\int_S \nabla_{\mathbf{x}'} \frac{1}{|\mathbf{x} - \mathbf{x}'|^{n-2}} \cdot \mathbf{n}' d\sigma' = n(n-2)\alpha(n)$$

where $\alpha(n)$ denotes the $(n - 1)$ -dimensional volume of the surface of the n -dimensional sphere.

Green's functions

Equation (W.10) is an important property of the function $\ln |\mathbf{x} - \mathbf{x}'|$. But the main problem with this function is that it doesn't vanish on the boundary S of Ω . To remedy this, we adjust it as follows. Suppose that we can find a solution $h(\mathbf{x}, \mathbf{x}')$ to Laplace's equation

$$\nabla_{\mathbf{x}}^2 h(\mathbf{x}, \mathbf{x}') = 0 \quad (\text{W.13})$$

on Ω , with boundary conditions

$$h(\mathbf{x}, \mathbf{x}') = \frac{1}{2\pi} \ln |\mathbf{x} - \mathbf{x}'| \quad (\text{W.14})$$

for \mathbf{x}' on S . That is, we insist that $h(\mathbf{x}, \mathbf{x}')$ is defined even when $\mathbf{x} = \mathbf{x}'$ (in the interior of Ω). Then the function

$$G(\mathbf{x}, \mathbf{x}') = h(\mathbf{x}, \mathbf{x}') - \frac{1}{2\pi} \ln |\mathbf{x} - \mathbf{x}'|$$

still satisfies

$$\nabla_{\mathbf{x}'}^2 G(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}') \quad (\text{W.15})$$

for \mathbf{x}' in the interior of Ω , but it now also satisfies $G(\mathbf{x}, \mathbf{x}') = 0$ for \mathbf{x}' on S . The function $G(\mathbf{x}, \mathbf{x}')$ defined this way is called the Green's function for the Laplace operator ∇^2 .

LEMMA W.3. *The Green function, if it exists, satisfies the symmetry relation $G(\mathbf{x}, \mathbf{x}') = G(\mathbf{x}', \mathbf{x})$.*

PROOF. Using Lemma W.10, we have

$$\begin{aligned} G(\mathbf{x}, \mathbf{x}') &= \int_{\Omega} G(\mathbf{x}, \mathbf{x}'') \delta(\mathbf{x}' - \mathbf{x}'') d\mathbf{x}'' = \int_{\Omega} G(\mathbf{x}, \mathbf{x}'') \nabla_{\mathbf{x}''}^2 G(\mathbf{x}', \mathbf{x}'') d\mathbf{x}'' \\ &= \int_{\Omega} G(\mathbf{x}', \mathbf{x}'') \nabla_{\mathbf{x}''}^2 G(\mathbf{x}, \mathbf{x}'') d\mathbf{x}'' = \int_{\Omega} G(\mathbf{x}, \mathbf{x}'') \delta(\mathbf{x}' - \mathbf{x}'') d\mathbf{x}'' = G(\mathbf{x}', \mathbf{x}). \end{aligned}$$

□

The construction of the Green's function $G(\mathbf{x}, \mathbf{x}')$ depends on solving Laplace's equation (W.13) with boundary conditions (W.14). We do this using Fredholm theory.

Hilbert space

A *Hilbert space* V is a (usually infinite dimensional) complex vector space with inner product $\langle \cdot, \cdot \rangle$ satisfying

- (i) $\langle x, \lambda y_1 + \mu y_2 \rangle = \lambda \langle x, y_1 \rangle + \mu \langle x, y_2 \rangle$,
- (ii) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (and in particular $\langle x, x \rangle$ is real), and
- (iii) $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0$ if and only if $x = 0$,

(iv) Writing $|x|$ for $\sqrt{\langle x, x \rangle}$, the metric with distance function $|x - y|$ is complete. In other words, every Cauchy sequence has a limit.

For example, if D is a compact domain in \mathbb{R}^n then the space $L^2(D)$ of square integrable functions on D is a Hilbert space, with inner product

$$\langle f, g \rangle = \int_{\Omega} \bar{f} g \, d\mathbf{x}.$$

In this example, the completeness is a standard fact from Lebesgue integration theory. In order to satisfy (iii), we stipulate that two functions are identified if they agree except on a set of measure zero. Of course, this never identifies two continuous functions.

LEMMA W.4 (Schwartz's inequality). *For vectors x and y in Hilbert space, we have $\langle x, y \rangle \leq |x||y|$.*

PROOF. Consider the quantity

$$\langle x - ty, x - ty \rangle = |x|^2 - 2t\langle x, y \rangle + t^2|y|^2 \geq 0.$$

Differentiating with respect to t , we see that this expression is minimized by setting $t = \langle x, y \rangle / |y|^2$. With this value of t , we get

$$|x|^2 - 2\langle x, y \rangle^2 / |y|^2 + \langle x, y \rangle^2 / |y|^2 \geq 0,$$

or $\langle x, y \rangle^2 / |y|^2 \leq |x|^2$. □

Elements x and y satisfying $\langle x, y \rangle = 0$ are said to be *orthogonal*. If W is a subspace of V , we write W^\perp for the subspace consisting of vectors v such that for all $w \in W$ we have $\langle v, w \rangle = 0$. If W is finite dimensional, then any vector v in V can be written in a unique way as $v = w + x$ with w in W and x in W^\perp , so that

$$V = W \oplus W^\perp.$$

If \mathbf{K} is a linear operator on V , its *image* is

$$\text{Im } (\mathbf{K}) = \{\mathbf{K}v, v \in V\}$$

and its *kernel* is

$$\text{Ker } (\mathbf{K}) = \{v \in V \mid \mathbf{K}v = 0\}.$$

LEMMA W.5. *If \mathbf{K} and \mathbf{K}^* are adjoint linear operators on V (i.e., for all x and y , $\langle \mathbf{K}^*x, y \rangle = \langle x, \mathbf{K}y \rangle$) and the image of \mathbf{K} is finite dimensional, then*

(i) $V = \text{Im } \mathbf{K} \oplus \text{Ker } \mathbf{K}^*$, and

(ii) $V = \text{Im } \mathbf{K}^* \oplus \text{Ker } \mathbf{K}$

are orthogonal direct sum decompositions of V , and

$$\dim \text{Im } (\mathbf{K}) = \dim \text{Im } (\mathbf{K}^*).$$

PROOF. If $\mathbf{K}^*x \in \text{Im}(\mathbf{K}^*)$ and $y \in \text{Ker}(\mathbf{K})$ then

$$\langle \mathbf{K}^*x, y \rangle = \langle x, \mathbf{K}y \rangle = 0$$

so $\text{Im}(\mathbf{K}^*) \perp \text{Ker}(\mathbf{K})$. If $x \in \text{Im}(\mathbf{K}^*) \cap \text{Ker}(\mathbf{K})$ then $\langle x, x \rangle = 0$ and so $x = 0$. Thus

$$\text{Im}(\mathbf{K}^*) \oplus \text{Ker}(\mathbf{K}) \leq V. \quad (\text{W.16})$$

so we have

$$\dim \text{Im}(\mathbf{K}) = \dim(V/\text{Ker}(\mathbf{K})) \geq \dim \text{Im}(\mathbf{K}^*), \quad (\text{W.17})$$

with equality if and only if (W.16) is an equality. In particular, it follows that $\text{Im}(\mathbf{K}^*)$ is also finite dimensional. So we may repeat the above argument with the roles of \mathbf{K} and \mathbf{K}^* reversed, so that

$$\text{Im}(\mathbf{K}) \oplus \text{Ker}(\mathbf{K}^*) \leq V \quad (\text{W.18})$$

and

$$\dim \text{Im}(\mathbf{K}^*) \geq \dim \text{Im}(\mathbf{K}) \quad (\text{W.19})$$

with equality if and only if (W.18) is an equality. Comparing (W.17) with (W.19), we see that both must be equalities, so (W.16) and (W.18) are equalities. \square

LEMMA W.6. *If \mathbf{K} and \mathbf{K}^* are adjoint operators and $\text{Im}(\mathbf{K})$ is finite dimensional then*

$$(i) \ V = \text{Im}(\mathbf{I} - \mathbf{K}) \oplus \text{Ker}(\mathbf{I} - \mathbf{K}^*) \text{ and}$$

$$(ii) \ V = \text{Im}(\mathbf{I} - \mathbf{K}^*) \oplus \text{Ker}(\mathbf{I} - \mathbf{K})$$

are orthogonal decompositions of V , and $\dim \text{Im}(\mathbf{I} - \mathbf{K}) = \dim \text{Im}(\mathbf{I} - \mathbf{K}^)$ is finite.*

PROOF. By Lemma W.5, $\text{Im}(\mathbf{K}^*)$ is finite dimensional, so $V_1 = \text{Im}(\mathbf{K}) + \text{Im}(\mathbf{K}^*) \leq V$ is also finite dimensional. So $V = V_1 \oplus V_2$ where

$$V_2 = V_1^\perp = \text{Ker}(\mathbf{K}) \cap \text{Ker}(\mathbf{K}^*).$$

So $\mathbf{I} - \mathbf{K}$ and $\mathbf{I} - \mathbf{K}^*$ send V_1 into V_1 and act as the identity map on V_2 . Applying Lemma W.5 with $\mathbf{I} - \mathbf{K}$ instead of \mathbf{K} and V_1 in place of V , we see that V_1 decomposes in the way described in the lemma. Since $\mathbf{I} - \mathbf{K}$ and $\mathbf{I} - \mathbf{K}^*$ act as the identity on V_2 , this just contributes another summand to $\text{Im}(\mathbf{I} - \mathbf{K})$ and $\text{Im}(\mathbf{I} - \mathbf{K}^*)$, so the decomposition holds for V . \square

The Fredholm alternative

Now let V be the vector space $L^2(D)$ of Lebesgue square integrable functions on a compact domain D in \mathbb{R}^n . Suppose that $K(\mathbf{x}, \mathbf{x}')$ is a continuous complex valued function of two variables \mathbf{x} and \mathbf{x}' in D . We are interested in the operator \mathbf{K} on $L^2(D)$ given by

$$\mathbf{K}\psi(\mathbf{x}) = \int_D \psi(\mathbf{x}') K(\mathbf{x}, \mathbf{x}') d\mathbf{x}'. \quad (\text{W.20})$$

Such an operator is called a *Fredholm operator*. Its adjoint is given by

$$\mathbf{K}^* \psi(\mathbf{x}) = \int_D \psi(\mathbf{x}') \overline{K(\mathbf{x}', \mathbf{x})} d\mathbf{x}'. \quad (\text{W.21})$$

In general, the image of a Fredholm operator is not finite dimensional, so we can't apply Lemma W.6 directly. However, a function of the form $K(\mathbf{x}, \mathbf{x}') = g(\mathbf{x})h(\mathbf{x}')$ gives rise to an operator \mathbf{K} with one dimensional image spanned by $g(\mathbf{x})$. Any polynomial function of \mathbf{x} and \mathbf{x}' can be written as a finite sum of monomials, each of which has this form. So if $K(\mathbf{x}, \mathbf{x}')$ is a polynomial function, we may apply Lemma W.6.

The Weierstrass approximation theorem states that any continuous function on a compact domain in \mathbb{R}^n may be uniformly approximated by polynomial functions. Applying this to $K(\mathbf{x}, \mathbf{x}')$ on $D \times D$, we may write $K = K_1 + K_2$ where K_1 is a polynomial function and K_2 satisfies $B < 1$, where B is defined by

$$B = \iint_D |K_2(\mathbf{x}, \mathbf{x}')|^2 d\mathbf{x} d\mathbf{x}'.$$

For any function $\psi(\mathbf{x})$ in $L^2(D)$, Schwartz's inequality (Lemma W.4) implies that

$$|\mathbf{K}_2 \psi(\mathbf{x})|^2 \leq \langle \psi, \psi \rangle \int_D |K_2(\mathbf{x}, \mathbf{x}')|^2 d\mathbf{x}'.$$

Integrating with respect to \mathbf{x} gives

$$\langle \mathbf{K}_2 \psi, \mathbf{K}_2 \psi \rangle \leq B \langle \psi, \psi \rangle.$$

It follows by comparing with the geometric series

$$1 + B + B^2 + B^3 + \dots$$

that the sequence whose n th term is

$$\sum_{i=0}^n \mathbf{K}_2^i \psi$$

forms a Cauchy sequence in $L^2(D)$. Since $L^2(D)$ is complete, it follows that this Cauchy sequence has a limit; in other words, the infinite sum

$$\sum_{i=0}^{\infty} \mathbf{K}_2^i \psi = \psi + \mathbf{K}_2 \psi + \mathbf{K}_2^2 \psi + \mathbf{K}_2^3 \psi + \dots$$

converges in $L^2(D)$. It is now easy to check that the operator

$$\mathbf{I} + \mathbf{K}_2 + \mathbf{K}_2^2 + \mathbf{K}_2^3 + \dots$$

is an inverse to $\mathbf{I} - \mathbf{K}_2$ on $L^2(D)$. So we write $(\mathbf{I} - \mathbf{K}_2)^{-1}$ for this inverse.

Now we have

$$\mathbf{I} - \mathbf{K} = \mathbf{I} - (\mathbf{K}_1 + \mathbf{K}_2) = (\mathbf{I} - \mathbf{K}_2)(\mathbf{I} - (\mathbf{I} - \mathbf{K}_2)^{-1} \mathbf{K}_1).$$

The operator $(\mathbf{I} - \mathbf{K}_2)^{-1} \mathbf{K}_1$ has finite dimensional image, because \mathbf{K}_1 does. So Lemma W.6 enables us to write $L^2(D)$ as a direct sum of the image of

$\mathbf{I} - (\mathbf{I} - \mathbf{K}_2)^{-1} \mathbf{K}_1$ and the kernel of its adjoint. The invertibility of $\mathbf{I} - \mathbf{K}_2$ then gives us the following theorem, which is known as the *Fredholm alternative*.

THEOREM W.7. *With \mathbf{K} and \mathbf{K}^* defined by equations (W.20) and (W.21), the kernels of $\mathbf{I} - \mathbf{K}$ and $\mathbf{I} - \mathbf{K}^*$ are finite dimensional, and have the same dimension. If this dimension is zero, then $\mathbf{I} - \mathbf{K}$ is invertible, so that the equation*

$$\psi - \mathbf{K}\psi = f$$

has a unique solution ψ for any given element f of $L^2(D)$. \square

Solving Laplace's equation

In the section on Green's functions (page 311), we saw that if we can solve Laplace's equation (W.13) with boundary conditions (W.14) then we can construct a Green's function $G(\mathbf{x}, \mathbf{x}')$ satisfying equation (W.15) and zero on the boundary S . In this section we use Fredholm theory to solve Laplace's equation

$$\nabla^2 \phi(\mathbf{x}) = 0 \tag{W.22}$$

subject to twice continuously differentiable boundary conditions $\phi(\mathbf{x}) = f(\mathbf{x})$ on S .

We begin with uniqueness. We define the *potential energy* of a continuously differentiable function ϕ on Ω by

$$E = \rho c^2 \int_{\Omega} \nabla \phi \cdot \nabla \phi \, d\mathbf{x}.$$

So $E \geq 0$, and if $E = 0$ then $\nabla \phi = 0$, so that ϕ is constant. If ϕ_1 and ϕ_2 are solutions of (W.22) satisfying the same boundary conditions, then $\phi = \phi_1 - \phi_2$ satisfies (W.22) and is zero on the boundary. By Green's first identity (W.4) with $f = g = \phi$, we see that we have $E = 0$, so ϕ is constant; since $\phi = 0$ on the boundary, this constant is zero. We conclude that if a solution to Laplace's equation (W.22) with given values on the boundary exists, then it is unique.

The same method can also be used for solutions of Laplace's equation (W.22) for the unbounded region Ω' obtained by removing the interior of Ω from \mathbb{R}^2 , but we need to be careful about the behavior of ϕ as \mathbf{x} goes off to infinity. The point is that we need to apply Green's first identity (W.4) for a region with a hole, bounded by S and a large circle S' of radius R surrounding Ω , and then let $R \rightarrow \infty$. The extra term we get from the second boundary component is $\int_{S'} \phi \nabla \phi \cdot \left(\frac{\mathbf{x}}{R}\right) d\sigma$, because the unit normal vector is \mathbf{x}/R . The length of S' is $2\pi R$, so we need to check that $2\pi R |\phi \nabla \phi \cdot (\frac{\mathbf{x}}{R})| \rightarrow 0$ as $|\mathbf{x}| \rightarrow \infty$. So we have proved the following theorem.

THEOREM W.8. (i) *If $\nabla^2 \phi = 0$ has a solution on Ω with specified values on S , then the solution is unique.*

(ii) If $\nabla^2 \phi = 0$ has a solution on Ω' with specified values on S , and satisfying

$$\lim_{|\mathbf{x}| \rightarrow \infty} |\phi \nabla \phi \cdot \mathbf{x}| = 0$$

then that solution is unique. \square

We now examine the question of existence of solutions. To this end, we look for solutions of equation (W.22) of the form

$$\phi(\mathbf{x}) = \int_S \psi(\mathbf{x}') \nabla_{\mathbf{x}'} \ln |\mathbf{x} - \mathbf{x}'| \cdot \mathbf{n}' d\sigma', \quad (\text{W.23})$$

with ψ a twice continuously differentiable function defined on S .

Any twice continuously differentiable function ψ on S can be extended to a twice continuously differentiable function on Ω , which we also denote by ψ . So we can use Green's first identity (W.4) to write

$$\phi(\mathbf{x}) = \int_{\Omega} (\psi(\mathbf{x}') \nabla_{\mathbf{x}'}^2 \ln |\mathbf{x} - \mathbf{x}'| + \nabla \psi(\mathbf{x}') \cdot \nabla_{\mathbf{x}'} \ln |\mathbf{x} - \mathbf{x}'|) d\mathbf{x}'.$$

By equation (W.12), we have

$$\phi(\mathbf{x}) = p(\mathbf{x})\psi(\mathbf{x}) + \int_{\Omega} \nabla \psi(\mathbf{x}') \cdot \nabla_{\mathbf{x}'} \ln |\mathbf{x} - \mathbf{x}'| d\mathbf{x}'. \quad (\text{W.24})$$

In this formula, it can be shown using some elementary estimates that the integral term is continuous as \mathbf{x} crosses the boundary S . It follows that $\phi(\mathbf{x})$ is discontinuous at S , so to solve Laplace's equation (W.22) using ϕ , we should use the limiting value at the boundary. Namely, for \mathbf{x}_0 in S and \mathbf{x} in Ω but not in S , we have

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_0} \phi(\mathbf{x}) = 2\pi\psi(\mathbf{x}_0) + \int_{\Omega} \nabla \psi(\mathbf{x}') \cdot \nabla_{\mathbf{x}'} \ln |\mathbf{x}_0 - \mathbf{x}'| d\mathbf{x}',$$

whereas except at the corners, the value of ϕ on S is given by

$$\phi(\mathbf{x}_0) = \pi\psi(\mathbf{x}_0) + \int_{\Omega} \nabla \psi(\mathbf{x}') \cdot \nabla_{\mathbf{x}'} \ln |\mathbf{x}_0 - \mathbf{x}'| d\mathbf{x}'.$$

So we have

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_0} \phi(\mathbf{x}) = \phi(\mathbf{x}_0) + \pi\psi(\mathbf{x}_0).$$

In order to satisfy the boundary condition we want

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_0} \phi(\mathbf{x}) = f(\mathbf{x}_0).$$

So we must solve the equation

$$\phi(\mathbf{x}) + \pi\psi(\mathbf{x}) = f(\mathbf{x}) \quad (\text{W.25})$$

on S . Notice that the value of ψ at corners is irrelevant to the integral (W.23), so we just ignore the anomalous values of ϕ at corners and solve (W.25) for all \mathbf{x} in S .

We rewrite equation (W.25) as

$$\psi(\mathbf{x}) + \frac{1}{\pi} \int_S \psi(\mathbf{x}') \nabla_{\mathbf{x}'} \ln |\mathbf{x} - \mathbf{x}'| \cdot \mathbf{n}' d\sigma' = \frac{1}{\pi} f(\mathbf{x}). \quad (\text{W.26})$$

Setting

$$K(\mathbf{x}, \mathbf{x}') = -\frac{1}{\pi} \nabla_{\mathbf{x}'} \ln |\mathbf{x} - \mathbf{x}'| \cdot \mathbf{n}' = \frac{(\mathbf{x} - \mathbf{x}') \cdot \mathbf{n}'}{\pi |\mathbf{x} - \mathbf{x}'|^2}$$

and $D = S$, we use equation (W.20) to obtain an operator \mathbf{K} on $L^2(S)$ given by

$$\mathbf{K}\psi(\mathbf{x}) = -\frac{1}{\pi} \int_S \psi(\mathbf{x}') \nabla_{\mathbf{x}'} \ln |\mathbf{x} - \mathbf{x}'| \cdot \mathbf{n}' d\sigma'.$$

Equation (W.26) then becomes

$$\psi - \mathbf{K}\psi = \frac{1}{\pi} f.$$

Applying Fredholm theory (Theorem W.7), we see that this equation always has a solution provided we can prove that the only solution of the equation

$$\psi - \mathbf{K}\psi = 0$$

is the zero function. So assume that ψ satisfies this equation, and define $\phi(\mathbf{x})$ by equation (W.23). Then $\nabla^2 \phi = 0$, and $\phi(\mathbf{x}) \rightarrow 0$ as \mathbf{x} approaches the boundary from inside Ω . So by Theorem W.8 (i), we have $\phi(\mathbf{x}) = 0$ for \mathbf{x} in Ω . Similarly, we define $\phi(\mathbf{x})$ by equation (W.23) on Ω' . Then using equation (W.6) we find that $|\phi \nabla \phi \cdot \mathbf{x}| \rightarrow 0$ as $R \rightarrow \infty$. So by Theorem W.8 (ii), we have $\phi(\mathbf{x}) = 0$ in Ω' . Now it follows from equation (W.24) that for a point \mathbf{x}_0 on S which is not a corner,

$$\lim_{\substack{\mathbf{x} \rightarrow \mathbf{x}_0 \\ \text{in } \Omega}} \phi(\mathbf{x}) - \lim_{\substack{\mathbf{x} \rightarrow \mathbf{x}_0 \\ \text{in } \Omega'}} \phi(\mathbf{x}) = 2\pi\psi(\mathbf{x}_0).$$

It follows that $\psi(\mathbf{x}_0) = 0$. Since we were only interested in ψ at points which are not corners, this completes the proof that the only solution of $\psi - \mathbf{K}\psi = 0$ is $\psi = 0$. Applying Fredholm theory as mentioned above, this completes the proof of existence of solutions of Laplace's equation.

Conservation of energy

We are now ready to begin proving existence and uniqueness for solutions of the wave equation (W.1). The basic tool for proving uniqueness of solutions is the conservation of energy. We define the *energy* $E(t)$ of a continuously differentiable function z of \mathbf{x} and t to be the quantity

$$E(t) = \rho \int_{\Omega} \left(\left(\frac{\partial z}{\partial t} \right)^2 + c^2 \nabla z \cdot \nabla z \right) d\mathbf{x}. \quad (\text{W.27})$$

The two terms in this integral correspond to kinetic and potential energy respectively. Since $E(t)$ is obtained by integrating a sum of squares, it satisfies $E(t) \geq 0$. Furthermore, $E(t) = 0$ can only occur if the integrand is zero; namely if $\frac{\partial z}{\partial t}$ and ∇z are zero.

Suppose that z satisfies the wave equation (W.1). Differentiating, and using the divergence theorem (W.2), we get

$$\begin{aligned}\frac{dE}{dt} &= \int_{\Omega} \rho \left(2 \frac{\partial z}{\partial t} \frac{\partial^2 z}{\partial t^2} + 2c^2 \nabla z \cdot \frac{\partial \nabla z}{\partial t} \right) d\mathbf{x} \\ &= \int_{\Omega} \rho \left(2 \frac{\partial z}{\partial t} c^2 \nabla^2 z + 2c^2 \nabla z \cdot \nabla \frac{\partial z}{\partial t} \right) d\mathbf{x} \\ &= \int_{\Omega} 2\rho c^2 \nabla \cdot \left(\frac{\partial z}{\partial t} \nabla z \right) d\mathbf{x} \\ &= \int_S 2\rho c^2 \left(\frac{\partial z}{\partial t} \nabla z \right) \cdot \mathbf{n} d\sigma.\end{aligned}$$

Since $\frac{\partial z}{\partial t} = 0$ on S , we obtain

$$\frac{dE}{dt} = 0$$

so that E is a constant, independent of t . This is the statement of the conservation of energy for solutions of the wave equation.

Uniqueness of solutions

We now prove the uniqueness theorem for solutions to the wave equation. Suppose that z_1 and z_2 are solutions to the wave equation (W.1) on Ω , with the same initial conditions (i.e., the same values of z and $\frac{\partial z}{\partial t}$ for $t = 0$), and both vanishing on S . Then $z = z_1 - z_2$ satisfies the initial conditions $z = 0$ and $\frac{\partial z}{\partial t} = 0$ at $t = 0$. Equation (W.27) then shows that $E(0) = 0$. Conservation of energy implies that $E(t) = 0$ for all t . So $\frac{\partial z}{\partial t} = 0$ for all t , which implies that z is independent of t . Since it is zero at $t = 0$, we deduce that $z = 0$ for all values of t . Thus z_1 and z_2 are equal. It follows that there is at most one solution to the wave equation (W.1) for a given set of initial conditions for z and $\frac{\partial z}{\partial t}$.

It is less easy to prove existence of solutions. For this, we use the eigenvalue method. This will occupy the rest of the appendix.

Eigenvalues are nonnegative and real

We now prove that the eigenvalues of the Laplace operator ∇^2 are nonnegative and real—even if we allow f to take complex values (for real valued functions, ignore the bars in the proof of the lemma).

LEMMA W.9. *Let Ω be a closed bounded region. If f is a nonzero (complex valued) twice differentiable function satisfying $\nabla^2 f = -\lambda f$ in Ω and $f = 0$ on the boundary S of Ω , then λ is a nonnegative real number.*

PROOF. Let \bar{f} be the complex conjugate of f . Then using Green's first identity (W.4), we have

$$\int_S (\bar{f} \nabla f) \cdot \mathbf{n} d\sigma = \int_{\Omega} \nabla \bar{f} \cdot \nabla f d\mathbf{x} + \int_{\Omega} \bar{f} (\nabla^2 f) d\mathbf{x}$$

$$= \int_{\Omega} |\nabla f|^2 d\mathbf{x} - \lambda \int_{\Omega} |f|^2 d\mathbf{x},$$

Since f is zero on S , the left hand side is zero. Since $\int_{\Omega} |f|^2 d\mathbf{x} > 0$ and $\int_{\Omega} |\nabla f|^2 d\mathbf{x} \geq 0$, this means that

$$\lambda = \frac{\int_{\Omega} |\nabla f|^2 d\mathbf{x}}{\int_{\Omega} |f|^2 d\mathbf{x}} \geq 0$$

so that λ is a nonnegative real number. This expression for λ is called *Rayleigh's quotient*. \square

Orthogonality

The relationship between ∇^2 and the inner product for functions on Ω is expressed in the following lemma, which says that ∇^2 is *self-adjoint* with respect to the inner product, for functions vanishing on the boundary.

LEMMA W.10. *For twice continuously differentiable functions f and g on Ω vanishing on the boundary S , we have*

$$\langle f, \nabla^2 g \rangle = \langle \nabla^2 f, g \rangle.$$

PROOF. This follows from Green's second identity (W.5) (replacing f by \bar{f}) and the fact that $f(\mathbf{x})$ and $g(\mathbf{x})$ vanish on the boundary S . The left hand side of equation (W.5) is zero, while the right hand side is equal to $\langle f, \nabla^2 g \rangle - \langle \nabla^2 f, g \rangle$. \square

This allows us to see easily why the eigenvalues of ∇^2 are real numbers (Lemma W.9). Namely if $\nabla^2 f = -\lambda f$, and $f(\mathbf{x}) = 0$ on the boundary S , then we have

$$\bar{\lambda} \langle f, f \rangle = \langle \lambda f, f \rangle = -\langle \nabla^2 f, f \rangle = -\langle f, \nabla^2 f \rangle = \langle f, \lambda f \rangle = \lambda \langle f, f \rangle.$$

Since $\langle f, f \rangle \neq 0$, we have $\lambda = \bar{\lambda}$. However, positivity is less easy to see from this point of view.

A similar argument shows that eigenfunctions with distinct eigenvalues are orthogonal, as in the following lemma.

LEMMA W.11. *Let f and g be Dirichlet eigenfunctions on Ω with eigenvalues λ and μ respectively. If $\lambda \neq \mu$ Then*

$$\langle f, g \rangle = 0.$$

PROOF. Using the fact that ∇^2 is self-adjoint (see Lemma W.10), we have

$$\lambda \langle f, g \rangle = \langle \nabla^2 f, g \rangle = \langle f, \nabla^2 g \rangle = \mu \langle f, g \rangle,$$

and so $(\lambda - \mu) \langle f, g \rangle = 0$. If $\lambda \neq \mu$, it follows that $\langle f, g \rangle = 0$. \square

Inverting ∇^2

The key to understanding the eigenvalues and eigenfunctions of ∇^2 is to find an inverse \mathbf{K} for the operator ∇^2 using Green's functions. The inverse is an integral operator with a wider domain of definition, and whose eigenvalues are the reciprocals of those for ∇^2 . The operator \mathbf{K} is an example of a *compact operator*, which is what makes the eigenvalue theory easier.

The construction of the inverse goes as follows. If $f(\mathbf{x})$ satisfies

$$\nabla^2 f(\mathbf{x}) = -\lambda f(\mathbf{x}) \quad (\text{W.28})$$

on Ω and $f(\mathbf{x}) = 0$ on S , then we have

$$\begin{aligned} f(\mathbf{x}) &= \int_{\Omega} f(\mathbf{x}') \delta(\mathbf{x} - \mathbf{x}') d\mathbf{x}' = \int_{\Omega} f(\mathbf{x}') \nabla^2 G(\mathbf{x}, \mathbf{x}') d\mathbf{x}' \\ &= \int_{\Omega} G(\mathbf{x}, \mathbf{x}') \nabla^2 f(\mathbf{x}') d\mathbf{x}' = -\lambda \int_{\Omega} f(\mathbf{x}') G(\mathbf{x}, \mathbf{x}') d\mathbf{x}'. \end{aligned}$$

In particular, $f(\mathbf{x}) \neq 0$ implies $\lambda \neq 0$, so zero is not an eigenvalue of ∇^2 .

We write \mathbf{K} for the operator defined by

$$\mathbf{K}f(\mathbf{x}) = - \int_{\Omega} f(\mathbf{x}') G(\mathbf{x}, \mathbf{x}') d\mathbf{x}'.$$

Then the above calculation shows that if $f(\mathbf{x})$ satisfies (W.28) then

$$\mathbf{K}f(\mathbf{x}) = \frac{1}{\lambda} f(\mathbf{x}).$$

So $f(\mathbf{x})$ is an eigenfunction of \mathbf{K} with eigenvalue $1/\lambda$. Conversely, if $f(\mathbf{x})$ is an eigenfunction of \mathbf{K} with *nonzero* eigenvalue μ , and f is twice continuously differentiable, then $f(\mathbf{x})$ is also an eigenfunction of ∇^2 with eigenvalue $\lambda = 1/\mu$.

Compact operators

Let V be a Hilbert space. We say that a sequence of elements x_1, x_2, \dots of elements of V is *bounded* if there is some positive constant M such that all the x_i satisfy $|x_i| \leq M$. A continuous operator \mathbf{K} on V is said to be *compact* if, given any bounded sequence x_1, x_2, \dots , the images $\mathbf{K}x_1, \mathbf{K}x_2, \dots$ has a convergent subsequence.

Example. If the image of \mathbf{K} is finite dimensional then the Bolzano–Weierstrass theorem implies that \mathbf{K} is compact. More generally, the Fredholm alternative can be expressed in terms of compact operators.

If \mathbf{K} is compact and self-adjoint then there is an upper bound to the values of $\langle \mathbf{K}x, x \rangle$ as x runs over the elements of V satisfying $|x| = 1$. This is because otherwise, there would be a sequence x_1, x_2, \dots such that $\langle \mathbf{K}x_i, x_i \rangle > i$, and then by Schwartz' lemma, $\langle \mathbf{K}x_i, \mathbf{K}x_i \rangle > i^2$, so that there could not exist a convergent subsequence; this would contradict the fact that \mathbf{K} is compact. Writing U for the least upper bound of the values for $\langle \mathbf{K}x, x \rangle$ for $|x| = 1$, we can find a sequence x_1, x_2, \dots of elements with $|x_i| = 1$, such

that $\langle \mathbf{K}x_1, x_1 \rangle, \langle \mathbf{K}x_2, x_2 \rangle, \dots$ converges to U . Using Schwartz' lemma again, we have

$$\begin{aligned} \langle \mathbf{K}x_i - Ux_i, \mathbf{K}x_i - Ux_i \rangle &= \langle \mathbf{K}x_i, \mathbf{K}x_i \rangle - 2U\langle \mathbf{K}x_i, x_i \rangle + U^2 \\ &\leq \langle \mathbf{K}x_i, x_i \rangle^2 - 2U\langle \mathbf{K}x_i, x_i \rangle + U^2 \\ &\leq 2U^2 - 2U\langle \mathbf{K}x_i, x_i \rangle \\ &= 2U(U - \langle \mathbf{K}x_i, x_i \rangle) \rightarrow 0 \quad \text{as } i \rightarrow \infty, \end{aligned}$$

and so $\mathbf{K}x_i - Ux_i \rightarrow 0$ as $i \rightarrow \infty$.

Since \mathbf{K} is compact, we can replace x_1, x_2, \dots by a subsequence with the property that $\mathbf{K}x_1, \mathbf{K}x_2, \dots$ converges. So Ux_1, Ux_2, \dots converges, and provided $U \neq 0$, this implies that x_1, x_2, \dots also converges. Setting $x = \lim_{i \rightarrow \infty} x_i$, the continuity of \mathbf{K} implies that $\mathbf{K}x = \lim_{i \rightarrow \infty} \mathbf{K}x_i$, so we have

$$\mathbf{K}x = Ux.$$

In other words, x is an eigenvector of \mathbf{K} with eigenvalue U . So if $U \neq 0$ then U is an eigenvalue of \mathbf{K} .

Eigenvalue stripping

In the last section, we saw a method for finding an eigenvalue and eigenvector for \mathbf{K} . Suppose that we have already found some eigenvalues μ_1, \dots, μ_n and corresponding eigenvectors ψ_1, \dots, ψ_n of \mathbf{K} , and we wish to find some more. The most convenient method is to form a new operator \mathbf{K}_n whose eigenvalues and eigenvectors are the same as \mathbf{K} except for the removal of the ones we have found. As a preliminary step, we make sure that if there are repeated eigenvalues, then the corresponding eigenvectors are orthogonal. This can be done using the Gram-Schmidt process of linear algebra. Then we define

$$K_n(\mathbf{x}, \mathbf{x}') = K(\mathbf{x}, \mathbf{x}') - \sum_{i=1}^n \frac{\psi_i(\mathbf{x})\overline{\psi_i(\mathbf{x}')}}{\mu_i}.$$

Then we define \mathbf{K}_n by

$$\mathbf{K}_n\psi = \int_{\Omega} K_n(\mathbf{x}, \mathbf{x}')\psi(\mathbf{x}') d\mathbf{x}',$$

so that \mathbf{K}_n takes value zero on ψ_1, \dots, ψ_n , and takes the same value as \mathbf{K} on any function orthogonal to ψ_1, \dots, ψ_n .

To be continued...

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CSound is a multiplatform *free* software synthesis program. It's hard to use at first, but is the most powerful thing around. Almost every synthesis technique you've ever heard of is implemented in a very flexible fashion. The first version came out in 1985, and it has been developing steadily since. This book contains separate articles by many authors, so there is something of a lack of overall coherence to the work. It comes with 2 CD-ROMs containing software for Mac, Linux and PC, hundreds of musical compositions, more than 3000 working instruments, and much more. There is a third CD-ROM available separately, called "The Csound Catalog with Audio," available from <http://www.csounds.com>. This CD-ROM contains over 2000 orchestra and score text files, and the corresponding audio files in mp3 format. It is also possible to order separately an updated version of the 2 CD-ROMs that came with the book, from the same web site, whether or not you own the book.

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Tom Johnson is a minimalist composer, whose work uses mathematical techniques such as the theory of automata to assist in the compositional process. Copies of this book may be obtained by writing to: Two Eighteen Press, PO Box 218, Village Station, New York, NY 10014, USA.
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Mobile instrument, Arthur Frick

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