APPENDIX B

Bessel functions

z	$J_0(z)$	$J_1(z)$	$J_2(z)$	$J_3(z)$	$J_4(z)$
0.00	1.00000 00000 00000	0.00000 00000	0.00000 00000	0.00000 000	0.00000 000
0.02	0.99990 00024 99972	0.00999 95000	0.00004 99983	0.00000 000	0.00000 000
0.05	0.99937 50976 49468	0.02499 21883	0.00031 24349	0.00000 260	0.00000 002
0.1	0.99750 15620 66040	0.04993 75260	0.00124 89587	0.00002 082	0.00000 026
0.2	0.99002 49722 39576	0.09950 08326	0.00498 33542	0.00016 625	0.00000 416
0.4	0.96039 82266 59563	0.19602 65780	0.01973 46631	0.00132 005	0.00006 614
0.6	0.91200 48634 97211	0.28670 09881	0.04366 50967	0.00439 966	0.00033 147
0.8	0.84628 73527 50480	0.36884 20461	0.07581 77625	0.01024 678	0.00103 299
1.0	0.76519 76865 57967	0.44005 05857	0.11490 34849	0.01956 335	0.00247 664
1.2	0.67113 27442 64363	0.49828 90576	0.15934 90183	0.03287 434	0.00502 267
1.4	0.56695 51203 74289	0.54194 77139	0.20735 58995	0.05049 771	0.00906 287
1.6	0.45540 21676 39381	0.56989 59353	0.25696 77514	0.07252 344	0.01499 516
1.8	0.33998 64110 42558	0.58151 69517	0.30614 35353	0.09880 202	0.02319 652
2.0	0.22389 07791 41236	0.57672 48078	0.35283 40286	0.12894 325	0.03399 572
2.2	0.11036 22669 22174	0.55596 30498	0.39505 86875	0.16232 547	0.04764 715
2.4	0.00250 76832 97244	0.52018 52682	0.43098 00402	0.19811 480	0.06430 696
2.6	-0.09680 49543 97038	0.47081 82665	0.45897 28517	0.23529 381	0.08401 287
2.8	-0.18503 60333 64387	0.40970 92469	0.47768 54954	0.27269 860	0.10666 866
3.0	-0.26005 19549 01933	0.33905 89585	0.48609 12606	0.30906 272	0.13203 418
3.5	-0.38012 77399 87263	0.13737 75274	0.45862 91842	0.38677 011	0.20440 529
4.0	-0.39714 98098 63847	-0.06604 33280	0.36412 81459	0.43017 147	0.28112 9
4.5	-0.32054 25089 85121	-0.23106 04319	0.21784 89837	0.42470 397	0.34842 3
5.0	-0.17759 67713 14338	-0.32757 91376	0.04656 51163	0.36483 123	0.39123 2
5.5	-0.00684 38694 17819	-0.34143 82154	-0.11731 54816	0.25611 8	0.39671 7
6.0	0.15064 52572 50997	-0.27668 38581	-0.24287 32100	0.11476 8	0.35764 2
6.5	0.26009 46055 81606	-0.15384 13014	-0.30743 03906	-0.03534 7	0.27480 3
7.0	0.30007 92705 19556	-0.00468 28235	-0.30141 72201	-0.16755 6	0.15779 8
8.0	0.17165 08071 37554	0.23463 63469	-0.11299 17204	-0.29113 2	-0.10535 7
9.0	-0.09033 36111 82876	0.24531 17866	0.14484 73415	-0.18093 5	-0.26547 1
10.0	-0.24593 57644 51348	0.04347 27462	0.25463 03137	0.05837 9	-0.21960 3
11.0	-0.17119 03004 07196	-0.17678 52990	0.13904 75188	0.22734 8	-0.01504 0
12.0	0.04768 93107 96834	-0.22344 71045	-0.08493 04949	0.19513 7	0.18249 9
13.0	0.20692 61023 77068	-0.07031 80521	-0.21774 42642	0.00332 0	0.21927 6
14.0	0.17107 34761 10459	0.13337 51547	-0.15201 98826	-0.17680 9	0.07624 4
15.0	-0.01422 44728 26781	0.20510 40386	0.04157 16780	-0.19401 8	-0.11917 9
16.0	-0.17489 90739 83629	0.09039 71757	0.18619 87209	-0.04384 7	-0.20264 2

z	$J_5(z)$	$J_6(z)$	$J_7(z)$	$J_8(z)$	$J_9(z)$	$J_{10}(z)$	$J_{11}(z)$	$J_{12}(z)$
0.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.5	0.000008	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
1.0	0.000250	0.000021	0.000002	0.000000	0.000000	0.000000	0.000000	0.000000
1.5	0.001799	0.000228	0.000025	0.000002	0.000000	0.000000	0.000000	0.000000
2.0	0.007040	0.001202	0.000175	0.000022	0.000002	0.000000	0.000000	0.000000
2.5	0.019502	0.004225	0.000777	0.000124	0.000018	0.000002	0.000000	0.000000
3.0	0.043028	0.011394	0.002547	0.000493	0.000084	0.000013	0.000002	0.000000
3.5	0.080442	0.025429	0.006743	0.001543	0.000311	0.000056	0.000009	0.000001
4.0	0.132087	0.049088	0.015176	0.004029	0.000939	0.000195	0.000037	0.000006
4.5	0.194715	0.084276	0.030022	0.009126	0.002425	0.000573	0.000122	0.000024
5.0	0.261141	0.131049	0.053376	0.018405	0.005520	0.001468	0.000351	0.000076
5.5	0.320925	0.186783	0.086601	0.033657	0.011309	0.003356	0.000893	0.000216
6.0	0.362088	0.245837	0.129587	0.056532	0.021165	0.006964	0.002048	0.000545
6.5	0.373565	0.299913	0.180121	0.088039	0.036590	0.013288	0.004297	0.001254
7.0	0.347896	0.339197	0.233584	0.127971	0.058921	0.023539	0.008335	0.002656
8.0	0.185775	0.337576	0.320589	0.223455	0.126321	0.060767	0.025597	0.009624
9.0	-0.055039	0.204317	0.327461	0.305067	0.214881	0.124694	0.062217	0.027393
10.0	-0.234062	-0.014459	0.216711	0.317854	0.291856	0.207486	0.123117	0.063370
11.0	-0.238286	-0.201584	0.018376	0.224972	0.308856	0.280428	0.201014	0.121600
12.0	-0.073471	-0.243725	-0.170254	0.045095	0.230381	0.300476	0.270412	0.195280
13.0	0.131620	-0.118031	-0.240571	-0.141046	0.066976	0.233782	0.292688	0.261537
14.0	0.220378	0.081168	-0.150805	-0.231973	-0.114307	0.085007	0.235745	0.285450
15.0	0.130456	0.206150	0.034464	-0.173984	-0.220046	-0.090072	0.099950	0.236666

Table of zeros of Bessel functions:

k	J_0	J_1	J_2	J_3	J_4	J_5	J_6	J_7
1	2.4048256	3.83171	5.13562	6.38016	7.58834	8.77148	9.93611	11.08637
2	5.5200781	7.01559	8.41724	9.76102	11.06471	12.33860	13.58929	14.82127
3	8.6537279	10.17347	11.61984	13.01520	14.37254	15.70017	17.00382	18.28758
4	11.7915344	13.32369	14.79595	16.22347	17.61597	18.98013	20.32079	21.64154
5	14.9309177	16.47063	17.95982	19.40942	20.82693	22.21780	23.58608	24.93493
6	18.0710640	19.61586	21.11700	22.58273	24.01902	25.43034	26.82015	28.19119
7	21.2116366	22.76008	24.27011	25.74817	27.19909	28.62662	30.03372	31.42279
8	24.3524715	25.90367	27.42057	28.90835	30.37101	31.81172	33.23304	34.63709
9	27.4934791	29.04683	30.56920	32.06485	33.53714	34.98878	36.42202	37.83872
10	30.6346065	32.18968	33.71652	35.21867	36.69900	38.15987	39.60324	41.03077
11	33.7758202	35.33231	36.86286	38.37047	39.85763	41.32638	42.77848	44.21541
12	36.9170984	38.47477	40.00845	41.52072	43.01374	44.48932	45.94902	47.39417

The kth zero of J_n is denoted $j_{n,k}$.

Fourier series

$$\sin(z\sin\theta) = 2\sum_{n=0}^{\infty} J_{2n+1}(z)\sin(2n+1)\theta$$

$$\cos(z\sin\theta) = J_0(z) + 2\sum_{n=1}^{\infty} J_{2n}(z)\cos 2n\theta$$

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - z\sin\theta)\,d\theta.$$

Differential equation

$$J_n''(z) + \frac{1}{z}J_n'(z) + \left(1 - \frac{n^2}{z^2}\right) J_n(z) = 0$$

Power series

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{z}{2})^{n+2k}}{k!(n+k)!}$$

Generating function

$$e^{\frac{1}{2}z(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} J_n(z)t^n$$

Limiting values

If n is constant, z is real and $|z| \to \infty$,

$$J_n(z) = \sqrt{\tfrac{2}{\pi z}}\cos(z - \tfrac{1}{2}(n + \tfrac{1}{2})\pi) + O(|z|^{-3/2}).$$

[Here, $O(|z|^{-3/2})$ represents an error term which is bounded by some constant multiple of $|z|^{-3/2}$]

If z is constant and $n \to \infty$, $J_n(z) \sim \frac{1}{\sqrt{2\pi n}} \left(\frac{ez}{2n}\right)^n$.

[The $\sim$ notation means that the ratio of these two quantities tends to one as $n \to \infty$]

For n fixed, as $k \to \infty$, $j_{n,k} \sim (k + \frac{1}{2}n - \frac{1}{4})\pi$.

Other formulas

$$\begin{aligned}
J_{-n}(z) &= (-1)^n J_n(z) \\
J_n'(z) &= \tfrac{1}{2}(J_{n-1}(z) - J_{n+1}(z)) \\
J_n(z) &= \tfrac{z}{2n}(J_{n-1}(z) + J_{n+1}(z)) \\
\frac{d}{dz}(z^n J_n(z)) &= z^n J_{n-1}(z) \\
\sum_{n=-\infty}^{\infty} J_n(z)^2 &= 1
\end{aligned}$$

In particular, $|J_n(z)| \le 1$ for all n and z, and if $n \neq 0$ then $|J_n(z)| \le \frac{1}{\sqrt{2}}$.

FM Synthesis

$$\sin(\phi + z\sin\theta) = \sum_{n=-\infty}^{\infty} J_n(z)\sin(\phi + n\theta)$$

The following table shows how index of modulation (I) varies as a function of operator output level (an integer in the range 0–99) on the Yamaha six operator synthesizers DX7, DX7IID, DX7IIFD, DX7S, DX5, DX1, TX7, TX816, TX216, TX802 and TF1:

	0	1	2	3	4	5	6	7	8	9
0	0.0002	0.0003	0.0005	0.0007	0.0010	0.0012	0.0016	0.0019	0.0023	0.0027
10	0.0032	0.0038	0.0045	0.0054	0.0064	0.0076	0.0083	0.0091	0.0108	0.0118
20	0.0140	0.0152	0.0166	0.0181	0.0198	0.0216	0.0235	0.0256	0.0280	0.0305
30	0.0332	0.0362	0.0395	0.0431	0.0470	0.0513	0.0559	0.0610	0.0665	0.0725
40	0.0791	0.0862	0.0940	0.1025	0.1118	0.1219	0.1330	0.1450	0.1581	0.1724
50	0.1880	0.2050	0.2236	0.2438	0.2659	0.2900	0.3162	0.3448	0.3760	0.4101
60	0.4472	0.4877	0.5318	0.5799	0.6324	0.6897	0.7521	0.8202	0.8944	0.9754
70	1.0636	1.1599	1.2649	1.3794	1.5042	1.6403	1.7888	1.9507	2.1273	2.3198
80	2.5298	2.7587	3.0084	3.2807	3.5776	3.9014	4.2545	4.6396	5.0595	5.5174
90	6.0168	6.5614	7.1552	7.8028	8.5090	9.2792	10.119	11.035	12.034	13.123

The following table shows how index of modulation (I) varies as a function of operator output level (an integer in the range 0–99) on the Yamaha four operator synthesizers DX11, DX21, DX27, DX27S, DX100 and TX81Z:

	0	1	2	3	4	5	6	7	8	9
0	0.0004	0.0006	0.0009	0.0013	0.0018	0.0024	0.0031	0.0036	0.0043	0.0052
10	0.0061	0.0073	0.0087	0.0103	0.0123	0.0146	0.0159	0.0174	0.0206	0.0225
20	0.0268	0.0292	0.0318	0.0347	0.0379	0.0413	0.0450	0.0491	0.0535	0.0584
30	0.0637	0.0694	0.0757	0.0826	0.0900	0.0982	0.1071	0.1168	0.1273	0.1388
40	0.1514	0.1651	0.1801	0.1963	0.2141	0.2335	0.2546	0.2777	0.3028	0.3302
50	0.3601	0.3927	0.4282	0.4670	0.5093	0.5554	0.6056	0.6604	0.7202	0.7854
60	0.8565	0.9340	1.0185	1.1107	1.2112	1.3209	1.4404	1.5708	1.7130	1.8680
70	2.0371	2.2214	2.4225	2.6418	2.8809	3.1416	3.4259	3.7360	4.0741	4.4429
80	4.8450	5.2835	5.7617	6.2832	6.8519	7.4720	8.1483	8.8858	9.6900	10.567
90	11.523	12.566	13.704	14.944	16.297	17.772	19.380	21.134	23.047	25.133

APPENDIX C

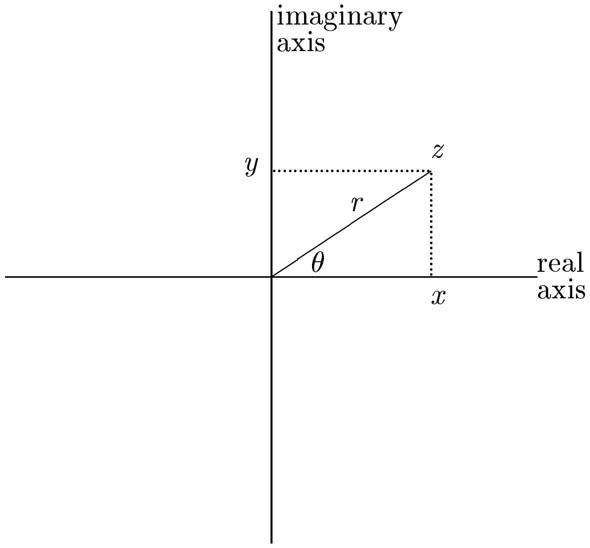
Complex numbers

We use i to denote $\sqrt{-1}$, and the general complex number is of the form $a + ib$ where a and b are real numbers. Addition and multiplication are given by

$$(a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$$
$$(a_1 + ib_1)(a_2 + ib_2) = (a_1a_2 - b_1b_2) + i(a_1b_2 + b_1a_2).$$

The real numbers a and b can be thought of as the Cartesian coordinates of the complex number $a + ib$, so that complex numbers correspond to points on the plane. In this language, the real numbers are contained in the complex numbers as the x axis, and the points on the y axis are called pure imaginary numbers.

For the purpose of multiplication, it is easier to work in polar coordinates. If $z = x + iy$ is a complex number, we define the *absolute value* of z to be $|z| = \sqrt{x^2 + y^2}$. The *argument* of z is the angle θ formed by the line from zero to z. Angle is measured counterclockwise from the x axis.



The *complex conjugate* of $z = x + iy$ is defined to be $\bar{z} = x - iy$, so that

$$z\bar{z} = |z|^2 = x^2 + y^2.$$

So division by a nonzero complex number z is achieved by multiplying by

$$\frac{\bar{z}}{|z|^2} = \frac{x}{x^2+y^2} - i\,\frac{y}{x^2+y^2},$$

which is the *multiplicative inverse* of z.

The *exponential function* is defined for a complex argument $z = x + iy$ by

$$e^z = e^x(\cos y + i \sin y).$$

This means that convertion from Cartesian coordinates to polar coordinates is given by

$$z = x + iy = re^{i\theta},$$

where $r = \sqrt{x^2+y^2}$ and $\tan\theta = y/x$. Translation in the other direction is given by $x = r\cos\theta$ and $y = r\sin\theta$. The trigonometric identities

$$\sin(A+B) = \sin A\cos B + \cos A \sin B$$
$$\cos(A+B) = \cos A\cos B - \sin A \sin B.$$

are equivalent to the statement that if z_1 and z_2 are complex numbers then

$$e^{z_1}e^{z_2} = e^{z_1+z_2}.$$

So we have Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{C.1}$$

and

$$\cos\theta = \tfrac{1}{2}(e^{i\theta} + e^{-i\theta}) \tag{C.2}$$
$$\sin\theta = \tfrac{1}{2i}(e^{i\theta} - e^{-i\theta}). \tag{C.3}$$

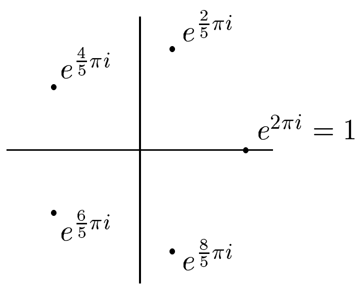
Using (C.1), the relation $(e^{i\theta})^n = e^{in\theta}$ translates as de Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta.$$

The complex nth roots of unity (i.e., of the number one) are the numbers

$$e^{2\pi i m/n} = \cos 2\pi m/n + i \sin 2\pi m/n$$

for $0 \le m \le n-1$. These are equally spaced around the unit circle in the complex plane. For example, here is a picture of the complex fifth roots of unity.



Remark. Engineers use the letter j instead of i.

Hyperbolic functions: In Section 3.7 the analysis of the xylophone involves the *hyperbolic functions* $\cosh x$ and $\sinh x$. These are defined by analogy with equations (C.2) and (C.3) via

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}) \tag{C.4}$$

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}). \tag{C.5}$$

The standard identities for these functions are

$$\cosh^2 x - \sinh^2 x = 1,$$

and

$$\sinh(A+B) = \sinh A \cosh B + \cosh A \sinh B$$
$$\cosh(A+B) = \cosh A \cosh B + \sinh A \sinh B.$$

The values at zero are given by

$$\sinh(0) = 0, \qquad \cosh(0) = 1.$$

The derivatives are given by

$$\frac{d}{dx}\sinh x = \cosh x, \qquad \frac{d}{dx}\cosh x = \sinh x.$$

Note the changes in sign from the corresponding trigonometric formulas.

APPENDIX D

Dictionary

As an aide to reading the literature on the subject in French, German, Italian, Latin and Spanish, as well as the literature on ancient Greek music, here is a dictionary of common terms.

Abklingen (G.), *decay*
Abgeleiteter Akkord (G.), *inversion of a chord*
Absatz (G.), *cadence*
Abstimmung (G.), *tuning*
accord (Fr.), *chord*
accordage (Fr.), accordatura (It.), *tuning, intonation*
accordo (It.), acorde (Sp.), *chord*
afinación (Sp.), *tuning*
affaiblissement (Fr.), *decay*
aigu (Fr.), *acute, high*
Akkord (G.), *chord*
allgemein (G.), *general*
amplificateur (Fr.), amplificatore (It.), amplificador (Sp.), *amplifier*
Anklang (G.), *tune, harmony, accord*
armoneggiare (It.), *to harmonize*
armonica (It.), armónico (Sp.), *harmonic*
atenuamiento (Sp.), attenuazione (It.), *decay*
audición (Sp.), audition (Fr.), *hearing*
auferions (archaic Eng.), *wire strings*
Aufhaltung (G.), *suspension (harmony)*
aulos (Gk.), *ancient Greek reed instrument*
Ausdruck (G.), *expression*

battements (Fr.), battimenti (It.), *beats*
bec (Fr.), becco (It.), *mouthpiece*
bécarre (Fr.), becuardo (Sp.), *natural* (♮)
Bedingung (G.), *condition*
Beispiel (G.), *example*
beliebig (G.), *arbitrary*
bémol (Fr.), bemol (Sp.), bemolle (It.), *flat* (♭)
bequadro (It.), *natural* (♮)
beweisen (G.), *to prove*
bruit (Fr.), *noise*
Bund (G.), *fret*

cadenza d'inganno (It.), *deceptive cadence*
caisse (Fr.), *drum*
Canonici, *followers of the Pythagorean system of music, where consonance is based on ratios, see also* Musici
chevalet (Fr.), *bridge of stringed instrument*
chiave (It.), clave (Sp.), clavis (L.), *clef, key*
chiffrage (Fr.), *time signature*
clavecin (Fr.), *harpsichord*
cloche (Fr.), *bell*
concento (It.), concentus (L.), *harmony*
controreazione (It.), *feedback*
conversio (L.), *inversion*
cor (Fr.), *horn*
cuarta (Sp.), *fourth*

Dach (G.), *sounding board*
daher (G.), *hence*
Darstellung (G.), *representation*
demi-ton (Fr.), *semitone*
denarius (L.), *numbers 1–10*
diapason (Fr., It.), diapasón (Sp.), *pitch*
diapason (Gk.), *octave*
diapente (Gk.), *fifth*
diastema (Gk.), *interval*
diatessaron (Gk.), *fourth*
diazeuxis (Gk.), *separation of two tetrachords by a tone*
dièse (Fr.), diesis (It.), *sharp* (♯)
disdiapason (Gk.), *two octaves*
dodécaphonique (Fr.), *twelve tone*

Doppelbee (G.), *double flat* (♭♭)
Doppelkreuz (G.), *double sharp* (𝄪)
Dreiklang (G.), *triad*
Dur (G.), *major*
durchgehend (G.), *transient*

échantilloneur (Fr.), *sampler*
échelle (Fr.), *scale*
einfach (G.), *simple*
Einheit (G.), *unity*
Einklang (G.), *consonance*
Einselement (G.), *identity element*
emmeleia (Gk.), *consonance*
enmascaramiento (Sp.), *masking*
ensemble (Fr.), *set*
entier (Fr.), *integer*
entonación (Sp.), *intonation*
entsprechen (G.), *to correspond to*
erhöhen (G.), *to raise, increase*
erweitern (G.), *to extend, augment*
escala (Sp.), *scale*
espectro (Sp.), *spectrum*
estribo (Sp.), étrier (Fr.), *stapes*

faux (Fr.), *out of tune*
Folge (G.), *sequence, series*

gama (Sp.), gamma (It.), gamme (Fr.), *scale*
ganancia (Sp.), *gain*
ganze Zahl (G.), *integer*
ganzer Ton (G.), *whole tone*
Gegenpunkt (G.), *counterpoint*
gerade (G.), *even, just, exactly*
Gesetz (G.), *law, rule*
giusto (It.), *just*
gleichschwebende (G.), *equal beating*
gleichstufige (G.), *equal (temperament)*
Gleichung (G.), *equation*
gleichzeitig (G.), *simultaneous*
Glied (G.), *term*
Grundton (G.), *fundamental*
guadagno (It.), *gain*

Halbton (G.), *semitone*
hautbois (Fr.), *oboe*
hauteur (Fr.), Höhe (G.), *pitch*
helicon (Gk.), *instrument used for calculating ratios*
hemiolios (Gk.), *ratio 3:2*
Hörbar (G.), *audible*
Hören (G.), *hearing*

impair (Fr.), *odd*
imparfait (Fr.), *imperfect*
Kettenbruch (G.), *continued fractions*
Klang (G.), *timbre*
Klangstufe (G.), *degree of scale*
klein (G.), *small, minor*
Kombinationston (G.), *combination tone*
Komma (G.), *comma*
Kraft (G.), *energy*
Kreuz (G.), *sharp* (♯)

laud (Sp.), Laute (G.), *lute*
Leistung (G.), *power*
leiten (G.), *to derive, deduce*
Leiter (G.), *scale*
ley (Sp.), *law*
limaçon (Fr.), *cochlea*
Lösung (G.), *solution*

maggiore (It.), majeur (Fr.), mayor (Sp.), *major*
marche d'harmonie (Fr.), *harmonic sequence*
Menge (G.), *set*
menor (Sp.), *minor*
mehrstimmig (G.), *polyphonic*
mésotonique (Fr.), *meantone*
minore (It.), *minor*
mitteltönig (G.), *meantone*
Moll (G.), *flat* (♭), *minor*
Musici, *followers of the Aristoxenian system of music, in which the ear is the judge of consonance, see also* Canonici
Muster (G.), *pattern*

Nachhall (G.), *reverberation*
Nenner (G.), *denominator*
neuvième (Fr.), *ninth*
Notenschlussel (G.), *clef*

Oberwelle (G.), *harmonic*
offen (G.), *open*
Ohr (G.), *ear*
Ohrmuschel (G.), *auricle*
oído (Sp.), *ear*
onda (It., Sp.), *wave*
onda portante (It.), onda portadora (Sp.), *carrier*
onde (Fr.), *wave*

orecchio (It.), oreille (Fr.), *ear*
ouïe (Fr.), *hearing; sound-hole*

padigione (It.), *auricle*
pair (Fr.), par (Sp.), *even*
parfait (Fr.), *perfect*
pavillon (Fr.), *auricle*
portée (Fr.), *staff, stave*
porteuse (Fr.), *carrier*
potencia (Sp.), potenza (It.), puissance (Fr.), *power*
profondeur (Fr.), *depth*
pulsaciones (Sp.), *beats*

Quadrat (G.), *natural* (♮)
quadrivium (L.), *The four disciplines: arithmetic, geometry, astronomy and music*
quarta (It., L.), quarte (Fr.), Quarte (G.), *fourth*
quaternarius (L.), *numbers 1–4*
quinta (It., L., Sp.), quinte (Fr.), Quinte (G.), *fifth*

réaction (Fr.), *feedback*
reine Stimmung (G.), *just intonation*
renversement (Fr.), *inversion*
retard (Fr.), *delay*
retroalimentación (Sp.), *feedback*
ronde (Fr.), *whole note* (USA), *semibreve* (GB)
Rückkopplung (G.), *feedback*

Saite (G.), *string*
Schall (G.), *sound*
Schlag (G.), *beat*
Schlüssel (G.), *clef*
Schnecke (G.), *cochlea*
Schwebungen (G.), *beats*
Schwingungen (G.), *vibrations*
senarius (L.), *numbers 1–6*
sensible (Fr.), *leading note*
septenarius (L.), *numbers 1–7*
septime (L.), *seventh*
Septimenakkord (G.), *chord of the seventh*
série de hauteurs (Fr.), *tone row*
sesquialtera (L.), *ratio 3:2*
sesquitertia (L.), *ratio 4:3*
Sext (G.), sexta (L.), *sixth*
sibilo (It.), sifflement (Fr.), silbo (Sp.), *hiss*
sillet (Fr.), *bridge*
Skala (G.), *scale*
son (Fr.), *sound*
son combiné (Fr.), *combination tone*
son différentiel (Fr.), *difference tone*
sonido (Sp.), *sound*
sonido de combinación (Sp.), *combination tone*
sonorità (It.), *harmony, resonance*
sonus (L.), *sound*
sostenido (Sp.), *sharp* (♯)
spectre (Fr.), *spectrum*
staffa (It.), *stapes*
stark (G.), *loud*
Stege (G.), *bridge*
Steigbügel (G.), *stapes*
Stufe (G.), *scale degree*
subsemitonia (L.), *split keys*
suono (It.), *sound*
suono di combinazione (It.), *combination tone*
synaphe (Gk.), *conjunction, or overlapping of two tetrachords*

Takt (G.), *time, measure, bar*
tambour (Fr.), tamburo (It.), tambor (Sp.), *drum*
Tastame (It.), Tastatur, Tastenbrett, Tastenleiter (G.), Tastatura, Tastiera (It.), *keyboard of piano or organ*
tasto (It.), tecla (Sp.), *fret*
teilbar (G.), *divisible*
Teilmenge (G.), *subset*
Teilung (G.), *division*
Temperatur (G.), *temperament*
temperiert (G.), *tempered*
temps (Fr.), *beat, measure*
tercera (Sp.), tertia (L.), Terz (G.), terza (It.), tierce (Fr.), *third*
ton (Fr.), *pitch, tone, key*
tonalité (Fr.), Tonart (G.), *key*
Tonausweichung (G.), *modulation*
Tonhöhe (G.), *pitch*
tono medio (It., Sp.), *meantone*
Tonschluss (G.), *cadence*
Tonstufe (G.), *scale degree*
touche (Fr.), *fret*

Träger (G.), *carrier*
tripla (L.), *ratio 3:1*
Trommel (G.), *drum*
tuyau à bouche (Fr.), *open pipe*
tuyau d'orgue (Fr.), *organ pipe*
tympan (Fr.), *eardrum*

Übereinstimmung (G.), *consonance, harmony*
Übermässig (G.), *augmented*
udibile (It.), *audible*
udito (It.), *hearing*
uguale (It.), *equal*
Umkehrung (G.), *inversion*
Unterdominant (G.), *subdominant*
Unterhalbton (G.), *leading note*
Unterleitton (G.), *dominant seventh*
Untergruppe (G.), *subgroup*

valeur propre (Fr.), *eigenvalue*
vent (Fr.), *wind*
Ventil (G.), ventile (It.), *valve, on wind instruments*
vents (Fr.), *wind instruments*
Verbindung (G.), *combination, union*
Verdeckung (G.), *masking*
Verhältnis (G.), *ratio, proportion*
Verknüpfung (G.), *operation*
vermindert (G.), *diminished*
versetzen (G.), *to transpose*
Versetzungszeichen (G.), *accidentals*
Verspätung (G.), *delay*
Verstärker (G.), *amplifier*
Verstärkung (G.), *gain*
verstimmt (G.), *out of tune*
verwandt (G.), *related*
Verzerrung (G.), *distortion*
Vollkommenheit (G.), *perfection*

Welle (G.), *wave*

Zahl (G.), *number*
Zeichen (G.), *sign, note*
Zischen (G.), *hiss*
Zuklang (G.), *unison, consonance*

APPENDIX E

Equal tempered scales

q	p_3	e_3	p_5	e_5	p_7	e_7	e_{35}	e_{357}	$e_5 . q^2$	$e_{35} . q^{\frac{3}{2}}$	$e_{357} . q^{\frac{4}{3}}$
2	1	+213.686	1	−101.955	2	+231.174	166.245	190.365	392	470	480
3	1	+13.686	2	+98.045	2	−168.826	70.000	112.993	882	**364**	489
4	1	−86.314	2	−101.955	3	−68.826	94.459	86.760	1631	756	551
5	2	+93.686	3	+18.045	4	−8.826	67.464	55.319	451	754	473
6	2	+13.686	4	+98.045	5	+31.174	70.000	59.922	3530	1029	653
7	2	−43.457	4	−16.241	6	+59.746	32.804	43.672	796	608	585
8	3	+63.686	5	+48.045	6	−68.826	56.410	60.831	3075	1276	973
9	3	+13.686	5	−35.288	7	−35.493	26.764	23.104	2858	723	433
10	3	−26.314	6	+18.045	8	−8.826	22.561	19.113	1804	713	412
11	4	+50.050	6	−47.410	9	+12.992	48.748	40.503	5737	1778	991
12	4	+13.686	7	−1.955	10	+31.174	9.776	19.689	**282**	**406**	541
13	4	−17.083	8	+36.507	10	−45.749	28.500	35.202	6170	1336	1076
14	5	+42.258	8	−16.241	11	−25.969	32.012	30.132	3183	1677	1017
15	5	+13.686	9	+18.045	12	−8.826	16.015	14.034	4060	930	519
16	5	−11.314	9	−26.955	13	+6.174	20.671	17.250	6900	1323	695
17	5	−33.373	10	+3.927	14	+19.409	23.761	22.404	1135	1665	979
18	6	+13.686	11	+31.378	15	+31.174	24.207	26.732	10167	1849	1261
19	6	−7.366	11	−7.218	15	−23.457	7.293	13.745	2606	604	697
20	6	−26.314	12	+18.045	16	−8.826	22.561	19.113	7218	2018	1038
21	7	+13.686	12	−16.241	17	+2.603	15.018	12.354	7162	1445	716
22	7	−4.496	13	+7.136	18	+12.992	5.964	8.943	3454	615	551
31	10	+0.783	18	−5.181	25	−1.084	3.705	3.089	4979	639	**301**
41	13	−5.826	24	+0.484	33	−2.972	4.134	3.786	814	1085	535
53	17	−1.408	31	−0.068	43	+4.759	0.997	2.866	**192**	**385**	570
65	21	+1.379	38	−0.417	52	−8.826	1.018	5.163	1760	534	1349
68	22	+1.922	40	+3.927	55	+1.762	3.092	2.722	18160	1734	755
72	23	−2.980	42	−1.955	58	−2.159	2.520	2.406	10135	1540	721
84	27	−0.599	49	−1.955	68	+2.603	1.446	1.911	13794	1113	703
99	32	+1.565	58	+1.075	80	−0.871	1.343	1.206	10539	1323	552
118	38	+0.127	69	−0.260	95	−2.724	0.205	1.582	3621	**262**	915
130	42	+1.379	76	−0.417	105	+0.405	1.018	0.864	7040	1509	569
140	45	−0.599	82	+0.902	113	−0.254	0.766	0.642	17682	1269	467
171	55	−0.349	100	−0.201	138	−0.405	0.285	0.330	5866	636	**313**
441	142	+0.081	258	+0.086	356	−0.118	0.083	0.096	16689	772	**324**
494	159	−0.079	289	+0.069	399	+0.405	0.074	0.241	16909	815	943
612	197	−0.039	358	+0.006	494	−0.198	0.028	0.117	2166	**424**	607
665	214	−0.148	389	−0.0001	537	+0.197	0.105	0.142	**50**	1798	825

This table shows how well the scales based around equal divisions of the octave approximate the 5:4 major third, the 3:2 perfect fifth and the 7:4 seventh harmonic. The first column (q) gives the number of divisions to the octave. The second column (p_3) shows the scale degree closest to the 5:4 major third (counting from zero for the tonic), and the next column (e_3) shows the error in cents:

$$e_3 = 1200\left(\frac{p_3}{q} - \log_2\left(\frac{5}{4}\right)\right).$$

Similarly, the next two columns (p_5 and e_5) show the scale degree closest to the 3:2 perfect fifth and the error in cents:

$$e_5 = 1200\left(\frac{p_5}{q} - \log_2\left(\frac{3}{2}\right)\right).$$

The two columns after that (p_7 and e_7) show the scale degree closest to the 7:4 seventh harmonic and the error in cents:

$$e_7 = 1200\left(\frac{p_7}{q} - \log_2\left(\frac{7}{4}\right)\right).$$

We write e_{35} for the root mean square (RMS) error of the major third and perfect fifth:

$$e_{35} = \sqrt{(e_3^2 + e_5^2)/2}$$

and e_{357} for the RMS error for the major third, perfect fifth and seventh harmonic:

$$e_{357} = \sqrt{(e_3^2 + e_5^2 + e_7^2)/3}.$$

Theorem 6.2.3 shows that the quantity $e_5.q^2$ is a good measure of how well the perfect fifth is approximated by p_5/q of an octave, with respect to the number of notes in the scale. This theorem shows that there are infinitely many values of q for which $e_5.q^2 < 1200$, while on average we should expect this quantity to grow linearly with q.

Similarly, Theorem 6.2.5 with $k = 2$ shows that the quantity $e_{35}.q^{\frac{3}{2}}$ is a good measure of how well the major third and perfect fifth are simultaneously approximated, and shows that there are infinitely many values of q for which $e_{35}.q^{\frac{3}{2}} < 1200$, while on average we should expect this quantity to grow like the square root of q. Theorem 6.2.5 with $k = 3$ shows that the quantity $e_{357}.q^{\frac{4}{3}}$ is a good measure of how well all three intervals: major third, perfect fifth and seventh harmonic are simultaneously approximated, and shows that there are infinitely many values of q for which $e_{357}.q^{\frac{4}{3}} < 1200$, while on average we should expect this quantity to grow like the cube root of q.

Particularly good values of $e_5.q^2$, $e_{35}.q^{\frac{3}{2}}$ and $e_{357}.q^{\frac{4}{3}}$ are indicated in bold face in the last three columns of the table.

APPENDIX I

Intervals

This is a table of intervals not exceeding one octave (or a tritave in the case of the Bohlen–Pierce, or BP scale). A much more extensive table may be found in Appendix XX to Helmholtz [**43**] (page 453), which was added by the translator, Alexander Ellis. Names of notes in the BP scale are denoted with a subscript BP, to save confusion with notes which may have the same name in the octave based scale.

The first column is equal to 1200 times the logarithm to base two of the ratio given in the second column. Logarithms to base two can be calculated by taking the natural logarithm and dividing by $\ln 2$. So the first column is equal to

$$\frac{1200}{\ln 2} \approx 1731.234$$

times the natural logarithm of the second column.

We have given all intervals to three decimal places for theoretical purposes. While intervals of less than a few cents are imperceptible to the human ear in a melodic context, in harmony very small changes can cause large changes in beats and roughness of chords. Three decimal places gives great enough accuracy that errors accumulated over several calculations should not give rise to perceptible discrepancies.

Cents	Interval ratio	Eitz	Name, etc.	Ref
0.000	1:1	C^0, C^0_{BP}	Fundamental	§4.1
1.000	$2^{\frac{1}{1200}}$:1		Cent	§5.4
1.805	$2^{\frac{1}{665}}$:1		Degree of 665 tone scale	§6.4
1.953	32805:32768	$B\sharp^{-1}$	Schisma	§5.6
3.986	$10^{\frac{1}{1000}}$:1		Savart	§5.4
14.191	245:243	C^{+1}_{BP}	BP-minor diesis	§6.7
19.553	2048:2025	$D\flat\flat^{+2}$	Diaschisma	§5.6
21.506	81:80	C^{+1}	Syntonic, or ordinary comma	§5.5
22.642	$2^{\frac{1}{53}}$:1		Degree of 53 tone scale	§6.3
23.460	3^{12}:2^{19}	$B\sharp^0$	Pythagorean comma	§5.2
27.264	64:63		Septimal comma	§5.6
35.099			Carlos' γ scale degree	§6.6
41.059	128:125	$D\flat\flat^{+3}$	Great diesis	§5.10
49.772	7^{13}:3^{23}	$D\flat\flat^0_{BP}$	BP 7/3 comma	§6.7
63.833			Carlos' β scale degree	§6.6
70.672	25:24	$C\sharp^{-2}$	Small (just) semitone	§5.5
77.965			Carlos' α scale degree	§6.6
90.225	256:243	$D\flat^0$	Diesis or Limma	§5.2
100.000	$2^{\frac{1}{12}}$:1	$\approx C\sharp^{-\frac{7}{11}}$	Equal semitone	§5.12
111.731	16:15	$D\flat^{+1}$	Just minor semitone (ti–do, mi–fa)	§5.5
113.685	2187:2048	$C\sharp^0$	Pythagorean apotomē	§5.2
133.238	27:25	$D\flat^{-2}_{BP}$		§6.7
146.304	$3^{\frac{1}{13}}$:1		BP-equal semitone	§6.7
182.404	10:9	D^{-1}	Just minor tone (re–mi, so–la)	§5.5
193.157	$\sqrt{5}$:2	$D^{-\frac{1}{2}}$	Meantone whole tone	§5.10
200.000	$2^{\frac{1}{6}}$:1	$\approx D^{-\frac{2}{11}}$	Equal whole tone	§5.12
203.910	9:8	D^0	Just major tone (do–re, fa–so, la–ti);	§5.5
			Pythagorean major tone;	§5.2
			Nineth harmonic	§4.1
294.135	32:27	$E\flat^0$	Pythagorean minor third	§5.2
300.000	$2^{\frac{1}{4}}$:1	$\approx E\flat^{+\frac{3}{11}}$	Equal minor third	§5.12
315.641	6:5	$E\flat^{+1}$	Just minor third (mi–so, la–do, ti–re)	§5.5
386.314	5:4	E^{-1}	Just major third (do–mi, fa–la, so–ti);	§5.5
			Meantone major third;	§5.10
			Fifth harmonic	§4.1
400.000	$2^{\frac{1}{3}}$:1	$\approx E^{-\frac{4}{11}}$	Equal major third	§5.12
407.820	81:64	E^0	Pythagorean major third	§5.2
498.045	4:3	F^0	Perfect fourth	§5.2

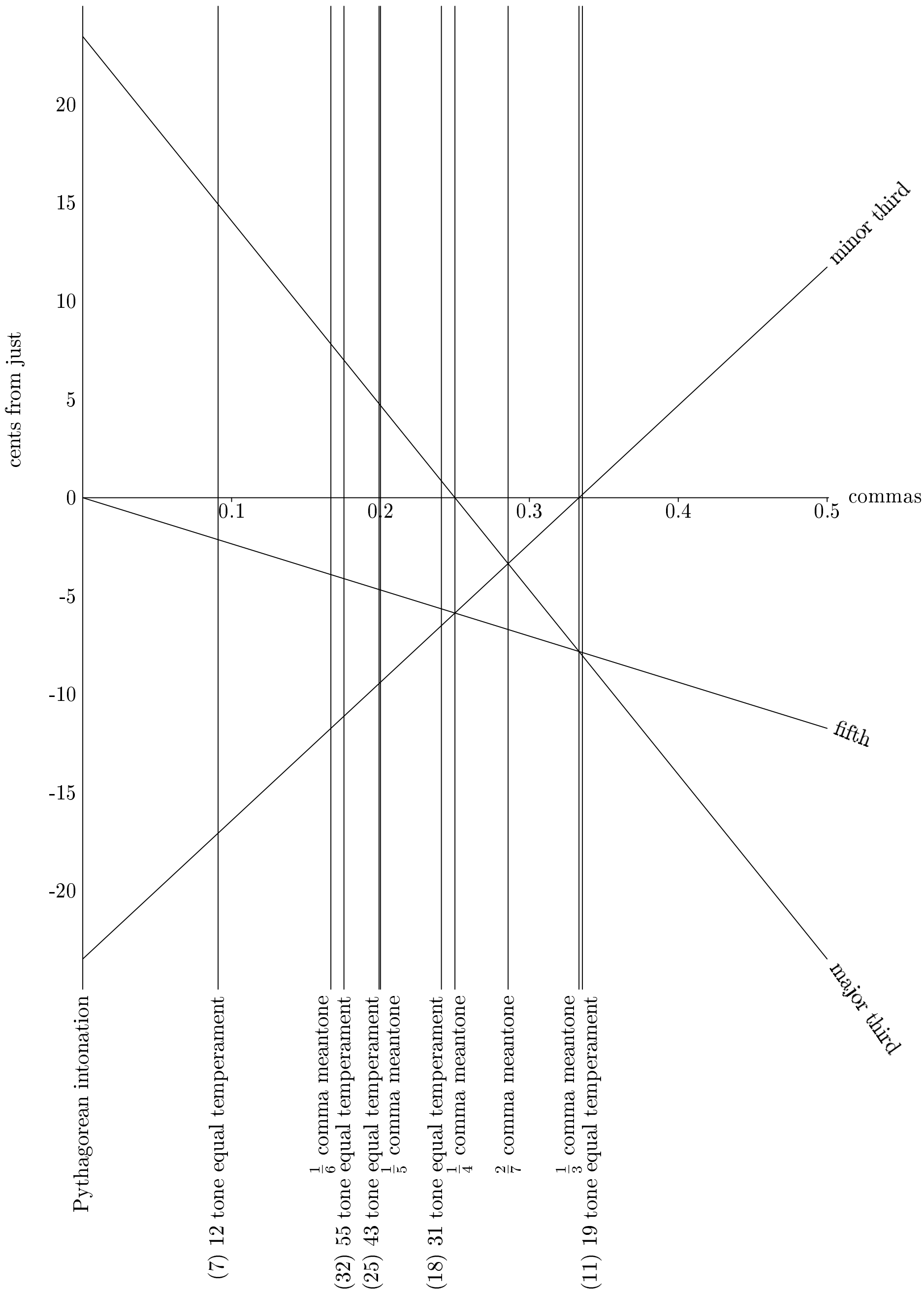
Cents	Interval ratio	Eitz	Name, etc.	Ref
500.000	$2^{\frac{5}{12}}:1$	$\approx \mathrm{F}^{+\frac{1}{11}}$	Equal fourth	§5.12
503.422	$2:5^{\frac{1}{4}}$	$\mathrm{F}^{+\frac{1}{4}}$	Meantone fourth	§5.10
551.318	11:8		Eleventh harmonic	§4.1
600.000	$\sqrt{2}:1$	$\approx \mathrm{F}\sharp^{-\frac{6}{11}}$	Equal tritone	§5.12
611.731	729:512	$\mathrm{F}\sharp^{0}$	Pythagorean tritone	§5.2
696.579	$5^{\frac{1}{4}}:1$	$\mathrm{G}^{-\frac{1}{4}}$	Meantone fifth	§5.10
700.000	$2^{\frac{7}{12}}:1$	$\approx \mathrm{G}^{-\frac{1}{11}}$	Equal fifth	§5.12
701.955	3:2	G^{0}	Just and Pythagorean (perfect) fifth;	§5.2
			Third harmonic	§4.1
792.180	128:81	$\mathrm{A}\flat^{0}$	Pythagorean minor sixth	§5.2
800.000	$2^{\frac{2}{3}}:1$	$\approx \mathrm{A}\flat^{+\frac{4}{11}}$	Equal minor sixth	§5.12
813.687	8:5	$\mathrm{A}\flat^{+1}$	Just minor sixth	§5.5
840.528	13:8		Thirteenth harmonic	§4.1
884.359	5:3	A^{-1}	Just major sixth	§5.5
889.735	$5^{\frac{3}{4}}:2$	$\mathrm{A}^{-\frac{3}{4}}$	Meantone major sixth	§5.10
900.000	$2^{\frac{3}{4}}:1$	$\approx \mathrm{A}^{-\frac{3}{11}}$	Equal major sixth	§5.12
905.865	27:16	A^{0}	Pythagorean major sixth	§5.2
968.826	7:4		Seventh harmonic	§4.1
996.091	16:9	$\mathrm{B}\flat^{0}$	Pythagorean minor seventh	§5.2
1000.000	$2^{\frac{5}{6}}:1$	$\approx \mathrm{B}\flat^{+\frac{2}{11}}$	Equal minor seventh	§5.12
1082.892	$5^{\frac{5}{4}}:4$	$\mathrm{B}^{-\frac{5}{4}}$	Meantone major seventh	§5.10
1088.269	15:8	B^{-1}	Just major seventh;	§5.5
			Fifteenth harmonic	§4.1
1100.000	$2^{\frac{11}{12}}:1$	$\approx \mathrm{B}^{-\frac{5}{11}}$	Equal major seventh	§5.12
1109.775	243:128	B^{0}	Pythagorean major seventh	§5.2
1200.000	2:1	C^{0}	Octave; Second harmonic	§4.1
1466.871	7:3	$\mathrm{A}^{0}_{\mathrm{BP}}$	BP-tenth	§6.7
1901.955	3:1	$\mathrm{C}^{0}_{\mathrm{BP}}$	BP-Tritave	§6.7

APPENDIX J

Just, equal and meantone scales compared

The figure on the next page has its horizontal axis measured in multiples of the (syntonic) comma, and the vertical axis measured in cents. Each vertical line represents a regular scale, generated by its fifth. The size of the fifth in the scale is equal to the Pythagorean fifth (ratio of 3:2, or 701.955 cents) minus the multiple of the comma given by the position along the horizontal axis. The three sloping lines show how far from the just values the fifth, major third and minor third are in these scales. This figure is relevant to Exercise **2** in §6.4.

It is worth noting that if $\frac{1}{11}$ comma meantone were drawn on this diagram, it would be indistinguishable from 12 tone equal temperament; see §5.12.

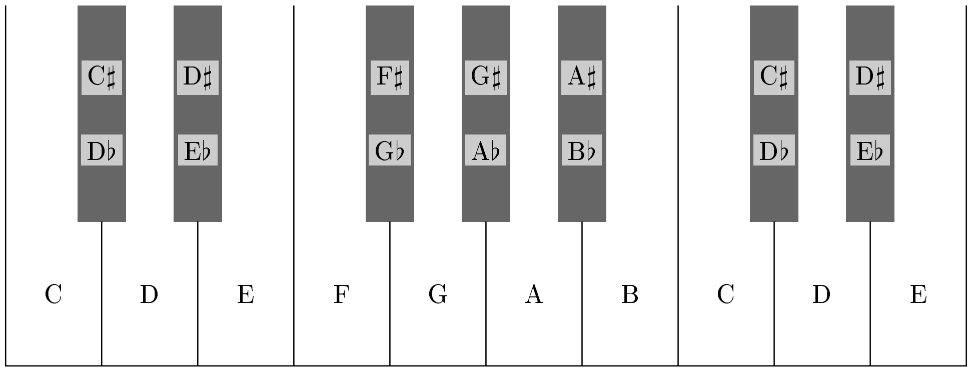


Regular scales and their deviations from just intonation

APPENDIX M

Music theory

This appendix consists of the background in elementary music theory needed to understand the main text. The emphasis is slightly different than that of a standard music text. We begin with the piano keyboard, as a convenient way to represent the modern scale.

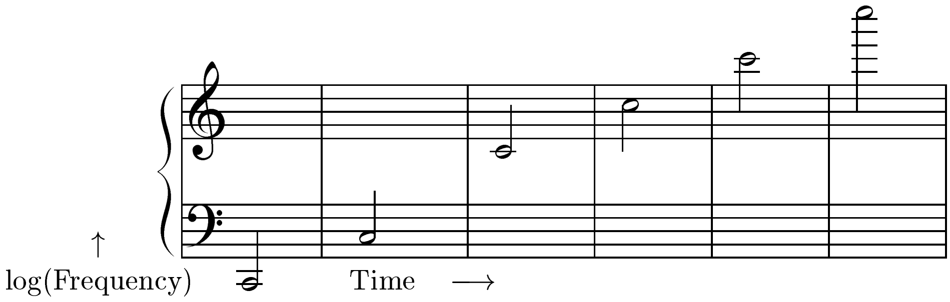


Both the black and the white keys represent notes. This keyboard is periodic in the horizontal direction, in the sense that it repeats after seven white notes and five black notes. The period is one *octave*, which represents doubling the frequency corresponding to the note. The principle of *octave equivalence* says that notes differing by a whole number of octaves are regarded as playing equivalent roles in harmony. In practice, this is almost but not quite completely true.

On a modern keyboard, each of the twelve intervals making up an octave represents the same frequency ratio, called a *semitone*. The name comes from the fact that two semitones make a *tone*. The twelfth power of the semitone's frequency ratio is a factor of 2:1, so a semitone represents a frequency ratio of $2^{\frac{1}{12}}$:1. The arrangement where all the semitones are equal in this way is called *equal temperament*. Frequency is an exponential function of position on the keyboard, and so the keyboard is really a *logarithmic* representation of frequency.

Because of this logarithmic scale, we talk about *adding* intervals when we want to *multiply* the frequency ratios. So when we add a semitone to another semitone, for example, we get a tone with a frequency ratio of $2^{1/12} \times 2^{1/12}$:1 or $2^{1/6}$:1. This transition between additive and multiplicative notation can be a source of great confusion.

Staff notation works in a similar way, except that the logarithmic frequency is represented vertically, and the horizontal direction represents time. So music notation paper can be regarded as graph paper with a linear horizontal time axis and a logarithmic vertical frequency axis.

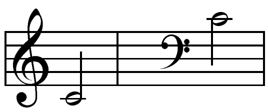


In the above diagram, each note is twice the frequency of the previous one, so they are equally spaced on the logarithmic frequency scale (except for the break between the bass and treble clefs). The gap between adjacent notes is one octave, so the gap between the lowest and highest note is described *additively* as five octaves, representing a *multiplicative* frequency ratio of 2^5:1.

There are two clefs on this diagram. The upper one is called the *treble clef*, with lines representing the notes E, G, B, D, F, beginning with the E two white notes above middle C and working up the lines. The spaces between them represent the notes F, A, C, E between them, so that this takes care of all the white notes between the E above middle C and the F an octave and a semitone above that. The black notes are represented in by using the line or space with the likewise lettered white note with a sharp (♯) or flat (♭) sign in front.

The lower clef is called the bass clef, with lines representing the notes G, B, D, F, A, with the last note representing the A two white notes below middle C and the first note representing the G an octave and a tone below that.

Middle C itself is represented using a *leger line*, either below the treble clef or above the bass clef.



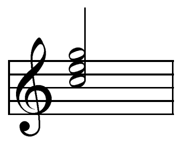
The frequency ratio represented by seven semitones, for example the interval from C to the G above it, is called a *perfect fifth.* Well, actually, this isn't quite true. A perfect fifth is supposed to be a frequency ratio of 3:2, or 1.5:1, whereas seven semitones on our modern equal tempered scale produce a frequency ratio of $2^{7/12}$:1 or roughly 1.4983:1. The perfect fifth is a consonant interval, just as the octave is, for reasons described in Chapter 4. So seven semitones is very close to a consonant interval. It is very difficult to discern the difference between a perfect fifth and an equal tempered fifth except by listening for beats; the difference is about one fiftieth of a semitone.

The perfect fourth represents the interval of 4:3, which is also consonant. The difference between a perfect fourth and the equal tempered fourth of five semitones is exactly the same as the difference between the perfect fifth and the equal tempered fifth, because they are obtained from the corresponding versions of a fifth by subtracting from an octave.

The frequency ratio represented by four semitones, for example the interval from C to the E above it, is called a *major third*. This represents a frequency ratio of $2^{4/12}$:1 or $\sqrt[3]{2}$:1, or roughly 1.25992:1. The *just major third* is defined to be the frequency ratio of 5:4 or 1.25:1. Again it is the just major third which represents the consonant interval, and the major third on our modern equal tempered scale is an approximation to it. The approximation is quite a bit worse than it was for the perfect fifth. The difference between a just major third and an equal tempered major third is quite audible; the difference is about one seventh of a semitone.

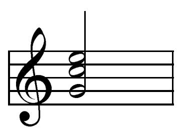
The frequency ratio represented by three semitones, for example the interval from E to the G above it, is called a *minor third*. This represents a frequency ratio of $2^{3/12}$:1 or $\sqrt[4]{2}$:1, or roughly 1.1892:1. The consonant *just minor third* is defined to be the frequency ratio of 6:5 or 1.2:1. The equal tempered minor third again differs from it by about a seventh of a semitone.

A major third plus a minor third makes up a fifth, either in the just/perfect versions or the equal tempered versions. So the intervals C to E (major third) plus E to G (minor third) make C to G (fifth). In the just/perfect versions, this gives ratios 4:5:6 for a *just major chord* C—E—G. We refer to C as the *root* of this chord. The chord is named after its root, so that this is a C major chord.



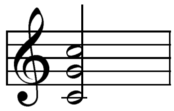
4:5:6

If we used the frequency ratios 3:4:5, it would just give an *inversion* of this chord, which is regarded as a variant form of the C major chord, because of the principle of octave equivalence.



3:4:5

while the frequency ratios 2:3:4 give a much simpler chord with a fifth and an octave.



2:3:4

So the just major chord 4:5:6 is the chord that is basic to the western system of musical harmony. On an equal tempered keyboard, this is approximated with the chord $2^{7/12}$:$2^{4/12}$:1, which is a good approximation except for the somewhat sharp major third.

The *major scale* is formed by taking three major chords on three notes separated by intervals of a fifth. So for example the scale of C major is formed from the notes of the chords F major, C major and G major. Between them, these account for the white notes on the keyboard, which make up the scale of C major. So in just intonation, the C major scale would have the following frequency ratios.

C	D	E	F	G	A	B	C	D
$\frac{1}{1}$	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	$\frac{2}{1}$	$\frac{9}{4}$
4	:	5	:	6	:		(8)	
			4	:	5	:	6	
	(3)	:		4	:	5	:	6

Here, we have made use of 2:1 octaves to transfer ratios between the right and left end of the diagram.

The basic problem with this scale is that the interval from D to A is almost, but not quite equal to a perfect fifth. It is just close enough that it sounds like a nasty, out of tune fifth. It is short of a perfect fifth by a ratio of 81:80. This interval is called a *syntonic comma.* In this text, when we use the word comma without further qualification, it will always mean the syntonic comma. This and other commas are investigated in Section 5.6.

The *meantone* scale addresses this problem by distributing the syntonic comma equally between the four fifths C–G–D–A–E. So in the meantone scale, the fifths are one quarter of a comma smaller than the perfect fifth, and the major thirds are just. In the meantone scale, a number of different keys work well, but the more remote keys do not. For further details, see Section 5.10.

To make all keys work well, the meantone scale must be bent to meet around the back. A number of different versions of this compromise have been used historically, the first ones being due to Werckmeister. Some of these well tempered scales are described in Section 5.11. Meantone and well tempered scales were in common use for about four centuries before equal temperament became widespread in the late nineteenth and early twentieth century.

APPENDIX O

Online papers

Several journals have good selections of papers available online. Access usually requires you to be logged on from an academic establishment which subscribes to the journal in question. Here is a selection of what is available from a typical academic institution.

From http://www.jstor.org you can obtain online copies of papers from the American Mathematics Monthly, a publication which concentrates on undergraduate level mathematics. Papers include the following, in chronological order.

J. M. Barbour, *Synthetic musical scales*, Amer. Math. Monthly 36 (3) (1929), 155–160.

J. M. Barbour, *A sixteenth century approximation for π*, Amer. Math. Monthly 40 (2) (1933), 69–73.

J. M. Barbour, *Music and ternary continued fractions*, Amer. Math. Monthly 55 (9) (1948), 545–555.

J. B. Rosser, *Generalized ternary continued fractions*, Amer. Math. Monthly 57 (8) (1950), 528–535.

J. M. Barbour, *A geometrical approximation to the roots of numbers*, Amer. Math. Monthly 64 (1) (1957), 1–9.

Mark Kac, *Can one hear the shape of a drum?* Amer. Math. Monthly 73 (4) (1966), 1–23.

John Rogers and Bary Mitchell, *A problem in mathematics and music*, Amer. Math. Monthly 75 (8) (1968), 871–873.

A. L. Leigh Silver, *Musimatics, or the nun's fiddle*, Amer. Math. Monthly 78 (4) (1971), 351–357.

G. D. Hasley and Edwin Hewitt, *More on the superparticular ratios in music*, Amer. Math. Monthly 79 (10) (1972), 1096–1100.

I. J. Schoenberg, *On the location of the frets on the guitar*, Amer. Math. Monthly 83 (7) (1976), 550–552.

David Gale, *Tone perception and decomposition of periodic function*, Amer. Math. Monthly 86 (1) (1979), 36–42.

Murray Schechter, *Tempered scales and continued fractions*, Amer. Math. Monthly 87 (1) (1980), 40–42.

David L. Reiner, *Enumeration in music theory*, Amer. Math. Monthly 92 (1) (1985), 51–54.

John Clough and Gerald Myerson, *Musical scales and the generalized circle of fifths*, Amer.

Math. Monthly 93 (9) (1986), 695–701.

S. J. Chapman, *Drums that sound the same*, Amer. Math. Monthly 102 (2) (1995), 124–138.

Rachel W. Hall and Krešimir Josić, *The mathematics of musical instruments*, Amer. Math. Monthly 108 (4) (2001), 347–357.

There are occasionally relevant articles in the SIAM[1] journals, also available from http://www.jstor.org. Examples include the following.

A. A. Goldstein, *Optimal temperament*, SIAM Review 19 (3) (1977), 554–562.

A. Inselberg, *Cochlear dynamics: the evolution of a mathematical model*, SIAM Review 20 (2) (1978), 301–351.

Robert Burridge, Jay Kappraff and Christine Mordeshi, *The Sitar string, a vibrating string with a one-sided inelastic constraint*, SIAM J. Appl. Math. 42 (6) (1982), 1231–1251.

M. H. Protter, *Can one hear the shape of a drum? Revisited*, SIAM Review 29 (2) (1987), 185–197.

Tobin A. Driscoll, *Eigenmodes of isospectral drums*, SIAM Review 39 (1) (1997), 1–17.

From http://ojps.aip.org/jasa/ (then hit “browse html” or “search”) you can obtain online copies of articles from the Journal of the Acoustical Society of America (JASA) from 1997 to the current issue. Here is a selection of some relevant articles that can be downloaded.

Donald L. Sullivan, *Accurate frequency tracking of timpani spectral lines*, JASA 101 (1) (1997), 530–538.

Antoine Chaigne and Vincent Doutaut, *Numerical simulations of xylophones. I. Time-domain modeling of the vibrating bars*, JASA 101 (1) (1997), 539–557.

Hugh J. McDermott and Colette M. McKay, *Musical pitch perception with electrical stimulation of the cochlea*, JASA 101 (3) (1997), 1622–1631.

John Sankey and William A. Sethares, *A consonance-based approach to the harpsichord tuning of Domenico Scarlatti*, JASA 101 (4) (1997), 2332–2337.

Knut Guettler and Anders Askenfelt, *Acceptance limits for the duration of pre-Helmholtz transients in bowed string attacks*, JASA 101 (5) (1997), 2903–2913.

Marc-Pierre Verge, Benoit Fabre, A. Hirschberg and A. P. J. Wijnands, *Sound production in recorderlike instruments. I. Dimensionless amplitude of the internal acoustic field*, JASA 101 (5) (1997), 2914–2924.

M. P. Verge, A. Hirschberg and R. Caussé, *Sound production in recorderlike instruments. II. A simulation model*, JASA 101 (5) (1997), 2925–2939.

David M. Mills, *Interpretation of distortion product otoacoustic emission measurements. I. Two stimulus tones*, JASA 102 (1) (1997), 413–429.

Eric Prame, *Vibrato extent and intonation in professional Western lyric singing*, JASA 102 (1) (1997), 616–621.

[1]Society for Industrial and Applied Mathematics

Guy Vandegrift and Eccles Wall, *The spatial inhomogeneity of pressure inside a violin at main air resonance*, JASA 102 (1) (1997), 622–627.

Harold A. Conklin, Jr., *Piano strings and "phantom" partials*, JASA 102 (1) (1997), 659.

I. Winkler, M. Tervaniemi and R. Näätänen, *Two separate codes for missing-fundamental pitch in the human auditory cortex*, JASA 102 (2) (1997), 1072–1082.

Alain de Cheveigné, *Harmonic fusion and pitch shifts of mistuned partials*, JASA 102 (2) (1997), 1083–1087.

Robert P. Carlyon, *The effects of two temporal cues on pitch judgments*, JASA 102 (2) (1997), 1097–1105.

N. Giordano, *Simple model of a piano soundboard*, JASA 102 (2) (1997), 1159–1168.

Ray Meddis and Lowel O'Mard, *A unitary model of pitch perception*, JASA 102 (3) (1997), 1811–1820.

Bruno H. Repp, *Acoustics, perception, and production of* legato *articulation on a computer-controlled grand piano*, JASA 102 (3) (1997), 1878–1890.

William A. Sethares, *Specifying spectra for musical scales*, JASA 102 (4) (1997), 2422–2431.

Eric D. Scheirer, *Tempo and beat analysis of acoustic musical signals*, JASA 103 (1) (1998), 588–601.

Myeong-Hwa Lee, Jeong-No Lee and Kwang-Sup Soh, *Chaos in segments from Korean traditional singing and Western singing*, JASA 103 (2) (1998), 1175–1182.

Alain de Cheveigné, *Cancellation model of pitch perception*, JASA 103 (3) (1998), 1261–1271.

Louise J. White and Christopher J. Plack, *Temporal processing of the pitch of complex tones*, JASA 103 (4) (1998), 2051–2063.

N. Giordano, *Mechanical impedance of a piano soundboard*, JASA 103 (4) (1998), 2128–2133.

Henry T. Bahnson, James F. Antaki and Quinter C. Beery, *Acoustical and physical dynamics of the diatonic harmonica*, JASA 103 (4) (1998), 2134–2144.

Jian-Yu Lin and William M. Hartmann, *The pitch of a mistuned harmonic: evidence for a template model*, JASA 103 (5) (1998), 2608–2617.

Shigeru Yoshikawa, *Jet-wave amplification in organ pipes*, JASA 103 (5) (1998), 2706–2717.

Teresa D. Wilson and Douglas H. Keefe, *Characterizing the clarinet tone: measurements of Lyapunov exponents, correlation dimension, and unsteadiness*, JASA 104 (1) (1998), 550–561.

Bruno H. Repp, *A microcosm of musical expression. I. Quantitative analysis of pianists' timing in the initial measures of Chopin's Etude in E major*, JASA 104 (2) (1998), 1085–1100.

Cornelis J. Nederveen, *Influence of a toroidal bend on wind instrument tuning*, JASA 104 (3) (1998), 1616–1626.

Joël Gilbert, Sylvie Ponthus and Jean-François Petiot, *Artificial buzzing lips and brass instruments: Experimental results*, JASA 104 (3) (1998), 1627–1632.

Vincent Doutant, Denis Matignon and Antoine Chaigne, *Numerical simulations of xylophones. II. Time-domain modeling of the resonator and of the radiated sound pressure*, JASA 104 (3) (1998), 1633–1647.

N. Giordano, *Sound production by a vibrating piano soundboard: Experiment*, JASA 104 (3) (1998), 1648–1653.

Jeffrey M. Brunstrom and Brian Roberts, *Profiling the perceptual suppression of partials in periodic complex tones: Further evidence for a harmonic template*, JASA 104 (6) (1998), 3511–3519.

George Bissinger, *A0 and A1 coupling, arching, rib height, and f-hole geometry dependence in the 2 degree-of-freedom network model of violin cavity modes*, JASA 104 (6) (1998), 3608–3615.

Harold A. Conklin, Jr., *Generation of partials due to nonlinear mixing in a stringed instrument*, JASA 105 (1) (1999), 536–545.

N. H. Fletcher and A. Tarnopolsky, *Blowing pressure, power, and spectrum in trumpet playing*, JASA 105 (2) (1999), 874–881.

Stephen McAdams, James W. Beauchamp and Suzanna Meneguzzi, *Discrimination of musical instrument sounds resynthesized with simplified spectrotemporal parameters*, JASA 105 (2) (1999), 882–897.

Judith C. Brown, *Computer identification of musical instruments using pattern recognition with cepstral coefficients as features*, JASA 105 (3) (1999), 1933–1941.

J. Bretos, C. Santamaría and J. Alonso Moral, *Vibrational patterns and frequency responses of the free plates and box of a violin obtained by finite element analysis*, JASA 105 (3) (1999), 1942–1950.

Daniel Pressnitzer and Stephen McAdams, *Two phase effects in roughness perception*, JASA 105 (5) (1999),2773–2782.

Seiji Adachi and Masashi Yamada, *An acoustical study of sound production in biphonic singing Xöömij*, JASA 105 (5) (1999), 2920–2932.

Xavier Boutillon and Gabriel Weinreich, *Three-dimensional mechanical admittance: Theory and new measurement method applied to the violin bridge*, JASA 105 (6) (1999), 3524–3533.

Eiji Hayashi, Masami Yamane and Hajime Mori, *Behavior of piano-action in a grand piano. I. Analysis of the motion of the hammer prior to string contact*, JASA 105 (6) (1999), 3534–3544.

Leïla Rhaouti, Antoine Chaigne and Patrick Joly, *Time-domain modeling and numerical simulation of a kettledrum*, JASA 105 (6) (1999), 3545–3562.

Sten Ternström, *Preferred self-to-other ratios in choir singing*, JASA 105 (6) (1999), 3563–3574.

Howard F. Pollard, *Tonal portrait of a pipe organ*, JASA 106 (1) (1999), 360–370.

Bruno H. Repp, *A microcosm of musical expression. III. Contributions of timing and* *dynamics to the aesthetic impression of pianists' performances of the initial measures of Chopin's Etude in E major*, JASA 106 (1) (1999), 469–478.

Alain de Cheveigné, *Pitch shifts of mistuned partials: A time-domain model*, JASA 106 (2) (1999), 887–897.

E. Obataya and M. Norimoto, *Acoustic properties of a reed (*Arundo donax *L.) used for the vibrating plate of a clarinet*, JASA 106 (2) (1999), 1106–1110.

George R. Plitnik and Bruce A. Lawson, *An investigation of correlations between geometry, acoustic variables, and psychoacoustic parameters for French horn mouthpieces*, JASA 106 (2) (1999), 1111–1125.

Valter Ciocca, *Evidence against an effect of grouping by spectral regularity on the perception of virtual pitch*, JASA 106 (5) (1999), 2746–2751.

Thomas D. Rossing and Gila Eban, *Normal modes of a radially braced guitar determined by electronic TV holography*, JASA 106 (5) (1999), 2991–2996.

Edward M. Burns and Adrianus J. M. Houtsma, *The influence of musical training on the perception of sequentially presented mistuned harmonics*, JASA 106 (6) (1999), 3564–3570.

Maureen Mellody and Gregory H. Wakefield, *The time-frequency characteristics of violin vibrato: modal distribution analysis and synthesis*, JASA 107 (1) (2000), 598–611.

Alpar Sevgen, *A principle of least complexity for musical scales*, JASA 107 (1) (2000), 665–667.

Huanping Dai, *On the relative influence of individual harmonics on pitch judgment*, JASA 107 (2) (2000), 953–959.

Jeffrey M. Brunstrom and Brian Roberts, *Separate mechanisms govern the selection of spectral components for perceptual fusion and for the computation of global pitch*, JASA 107 (3) (2000), 1566–1577.

N. Giordano and J. P. Winans II, *Piano hammers and their force compression characteristics: Does a power law make sense?*, JASA 107 (4) (2000), 2248–2255.

Richard J. Krantz and Jack Douthett, *Construction and interpretation of equal-tempered scales using frequency ratios, maximally even sets, and P-cycles*, JASA 107 (5) (2000), 2725–2734.

Anna Runnemalm, Nils-Erik Molin and Erik Jansson, *On operating deflection shapes of the violin body including in-plane motions*, JASA 107 (6) (2000), 3452–3459.

G. R. Plitnik, *Vibration characteristics of pipe organ reed tongues and the effect of the shallot, resonator, and reed curvature*, JASA 107 (6) (2000), 3460–3473.

Robert P. Carlyon, Brian C. J. Moore and Christophe Micheyl, *The effect of modulation rate on the detection of frequency modulation and mistuning of complex tones*, JASA 108 (1) (2000), 304–315.

J. Woodhouse, R. T. Schumacher and S. Garoff, *Reconstruction of bowing point friction force in a bowed string*, JASA 108 (1) (2000), 357–368.

M. J. Elejabarrieta, A. Ezcurra and C. Santamaría, *Evolution of the vibrational behavior* *of a guitar soundboard along successive construction phases by means of the modal analysis technique*, JASA 108 (1) (2000), 369–378.

Georg Essl and Perry R. Cook, *Measurements and efficient simulations of bowed bars*, JASA 108 (1) (2000), 379–388.

J. M. Harrison and N. Thompson-Allen, *Constancy of loudness of pipe organ sounds at different locations in an auditorium*, JASA 108 (1) (2000), 389–399.

A. Z. Tarnopolsky, N. H. Fletcher and J. C. S. Lai, *Oscillating reed valves—An experimental study*, JASA 108 (1) (2000), 400–406.

Thomas D. Rossing, Uwe J. Hansen and D. Scott Hampton, *Vibrational mode shapes in Caribbean steelpans. I. Tenor and double second*, JASA 108 (2) (2000), 803–812.

N. H. Fletcher, *A class of chaotic bird calls?*, JASA 108 (2) (2000), 821–826.

Gabriel Weinreich, Colin Holmes and Maureen Mellody, *Air-wood coupling and the Swiss-cheese violin*, JASA 108 (5) (2000), 2389–2402.

Robert P. Carlyon, Laurent Demany and John Deeks, *Temporal pitch perception and the binaural system*, JASA 109 (2) (2000), 686–700.

Hedwig Gockel, Brian C. J. Moore and Robert P. Carlyon, *Influence of rate of change of frequency on the overall pitch of frequency-modulated tones*, JASA 109 (2) (2000), 701–712.

Daniel Pressnitzer, Roy D. Patterson and Katrin Krumbholz, *The lower limit of melodic pitch*, JASA 109 (5) (2000), 2074–2084.

R. Ranvaud, W. F. Thompson, L. Silveira-Moriyama and L.-L. Balkwill, *The speed of pitch resolution in a musical context*, JASA 109 (6) (2001), 3021–3030.

Jeffrey M. Brunstrom and Brian Roberts, *Effects of asynchrony and ear of presentation on the pitch of mistuned partials in harmonic and frequency-shifted complex tones*, JASA 110 (1) (2001), 391–401.

Lily M. Wang and Courtney B. Burroughs, *Acoustic radiation from bowed violins*, JASA 110 (1) (2001), 543–555.

Michael W. Thompson and William J. Strong, *Inclusion of wave steepening in a frequency-domain model of trombone sound reproduction*, JASA 110 (1) (2001), 556–562.

Werner Goebl, *Melody lead in piano performance: Expressive device or artifact?*, JASA 110 (1) (2001), 563–572.

Michael A. Akeroyd, Brian C. J. Moore and Geoffrey A. Moore, *Melody recognition using three types of dichotic-pitch stimulus*, JASA 110 (3) (2001), 1498-1504.

Alexander Galembo, Anders Askenfelt, Lola L. Cuddy and Frank A. Russo, *Effects of relative phases on pitch and timbre in the piano bass range*, JASA 110 (3) (2001), 1649–1666.

From http://ojps.aip.org/chaos/ you can obtain online copies of papers from the journal "Chaos" from 1991 to the current issue. The only relevant article I've found is the following.

Diana S. Dabby, *Musical variations from a chaotic mapping*, Chaos 6 (2) (1996), 95–107.

From http://www.elsevier.com you can download the following papers.

R. C. Read, *Combinatorial problems in the theory of music*, Discrete Mathematics 167/168 (1997), 543–551.

Ján Haluška, *Equal temperament and Pythagorean tuning: a geometrical interpretation in the plane*, Fuzzy Sets and Systems 114 (2000), 261–269.

From http://www.idealibrary.com, you can obtain online copies of papers from a number of journals; for example, the following papers come from the Journal of Sound and Vibration.

F. Gautier and N. Tahani, *Vibroacoustic behaviour of a simplified musical wind instrument*, Journal of Sound and Vibration 213 (1) (1998), 107–125.

S. Gaudet, C. Gauthier and V. G. LeBlanc, *On the vibrations of an N-string*, Journal of Sound and Vibration 238 (1) (2000), 147–169.

APPENDIX P

Partial derivatives

Partial derivatives are what happens when we differentiate a function of more than one variable. For example, a geographical map which indicates height above sea level, by some device such as coloration or contours, can be regarded as describing a function $z = f(x, y)$. Here, x and y represent the two coordinates of the map, and z denotes height above sea level. If we move due east, which we take to be the direction of the x axis, then we are keeping y constant and changing x. So the slope in this direction would be the derivative of $z = f(x, y)$ with respect to x, regarding y as a constant. This derivative is denoted $\dfrac{\partial z}{\partial x}$. More formally,

$$\frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}.$$

Similarly, $\dfrac{\partial z}{\partial y}$ is the derivative of z with respect to y, regarding x as a constant. As an example, let $z = x^4 + x^2 y - 2y^2$. Then we have $\dfrac{\partial z}{\partial x} = 4x^3 + 2xy$, because $x^2 y$ is being regarded as a constant multiple of x^2, and $-2y^2$ is just a constant. Similarly, $\dfrac{\partial z}{\partial y} = x^2 - 4y$, because x^4 is a constant and $x^2 y$ is a constant multiple of y.

Second partial derivatives are defined similarly, but we now find that we can mix the variables. As well as $\dfrac{\partial^2 z}{\partial x^2}$ and $\dfrac{\partial^2 z}{\partial y^2}$, we can now form $\dfrac{\partial^2 z}{\partial x \partial y}$ by taking the partial derivative of $\dfrac{\partial z}{\partial y}$ with respect to x, regarding y as constant, and we can also form $\dfrac{\partial^2 z}{\partial y \partial x}$ by taking partial derivatives in the opposite order. So in the above example, we have

$$\frac{\partial^2 z}{\partial x^2} = 12x^2 + 2y, \qquad \frac{\partial^2 z}{\partial y^2} = -4, \qquad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = 2x.$$

In fact, the two mixed partial derivatives agree under some fairly mild hypotheses.

THEOREM P.1. *Suppose that the partial derivatives* $\dfrac{\partial^2 z}{\partial x \partial y}$ *and* $\dfrac{\partial^2 z}{\partial y \partial x}$ *both exist and are both continuous at some point (i.e., for some chosen values of x and y). Then they are equal at that point.*

PROOF. See any book on elementary analysis; for example, J. C. Burkhill, *A first course in mathematical analysis*, CUP, 1962, theorem 8.3. □

Partial derivatives work in exactly the same way for functions of more variables. So for example if $f(x, y, z) = xy^2 \sin z$ then we have $\dfrac{\partial f}{\partial x} = y^2 \sin z$, $\dfrac{\partial f}{\partial y} = 2xy \sin z$, and $\dfrac{\partial f}{\partial z} = xy^2 \cos z$. For each pair of variables, the two mixed partial derivatives with respect to those variables agree provided they are both continuous.

The chain rule for partial derivatives needs some care. Suppose, by way of example, that z is a function of u, v and w, and that each of u, v and w is a function of x and y. Then z can also be regarded as a function of x and y. A change in the value of x, keeping y constant, will result in a change of all of u, v and w, and each of these changes will result in a change in the value of z. These changes have to be added as follows:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial x}.$$

Similarly, we have

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial y}.$$

It is essential to keep track of which variables are independent, intermediate, and dependent. In this example, the independent variables are x and y, the intermediate ones are u, v and w, and the dependent variable is z.

A good illustration of the chain rule for partial derivatives is given by the conversion from Cartesian to polar coordinates. If z is a function of x and y then it can also be regarded as a function of r and θ. To convert from polar to Cartesian coordinates, we use $x = r\cos\theta$ and $y = r\sin\theta$, and to convert back we use $r = \sqrt{x^2 + y^2}$ and $\tan\theta = y/x$. Let us convert the quantity

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2},$$

into polar coordinates, assuming that all mixed second partial derivatives are continuous, so that the above theorem applies. This calculation will be needed in §3.5, where we investigate the vibrational modes of the drum. For this purpose, it is actually technically slightly easier to regard x and y as the intermediate variables and r and θ as the independent variables, although it would be quite permissible to interchange their roles. The dependent variable is z. We have

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial z}{\partial x} + \sin\theta\frac{\partial z}{\partial y}. \tag{P.1}$$

To take the second derivative, we do the same again.

$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} &= \cos\theta \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) + \sin\theta \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right) \\ &= \cos\theta\left(\cos\theta\frac{\partial^2 z}{\partial x^2} + \sin\theta\frac{\partial^2 z}{\partial y\partial x}\right) + \sin\theta\left(\cos\theta\frac{\partial^2 z}{\partial x\partial y} + \sin\theta\frac{\partial^2 z}{\partial y^2}\right) \\ &= \cos^2\theta\frac{\partial^2 z}{\partial x^2} + 2\sin\theta\cos\theta\frac{\partial^2 z}{\partial x\partial y} + \sin^2\theta\frac{\partial^2 z}{\partial y^2}. \qquad \text{(P.2)}\end{aligned}$$

Similarly, we have

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = (-r\sin\theta)\frac{\partial z}{\partial x} + (r\cos\theta)\frac{\partial z}{\partial y},$$

and

$$\begin{aligned}\frac{\partial^2 z}{\partial \theta^2} &= (-r\sin\theta)\frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial x}\right) + (-r\cos\theta)\frac{\partial z}{\partial x} \\ &\quad + (r\cos\theta)\frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial y}\right) + (-r\sin\theta)\frac{\partial z}{\partial y} \\ &= (-r\sin\theta)\left((-r\sin\theta)\frac{\partial^2 z}{\partial x^2} + (r\cos\theta)\frac{\partial^2 z}{\partial y\partial x}\right) + (-r\cos\theta)\frac{\partial z}{\partial x} \\ &\quad + (r\cos\theta)\left((-r\sin\theta)\frac{\partial^2 z}{\partial x\partial y} + (r\cos\theta)\frac{\partial^2 z}{\partial y^2}\right) + (-r\cos\theta)\frac{\partial z}{\partial y} \\ &= r^2\left(\sin^2\theta\frac{\partial^2 z}{\partial x^2} - 2\sin\theta\cos\theta\frac{\partial^2 z}{\partial x\partial y} + \cos^2\theta\frac{\partial^2 z}{\partial y^2}\right) \\ &\quad - r\left(\cos\theta\frac{\partial z}{\partial x} + \sin\theta\frac{\partial z}{\partial y}\right). \qquad \text{(P.3)}\end{aligned}$$

Comparing the formula (P.2) for $\dfrac{\partial^2 z}{\partial r^2}$ with the formula (P.3) for $\dfrac{\partial^2 z}{\partial \theta^2}$, and using the fact that $\sin^2\theta + \cos^2\theta = 1$, we see that

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} - \frac{1}{r}\left(\cos\theta\frac{\partial z}{\partial x} + \sin\theta\frac{\partial z}{\partial y}\right).$$

Finally, looking back at equation (P.1) for $\dfrac{\partial z}{\partial r}$, we obtain the formula we were looking for, namely

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}. \qquad \text{(P.4)}$$

APPENDIX R

Recordings

Go to the entry "compact discs" in the index to find the points in the text which refer to these recordings.

Johann Sebastian Bach, *The Complete Organ Music*, recorded by Hans Fagius, Volumes 6 and 8, BIS-CD-397/398 (1989) and BIS-CD-443/444 (1989 & 1990). These recordings are played on the reconstructed 1764 Wahlberg organ, Fredrikskyrkan, Karlskrona, Sweden. This organ was reconstructed using the original temperament, which was Neidhardt's Circulating Temperament No. 3 "für eine grosse Stadt" (for a large town).

Clarence Barlow's "OTOdeBLU" is in 17 tone equal temperament, played on two pianos. This piece was composed in celebration of John Pierce's eightieth birthday, and appeared as track 15 on the Computer Music Journal's Sound Anthology CD, 1995, to accompany volumes 15–19 of the journal. The CD can be obtained from MIT press for $15.

Easley Blackwood has composed a set of microtonal compositions in each of the equally tempered scales from 13 tone to 24 tone, as part of a research project funded by the National Endowment for the Humanities to explore the tonal and modal behavior of these temperaments. He devised notations for each tuning, and his compositions were designed to illustrate chord progressions and practical application of his notations. The results are available on compact disc as Cedille Records CDR 90000 018, Easley Blackwood: *Microtonal Compositions* (1994). Copies of the scores of the works can be obtained from Blackwood Enterprises, 5300 South Shore Drive, Chicago, IL 60615, USA for a nominal cost.

Dietrich Buxtehude, *Orgelwerke*, Volumes 1–7, recorded by Harald Vogel, published by Dabringhaus and Grimm. These works are recorded on a variety of European organs in different temperaments. Extensive details are given in the liner notes.

CD1 Tracks 1–8: Norden – St. Jakobi/Kleine organ in Werckmeister III;

Tracks 9–15: Norden – St. Ludgeri organ in modified $\frac{1}{5}$ Pythagorean comma meantone with $\mathrm{C}\sharp^{-\frac{6}{5}p}$, $\mathrm{G}\sharp^{-\frac{6}{5}p}$, $\mathrm{B}\flat^{+\frac{1}{5}p}$ and $\mathrm{E}\flat^{0}$;

CD2 Tracks 1–6: Stade – St. Cosmae organ in modified quarter comma meantone with[1] $\mathrm{C}\sharp^{-\frac{3}{2}}$, $\mathrm{G}\sharp^{-\frac{3}{2}}$, F^{0}, $\mathrm{B}\flat^{0}$, $\mathrm{E}\flat^{-\frac{1}{5}}$;

Tracks 7–15: Weener – Georgskirche organ in Werckmeister III;

CD3 Tracks 1–10: Grasberg organ in Neidhardt No. 3;

Tracks 11–14: Damp – Herrenhaus organ in modified meantone with pitches taken from original pipe lengths;

[1]The liner notes are written as though $\mathrm{G}\sharp^{-\frac{3}{2}}$ were equal to $\mathrm{A}\flat^{-\frac{2}{5}}$, which is not quite true. But the discrepancy is only about 0.2 cents.

CD4 Tracks 1–8: Noordbroeck organ in Werckmeister III;

Tracks 9–15: Groningen – Aa-Kerk orgen in (almost) equal temperament;

CD5 Tracks 1–5: Pilsum organ in modified $\frac{1}{5}$ Pythagorean comma meantone (the same as the Norden – St. Ludgeri organ described above);

Tracks 6–7: Buttforde organ;

Tracks 8–10: Langwarden organ in modified quarter comma meantone with $\mathrm{G}\sharp^{-\frac{7}{4}}$, $\mathrm{B}\flat^{-\frac{1}{4}}$, $\mathrm{E}\flat^{-\frac{1}{4}}$;

Tracks 11–13: Basedow organ in quarter comma meantone;

Tracks 14–15: Groß Eichsen organ in quarter comma meantone;

CD6 Tracks 1–10: Roskilde organ in Neidhardt (no. 3?);

Track 11: Helsingør organ (unspecified temperament);

Tracks 12–15: Torrlösa organ (unspecified temperament);

CD7 Tracks 1–10 modified $\frac{1}{5}$ comma meantone with[2] $\mathrm{C}\sharp^{-\frac{6}{5}}$, $\mathrm{G}\sharp^{-\frac{6}{5}}$, $\mathrm{B}\flat^{+\frac{1}{5}}$ and $\mathrm{E}\flat^{\frac{1}{5}-\frac{1}{10}p}$.

William Byrd, *Cantones Sacrae 1575, The Cardinall's Music*, conducted by David Skinner. Track 12, *Diliges Dominum*, exhibits temporal reflectional symmetry, so that it is a perfect palindrome.

Wendy Carlos, *Switched-On Bach 2000*, Audio CD, Telarc, 1992. CD-80323. Carlos' original "Switched-On Bach" recording was performed on a Moog analog synthesizer, back in the late 1960s. The Moog is only capable of playing in equal temperament. Improvements in technology inspired her to release this new recording, using a variety of temperaments and modern methods of digital synthesis. The temperaments used are $\frac{1}{5}$ and $\frac{1}{4}$ comma meantone, and various circular (irregular) temperaments.

Wendy Carlos, *Beauty in the Beast*, Audion, 1986, Passport Records, Inc., SYNCD 200. Tracks 4 and 5 make use of super just scales.

Charles Carpenter has two CDs, titled *Frog à la Pêche* (Caterwaul Records, CAT8221, 1994) and *Splat* (Caterwaul Records, CAT4969, 1996), composed using the Bohlen–Pierce scale, and played in a progressive rock/jazz style. These recordings can be ordered directly from http://www.kspace.com/carpenter for $13.95 each. Although Carpenter does not restrict himself to sounds composed mainly of odd harmonics, his compositions are nonetheless compelling.

Perry Cook (ed.), *Music, congnition and computerized sound. An introduction to psychoacoustics* [**15**] comes with an accompanying CD full of sound examples.

Michael Harrison, *From Ancient Worlds*, for Harmonic Piano, New Albion Records, Inc., 1992. NA 042 CD. The pieces on this recording all make use of his 24 tone super just scale.

In Joseph Haydn's *Sonata 41* in A (Hob. XVI:26), the movement *Menuetto al rovescio* is a perfect palindrome. This piece can be found as track 16 on the Naxos CD number 8.553127, Haydn, *Piano sonatas, Vol. 4*, with Jenõ Jandó at the piano.

A. J. M. Houtsma and T. D. Rossing and W. M. Wagenaars, *Auditory Demonstrations*, Audio CD and accompanying booklet, Philips, 1987. This classic collection of sound examples illustrates a number of acoustic and psychoacoustic phenomena. It can be obtained from the Acoustical Society of America at http://asa.aip.org/discs.html for $26 + shipping.

[2]The liner notes identify $\mathrm{A}\flat^{-\frac{1}{10}p}$ with $\mathrm{G}\sharp^{-\frac{6}{5}}$, in accordance with the approximation of Kirnberger and Farey described in §5.12.

Enid Katahn, *Beethoven in the Temperaments* (Gasparo GSCD-332, 1997). Katahn plays Beethoven's Sonatas Op. 13, *Pathétique* and Op. 14 Nr. 1 using the Prinz temperament, and Sonatas Op. 27 Nr. 2, *Moonlight* and Op. 53 *Waldstein* in Thomas Young's temperament. The instrument is a modern Steinway concert grand rather than a period instrument. The tuning and liner notes are by Edward Foote.

Enid Katahn and Edward Foote have also brought out a recording, *Six degrees of tonality* (Gasparo GSCD-344, 2000). This begins with Scarlatti's Sonata K. 96 in quarter comma meantone, followed by Mozart's *Fantasie* Kv. 397 in Prelleur temperament, a Haydn sonata in Kirnberger III, a Beethoven sonata in Young temperament, Chopin's *Fantaisie-Impromptu* in DeMorgan temperament, and Grieg's *Glochengeläute* in Coleman 11 temperament. Finally, and in many ways the most interesting part of this recording, the Mozart *Fantasie* is played in quarter comma meantone, Prelleur temperament and equal temperament in succession, which allows a very direct comparison to be made. Unfortunately, the tempi are slightly different, which makes this recording not very useful for a blind test.

Bernard Lagacé has recorded a CD of music of various composers on the C. B. Fisk organ at Wellesley College, Massachusetts, USA, tuned in quarter comma meantone temperament. This recording is available from Titanic Records Ti-207, 1991.

Guillaume de Machaut (1300–1377), *Messe de Notre Dame* and other works. The Hilliard Ensemble, Hyperíon, 1989, CDA66358. This recording is sung in Pythagorean intonation throughout. The mass alternates polyphonic with monophonic sections. The double leading-note cadences at the end of each polyphonic section are particularly striking in Pythagorean intonation. Track 19 of this recording is *Ma fin est mon commencement* (My end is my beginning). This is an example of retrograde canon, meaning that it exhibits temporal reflectional symmetry.

Mathews and Pierce, *Current directions in computer music research* [**66**] comes with a companion CD containing numerous examples; note that track 76 is erroneous, cf. Pierce [**84**], page 257.

Edward Parmentier, *Seventeenth Century French Harpsichord Music*, Wildboar, 1985, WLBR 8502. This collection contains pieces by Johann Jakob Froberger, Louis Couperin, Jacques Champion de Chambonnières, and Jean-Henri d'Anglebert. The recording was made using a Keith Hill copy of a 1640 harpsichord by Joannes Couchet, tuned in $\frac{1}{3}$ comma meantone temperament.

Many of Harry Partch's compositions have been rereleased on CD by Composers Recordings Inc., 73 Spring Street, Suite 506, New York, NY 10012-5800. As a starting point, I would recommend *The Bewitched*, CRI CD 7001, originally released on Partch's own label, Gate 5. This piece makes extensive use of his 43 tone super just scale.

A number of Robert Rich's recordings are in some form of super just scale. His basic scale is mostly 5-limit with a 7:5 tritone:

$$1{:}1,\ 16{:}15,\ 9{:}8,\ 6{:}5,\ 5{:}4,\ 4{:}3,\ 7{:}5,\ 3{:}2,\ 8{:}5,\ 5{:}3,\ 9{:}5,\ 15{:}8.$$

This appears throughout the CDs *Numena*, *Geometry*, *Rainforest*, and others. One of the nicest examples of this tuning is *The Raining Room* on the CD *Rainforest*, Hearts of Space HS11014-2. He also uses the 7-limit scale

$$1{:}1,\ 15{:}14,\ 9{:}8,\ 7{:}6,\ 5{:}4,\ 4{:}3,\ 7{:}5,\ 3{:}2,\ 14{:}9,\ 5{:}3,\ 7{:}4,\ 15{:}8.$$

This appears on *Sagrada Familia* on the CD *Gaudi*, Hearts of Space HS11028-2. See http://www.amoeba.com for a more complete discography of Robert Rich's work.

William Sethares, *Xentonality*, Music in 10-, 17- and 19-tet. See Frog Peak Music http://www.frogpeak.org to get hold of this recording.

Sethares, *Tuning, timbre, spectrum, scale* [**105**] comes with a CD full of examples.

Isao Tomita, *Pictures at an Exhibition* (Mussorgsky), BMG 60576-2-RG. This recording was made on analog synthesizers in 1974, and is remarkably sophisticated for that era.

Johann Gottfried Walther, *Organ Works*, Volumes 1 and 2, played by Craig Cramer on the organ of St. Bonifacius, Tröchtelborn, Germany. Naxos CD numbers 8.554316 and 8.554317. This organ was restored in Kellner's reconstruction of Bach's temperament, see §5.11. For more information about the organ (details are not given in the CD liner notes), see http://www.gdo.de/neurest/troechtelborn.html.

Aldert Winkelman, *Works by Mattheson, Couperin, and others.* Clavigram VRS 1735-2. This recording is hard to obtain. The pieces by Johann Mattheson, François Couperin, Johann Jakob Froberger, Joannes de Gruytters and Jacques Duphly are played on a harpsichord tuned to Werckmeister III. The pieces by Louis Couperin and Gottlieb Muffat are played on a spinet tuned in quarter comma meantone.

APPENDIX W

The wave equation

This appendix is a supplement to Section 3.6. Its purpose is to justify the method of separation of variables for the wave equation, and to explain why a drum has "enough" eigenvalues. The account of the solution of the wave equation given here is deliberately much more compressed than the account usually given in books on partial differential equations, to emphasize the shape of the reasoning rather than the more computational aspects usually emphasized. The level of mathematical sophistication needed to follow this appendix is rather greater than for the rest of the book, but it should be accessible to someone who has taken standard undergraduate courses in vector calculus, analysis and linear algebra.

We discuss solutions z of the two dimensional wave equation

$$\frac{\partial^2 z}{\partial t^2} = c^2 \nabla^2 z, \tag{W.1}$$

on a closed, bounded domain Ω. We assume that z is identically zero on the boundary S. Initial conditions are given by specifying the values of z and $\frac{\partial z}{\partial t}$ at $t = 0$.

Throughout this appendix, Ω is a closed, bounded, simply connected domain in $\mathbb{R}^2$ with piecewise twice continuously differentiable boundary S, such that the pieces of the boundary meet at nonzero interior angles. We write $\mathbf{x}$ for the position vector (x, y) on Ω, and $d\mathbf{x}$ for the element $dx\,dy$ of area on Ω. We write $\mathbf{n}$ for the outward normal vector to S, and $d\sigma$ denotes the element of length on S. With this notation, the divergence theorem states that if $f(\mathbf{x})$ is a continuously differentiable function on Ω then

$$\int_S f \,.\, \mathbf{n}\, d\sigma = \int_\Omega \nabla f\, d\mathbf{x}. \tag{W.2}$$

In order to solve the wave equation, we begin with a study of Laplace's equation

$$\nabla^2 \phi = 0$$

on Ω, with Dirichlet boundary conditions. In other words, the value of ϕ is given on the boundary S.

Green's Identities

Let Ω be a closed bounded region with boundary S. Suppose that $f(\mathbf{x})$ and $g(\mathbf{x})$ are functions on Ω. Then we have

$$\nabla.(f\nabla g) = f\nabla^2 g + \nabla f \,.\, \nabla g. \tag{W.3}$$

If Ω is a closed bounded region with boundary S, then integrating over Ω and using the divergence theorem (W.2), we get Green's first identity.

THEOREM W.1 (Green's First Identity). *Let $f(\mathbf{x})$ be continuously differentiable, and $g(\mathbf{x})$ be twice continuously differentiable on Ω. Then*

$$\int_S (f\nabla g) \,.\, \mathbf{n}\, d\sigma = \int_\Omega (f\nabla^2 g + \nabla f \,.\, \nabla g)\, d\mathbf{x}. \tag{W.4}$$

Reversing the roles of f and g and subtracting gives Green's second identity.

THEOREM W.2 (Green's Second Identity). *Let $f(\mathbf{x})$ and $g(\mathbf{x})$ be twice continuously differentiable on Ω. Then*

$$\int_S (f\nabla g - g\nabla f) \,.\, \mathbf{n}\, d\sigma = \int_\Omega (f\nabla^2 g - g\nabla^2 f)\, d\mathbf{x}. \tag{W.5}$$

Gauss' formula

We start with the function of two variables $\mathbf{x}$ and $\mathbf{x}'$ in Ω given by $z = \ln|\mathbf{x}-\mathbf{x}'|$. For functions of two variables, it makes sense to apply ∇ with respect to $\mathbf{x}$ keeping $\mathbf{x}'$ constant, or vice versa. These are analogs of partial differentiation. To distinguish between these two options, we write $\nabla_{\mathbf{x}}$ or $\nabla_{\mathbf{x}'}$.

An easy calculation in terms of coordinates shows that as long as $\mathbf{x} \neq \mathbf{x}'$, we have

$$\nabla_{\mathbf{x}'} \ln|\mathbf{x}-\mathbf{x}'| = -\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^2} \tag{W.6}$$

and

$$\nabla^2_{\mathbf{x}'} \ln|\mathbf{x}-\mathbf{x}'| = 0. \tag{W.7}$$

For $\mathbf{x} = \mathbf{x}'$, the quantity $\nabla^2_{\mathbf{x}'} \ln|\mathbf{x}-\mathbf{x}'|$ doesn't make sense, because the logarithm isn't defined. But if we pretend that it is continuously differentiable, and integrate using the divergence theorem (W.2) we get

$$\int_\Omega \nabla^2_{\mathbf{x}'} \ln|\mathbf{x}-\mathbf{x}'|\, d\mathbf{x}' = \int_S \nabla_{\mathbf{x}'} \ln|\mathbf{x}-\mathbf{x}'| \,.\, \mathbf{n}'\, d\sigma' = -\int_S \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^2} \,.\, \mathbf{n}'\, d\sigma', \tag{W.8}$$

where $\mathbf{n}'$ and σ' are with respect to $\mathbf{x}'$. The shape of the region Ω doesn't matter in this calculation, as long as $\mathbf{x}'$ is in the interior, because of equation (W.7). If we measure using $\mathbf{x}$ as the origin and make the region a unit disk centered at the origin, then the calculation reduces to $\int_S \mathbf{x}'.\mathbf{n}'\, d\sigma'$. But in this case $\mathbf{x}'$ and $\mathbf{n}'$ are unit vectors in the same direction, so $\mathbf{x}'.\mathbf{n}' = 1$. Since the circumference of the unit circle is 2π, the integral gives 2π,

$$\int_S \nabla_{\mathbf{x}'} \ln|\mathbf{x}-\mathbf{x}'| \,.\, \mathbf{n}' \, d\sigma' = 2\pi. \tag{W.9}$$

The interpretation of this calculation is that although $\ln|\mathbf{x}-\mathbf{x}'|$ is not differentiable with respect to $\mathbf{x}'$ at $\mathbf{x}'=\mathbf{x}$, we can think of $\nabla^2_{\mathbf{x}'} \ln|\mathbf{x}-\mathbf{x}'|$ as a distribution, in the sense in which we introduced the term in Section 2.15. We have to replace $\int_{-\infty}^{\infty}$ with $\int_\Omega$, so that the delta function $\delta(\mathbf{x})$ is defined to be zero for $\mathbf{x} \neq \mathbf{0}$, and $\int_\Omega \delta(\mathbf{x})\, d\mathbf{x} = 1$. In terms of this delta function, the above calculation can be expressed as saying that

$$\nabla^2_{\mathbf{x}'} \ln|\mathbf{x}-\mathbf{x}'| = 2\pi\delta(\mathbf{x}-\mathbf{x}'). \tag{W.10}$$

So far, we have assumed that $\mathbf{x}'$ is in the interior of Ω. For a point $\mathbf{x}'$ outside Ω, the integrand in equation (W.8) is zero so the integral is zero. If $\mathbf{x}'$ is on the boundary S, and it is a point where S is continuously differentiable, then instead of a circle, in the above calculation we have to integrate over a semicircle. So the integral is π instead of 2π. At a corner with angle θ, we are integrating over a sector of a circle with angle θ, so the integral is θ. So we define a function $p(\mathbf{x})$ on $\mathbb{R}^2$ by

$$p(\mathbf{x}) = \begin{cases} 2\pi & \text{if } \mathbf{x} \text{ is in the interior of } \Omega, \\ 0 & \text{if } \mathbf{x} \text{ is not in } \Omega, \\ \pi & \text{if } \mathbf{x} \text{ is a continuously differentiable point on } S, \\ \theta & \text{if } \mathbf{x} \text{ is a corner of } S \text{ with interior angle } \theta. \end{cases}$$

Then the extension of equation (W.9) to the plane is **Gauss' formula**

$$\int_S \nabla_{\mathbf{x}'} \ln|\mathbf{x}-\mathbf{x}'| \,.\, \mathbf{n}' \, d\sigma' = p(\mathbf{x}). \tag{W.11}$$

If $f(\mathbf{x})$ is any continuous function on Ω, then we have

$$\int_\Omega f(\mathbf{x}')\nabla^2_{\mathbf{x}'} \ln|\mathbf{x}-\mathbf{x}'|\, d\mathbf{x}' = p(\mathbf{x})f(\mathbf{x}). \tag{W.12}$$

This is because the integrand is zero except near $\mathbf{x}=\mathbf{x}'$, so $f(\mathbf{x}')$ may as well be replaced by $f(\mathbf{x})$ and taken out of the integral before applying the divergence theorem.

Remark. The above calculation was performed in two dimensions. The corresponding calculation in three dimensions uses the function $1/|\mathbf{x}-\mathbf{x}'|$ instead of $\ln|\mathbf{x}-\mathbf{x}'|$. The unit circle is replaced by the unit sphere, of surface area 4π, and the analog of equation (W.9) is

$$\int_S \nabla_{\mathbf{x}'} \frac{1}{|\mathbf{x}-\mathbf{x}'|} \,.\, \mathbf{n}' \, d\sigma' = 4\pi.$$

The definition of $h(\mathbf{x},\mathbf{x}')$ and $G(\mathbf{x},\mathbf{x}')$ below are adjusted accordingly.

Similarly, in n dimensions ($n \geq 3$), the corresponding formula is

$$\int_S \nabla_{\mathbf{x}'} \frac{1}{|\mathbf{x}-\mathbf{x}'|^{n-2}} \,.\, \mathbf{n}' \, d\sigma' = n(n-2)\alpha(n)$$

where $\alpha(n)$ denotes the $(n-1)$-dimensional volume of the surface of the n-dimensional sphere.

Green's functions

Equation (W.10) is an important property of the function $\ln|\mathbf{x}-\mathbf{x}'|$. But the main problem with this function is that it doesn't vanish on the boundary S of Ω. To remedy this, we adjust it as follows. Suppose that we can find a solution $h(\mathbf{x},\mathbf{x}')$ to Laplace's equation

$$\nabla^2_{\mathbf{x}'} h(\mathbf{x},\mathbf{x}') = 0 \tag{W.13}$$

on Ω, with boundary conditions

$$h(\mathbf{x},\mathbf{x}') = \frac{1}{2\pi}\ln|\mathbf{x}-\mathbf{x}'| \tag{W.14}$$

for $\mathbf{x}'$ on S. That is, we insist that $h(\mathbf{x},\mathbf{x}')$ is defined even when $\mathbf{x}=\mathbf{x}'$ (in the interior of Ω). Then the function

$$G(\mathbf{x},\mathbf{x}') = h(\mathbf{x},\mathbf{x}') - \frac{1}{2\pi}\ln|\mathbf{x}-\mathbf{x}'|$$

still satisfies

$$\nabla^2_{\mathbf{x}'} G(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}') \tag{W.15}$$

for $\mathbf{x}'$ in the interior of Ω, but it now also satisfies $G(\mathbf{x},\mathbf{x}')=0$ for $\mathbf{x}'$ on S. The function $G(\mathbf{x},\mathbf{x}')$ defined this way is called the Green's function for the Laplace operator ∇^2.

LEMMA W.3. *The Green function, if it exists, satisfies the symmetry relation* $G(\mathbf{x},\mathbf{x}') = G(\mathbf{x}',\mathbf{x})$.

PROOF. Using Lemma W.10, we have

$$\begin{aligned} G(\mathbf{x},\mathbf{x}') &= \int_\Omega G(\mathbf{x},\mathbf{x}'')\delta(\mathbf{x}'-\mathbf{x}'')\,d\mathbf{x}'' = \int_\Omega G(\mathbf{x},\mathbf{x}'')\nabla^2_{\mathbf{x}''}G(\mathbf{x}',\mathbf{x}'')\,d\mathbf{x}'' \\ &= \int_\Omega G(\mathbf{x}',\mathbf{x}'')\nabla^2_{\mathbf{x}''}G(\mathbf{x},\mathbf{x}'')\,d\mathbf{x}'' = \int_\Omega G(\mathbf{x},\mathbf{x}'')\delta(\mathbf{x}'-\mathbf{x}'')\,d\mathbf{x}'' = G(\mathbf{x}',\mathbf{x}). \end{aligned}$$

□

The construction of the Green's function $G(\mathbf{x},\mathbf{x}')$ depends on solving Laplace's equation (W.13) with boundary conditions (W.14). We do this using Fredholm theory.

Hilbert space

A *Hilbert space* V is a (usually infinite dimensional) complex vector space with inner product $\langle\ ,\ \rangle$ satisfying

(i) $\langle x, \lambda y_1 + \mu y_2\rangle = \lambda\langle x,y_1\rangle + \mu\langle x,y_2\rangle$,

(ii) $\langle x,y\rangle = \overline{\langle y,x\rangle}$ (and in particular $\langle x,x\rangle$ is real), and

(iii) $\langle x,x\rangle \geq 0$, and $\langle x,x\rangle = 0$ if and only if $x=0$,

(iv) Writing $|x|$ for $\sqrt{\langle x,x\rangle}$, the metric with distance function $|x-y|$ is complete. In other words, every Cauchy sequence has a limit.

For example, if D is a compact domain in $\mathbb{R}^n$ then the space $L^2(D)$ of square integrable functions on D is a Hilbert space, with inner product

$$\langle f,g\rangle = \int_\Omega \bar{f}\, g\, d\mathbf{x}.$$

In this example, the completeness is a standard fact from Lebesgue integration theory. In order to satisfy (iii), we stipulate that two functions are identified if they agree except on a set of measure zero. Of course, this never identifies two continuous functions.

LEMMA W.4 (Schwartz's inequality). *For vectors x and y in Hilbert space, we have $\langle x,y\rangle \le |x||y|$.*

PROOF. Consider the quantity

$$\langle x-ty, x-ty\rangle = |x|^2 - 2t\langle x,y\rangle + t^2|y|^2 \ge 0.$$

Differentiating with respect to t, we see that this expression is minimized by setting $t = \langle x,y\rangle/|y|^2$. With this value of t, we get

$$|x|^2 - 2\langle x,y\rangle^2/|y|^2 + \langle x,y\rangle^2/|y|^2 \ge 0,$$

or $\langle x,y\rangle^2/|y|^2 \le |x|^2$. □

Elements x and y satisfying $\langle x,y\rangle = 0$ are said to be *orthogonal*. If W is a subspace of V, we write $W^\perp$ for the subspace consisting of vectors v such that for all $w \in W$ we have $\langle v,w\rangle = 0$. If W is finite dimensional, then any vector v in V can be written in a unique way as $v = w + x$ with w in W and x in $W^\perp$, so that

$$V = W \oplus W^\perp.$$

If $\mathbf{K}$ is a linear operator on V, its *image* is

$$\mathrm{Im}\,(\mathbf{K}) = \{\mathbf{K}v,\ v \in V\}$$

and its *kernel* is

$$\mathrm{Ker}\,(\mathbf{K}) = \{v \in V \mid Kv = 0\}.$$

LEMMA W.5. *If $\mathbf{K}$ and $\mathbf{K}^*$ are adjoint linear operators on V (i.e., for all x and y, $\langle \mathbf{K}^*x, y\rangle = \langle x, \mathbf{K}y\rangle$) and the image of $\mathbf{K}$ is finite dimensional, then*

(i) $V = \mathrm{Im}\,\mathbf{K} \oplus \mathrm{Ker}\,\mathbf{K}^*$, *and*

(ii) $V = \mathrm{Im}\,\mathbf{K}^* \oplus \mathrm{Ker}\,\mathbf{K}$

are orthogonal direct sum decompositions of V, and

$$\dim \mathrm{Im}\,(\mathbf{K}) = \dim \mathrm{Im}\,(\mathbf{K}^*).$$

PROOF. If $\mathbf{K}^*x \in \mathrm{Im}\,(\mathbf{K}^*)$ and $y \in \mathrm{Ker}\,(\mathbf{K})$ then

$$\langle \mathbf{K}^*x, y\rangle = \langle x, \mathbf{K}y\rangle = 0$$

so $\mathrm{Im}\,(\mathbf{K}^*) \perp \mathrm{Ker}\,(\mathbf{K})$. If $x \in \mathrm{Im}\,(\mathbf{K}^*) \cap \mathrm{Ker}\,(\mathbf{K})$ then $\langle x, x\rangle = 0$ and so $x = 0$. Thus

$$\mathrm{Im}\,(\mathbf{K}^*) \oplus \mathrm{Ker}\,(\mathbf{K}) \leq V. \tag{W.16}$$

so we have

$$\dim \mathrm{Im}\,(\mathbf{K}) = \dim(V/\mathrm{Ker}\,(\mathbf{K})) \geq \dim \mathrm{Im}\,(\mathbf{K}^*), \tag{W.17}$$

with equality if and only if (W.16) is an equality. In particular, it follows that $\mathrm{Im}\,(\mathbf{K}^*)$ is also finite dimensional. So we may repeat the above argument with the roles of $\mathbf{K}$ and $\mathbf{K}^*$ reversed, so that

$$\mathrm{Im}\,(\mathbf{K}) \oplus \mathrm{Ker}\,(\mathbf{K}^*) \leq V \tag{W.18}$$

and

$$\dim \mathrm{Im}\,(\mathbf{K}^*) \geq \dim \mathrm{Im}\,(\mathbf{K}) \tag{W.19}$$

with equality if and only if (W.18) is an equality. Comparing (W.17) with (W.19), we see that both must be equalities, so (W.16) and (W.18) are equalities. □

LEMMA W.6. *If* $\mathbf{K}$ *and* $\mathbf{K}^*$ *are adjoint operators and* $\mathrm{Im}\,(\mathbf{K})$ *is finite dimensional then*

(i) $V = \mathrm{Im}\,(\mathbf{I} - \mathbf{K}) \oplus \mathrm{Ker}\,(\mathbf{I} - \mathbf{K}^*)$ *and*

(ii) $V = \mathrm{Im}\,(\mathbf{I} - \mathbf{K}^*) \oplus \mathrm{Ker}\,(\mathbf{I} - \mathbf{K})$

are orthogonal decompositions of V, *and* $\dim \mathrm{Im}\,(\mathbf{I} - \mathbf{K}) = \dim \mathrm{Im}\,(\mathbf{I} - \mathbf{K}^*)$ *is finite.*

PROOF. By Lemma W.5, $\mathrm{Im}\,(\mathbf{K}^*)$ is finite dimensional, so $V_1 = \mathrm{Im}\,(\mathbf{K}) + \mathrm{Im}\,(\mathbf{K}^*) \leq V$ is also finite dimensional. So $V = V_1 \oplus V_2$ where

$$V_2 = V_1^{\perp} = \mathrm{Ker}\,(\mathbf{K}) \cap \mathrm{Ker}\,(\mathbf{K}^*).$$

So $\mathbf{I} - \mathbf{K}$ and $\mathbf{I} - \mathbf{K}^*$ send V_1 into V_1 and act as the identity map on V_2. Applying Lemma W.5 with $\mathbf{I} - \mathbf{K}$ instead of $\mathbf{K}$ and V_1 in place of V, we see that V_1 decomposes in the way described in the lemma. Since $\mathbf{I} - \mathbf{K}$ and $\mathbf{I} - \mathbf{K}^*$ act as the identity on V_2, this just contributes another summand to $\mathrm{Im}\,(\mathbf{I} - \mathbf{K})$ and $\mathrm{Im}\,(\mathbf{I} - \mathbf{K}^*)$, so the decomposition holds for V. □

The Fredholm alternative

Now let V be the vector space $L^2(D)$ of Lebesgue square integrable functions on a compact domain D in $\mathbb{R}^n$. Suppose that $K(\mathbf{x}, \mathbf{x}')$ is a continuous complex valued function of two variables $\mathbf{x}$ and $\mathbf{x}'$ in D. We are interested in the operator $\mathbf{K}$ on $L^2(D)$ given by

$$\mathbf{K}\psi(\mathbf{x}) = \int_D \psi(\mathbf{x}')K(\mathbf{x}, \mathbf{x}')\, d\mathbf{x}'. \tag{W.20}$$

Such an operator is called a *Fredholm operator*. Its adjoint is given by

$$\mathbf{K}^*\psi(\mathbf{x}) = \int_D \psi(\mathbf{x}')\overline{K(\mathbf{x}',\mathbf{x})}\,d\mathbf{x}'. \tag{W.21}$$

In general, the image of a Fredholm operator is not finite dimensional, so we can't apply Lemma W.6 directly. However, a function of the form $K(\mathbf{x},\mathbf{x}') = g(\mathbf{x})h(\mathbf{x}')$ gives rise to an operator $\mathbf{K}$ with one dimensional image spanned by $g(\mathbf{x})$. Any polynomial function of $\mathbf{x}$ and $\mathbf{x}'$ can be written as a finite sum of monomials, each of which has this form. So if $K(\mathbf{x},\mathbf{x}')$ is a polynomial function, we may apply Lemma W.6.

The Weierstrass approximation theorem states that any continuous function on a compact domain in $\mathbb{R}^n$ may be uniformly approximated by polynomial functions. Applying this to $K(\mathbf{x},\mathbf{x}')$ on $D \times D$, we may write $K = K_1 + K_2$ where K_1 is a polynomial function and K_2 satisfies $B < 1$, where B is defined by

$$B = \iint_D |K_2(\mathbf{x},\mathbf{x}')|^2\,d\mathbf{x}\,d\mathbf{x}'.$$

For any function $\psi(\mathbf{x})$ in $L^2(D)$, Schwartz's inequality (Lemma W.4) implies that

$$|\mathbf{K}_2\psi(\mathbf{x})|^2 \le \langle\psi,\psi\rangle \int_D |K_2(\mathbf{x},\mathbf{x}')|^2\,d\mathbf{x}'.$$

Integrating with respect to $\mathbf{x}$ gives

$$\langle\mathbf{K}_2\psi,\mathbf{K}_2\psi\rangle \le B\langle\psi,\psi\rangle.$$

It follows by comparing with the geometric series

$$1 + B + B^2 + B^3 + \dots$$

that the sequence whose nth term is

$$\sum_{i=0}^{n} \mathbf{K}_2^i\psi$$

forms a Cauchy sequence in $L^2(D)$. Since $L^2(D)$ is complete, it follows that this Cauchy sequence has a limit; in other words, the infinite sum

$$\sum_{i=0}^{\infty} \mathbf{K}_2^i\psi = \psi + \mathbf{K}_2\psi + \mathbf{K}_2^2\psi + \mathbf{K}_2^3\psi + \cdots$$

converges in $L^2(D)$. It is now easy to check that the operator

$$\mathbf{I} + \mathbf{K}_2 + \mathbf{K}_2^2 + \mathbf{K}_2^3 + \dots$$

is an inverse to $\mathbf{I} - \mathbf{K}_2$ on $L^2(D)$. So we write $(\mathbf{I} - \mathbf{K}_2)^{-1}$ for this inverse.

Now we have

$$\mathbf{I} - \mathbf{K} = \mathbf{I} - (\mathbf{K}_1 + \mathbf{K}_2) = (\mathbf{I} - \mathbf{K}_2)(\mathbf{I} - (\mathbf{I} - \mathbf{K}_2)^{-1}\mathbf{K}_1).$$

The operator $(\mathbf{I} - \mathbf{K}_2)^{-1}\mathbf{K}_1$ has finite dimensional image, because $\mathbf{K}_1$ does. So Lemma W.6 enables us to write $L^2(D)$ as a direct sum of the image of

$\mathbf{I}-(\mathbf{I}-\mathbf{K}_2)^{-1}\mathbf{K}_1$ and the kernel of its adjoint. The invertibility of $\mathbf{I}-\mathbf{K}_2$ then gives us the following theorem, which is known as the *Fredholm alternative.*

THEOREM W.7. *With* $\mathbf{K}$ *and* $\mathbf{K}^*$ *defined by equations* (W.20) *and* (W.21), *the kernels of* $\mathbf{I}-\mathbf{K}$ *and* $\mathbf{I}-\mathbf{K}^*$ *are finite dimensional, and have the same dimension. If this dimension is zero, then* $\mathbf{I}-\mathbf{K}$ *is invertible, so that the equation*

$$\psi - \mathbf{K}\psi = f$$

has a unique solution ψ *for any given element* f *of* $L^2(D)$. □

Solving Laplace's equation

In the section on Green's functions (page 311), we saw that if we can solve Laplace's equation (W.13) with boundary conditions (W.14) then we can construct a Green's function $G(\mathbf{x},\mathbf{x}')$ satisfying equation (W.15) and zero on the boundary S. In this section we use Fredholm theory to solve Laplace's equation

$$\nabla^2\phi(\mathbf{x}) = 0 \tag{W.22}$$

subject to twice continuously differentiable boundary conditions $\phi(\mathbf{x}) = f(\mathbf{x})$ on S.

We begin with uniqueness. We define the *potential energy* of a continuously differentiable function ϕ on Ω by

$$E = \rho c^2 \int_\Omega \nabla\phi\,.\,\nabla\phi\,d\mathbf{x}.$$

So $E \geq 0$, and if $E = 0$ then $\nabla\phi = 0$, so that ϕ is constant. If ϕ_1 and ϕ_2 are solutions of (W.22) satisfying the same boundary conditions, then $\phi = \phi_1 - \phi_2$ satisfies (W.22) and is zero on the boundary. By Green's first identity (W.4) with $f = g = \phi$, we see that we have $E = 0$, so ϕ is constant; since $\phi = 0$ on the boundary, this constant is zero. We conclude that if a solution to Laplace's equation (W.22) with given values on the boundary exists, then it is unique.

The same method can also be used for solutions of Laplace's equation (W.22) for the unbounded region Ω' obtained by removing the interior of Ω from $\mathbb{R}^2$, but we need to be careful about the behavior of ϕ as $\mathbf{x}$ goes off to infinity. The point is that we need to apply Green's first identity (W.4) for a region with a hole, bounded by S and a large circle S' of radius R surrounding Ω, and then let $R \to \infty$. The extra term we get from the second boundary component is $\int_{S'} \phi\,\nabla\phi\,.\big(\frac{\mathbf{x}}{R}\big)\,d\sigma$, because the unit normal vector is $\mathbf{x}/R$. The length of S' is $2\pi R$, so we need to check that $2\pi R|\phi\,\nabla\phi\,.\big(\frac{\mathbf{x}}{R}\big)| \to 0$ as $|\mathbf{x}| \to 0$. So we have proved the following theorem.

THEOREM W.8. (i) *If* $\nabla^2\phi = 0$ *has a solution on* Ω *with specified values on* S, *then the solution is unique.*

(ii) *If* $\nabla^2\phi = 0$ *has a solution on* Ω' *with specified values on* S, *and satisfying*

$$\lim_{|\mathbf{x}|\to\infty} |\phi\,\nabla\phi\,.\,\mathbf{x}| = 0$$

then that solution is unique. □

We now examine the question of existence of solutions. To this end, we look for solutions of equation (W.22) of the form

$$\phi(\mathbf{x}) = \int_S \psi(\mathbf{x}')\nabla_{\mathbf{x}'}\ln|\mathbf{x}-\mathbf{x}'|\,.\,\mathbf{n}'\,d\sigma', \tag{W.23}$$

with ψ a twice continuously differentiable function defined on S.

Any twice continuously differentiable function ψ on S can be extended to a twice continuously differentiable function on Ω, which we also denote by ψ. So we can use Green's first identity (W.4) to write

$$\phi(\mathbf{x}) = \int_\Omega (\psi(\mathbf{x}')\nabla^2_{\mathbf{x}'}\ln|\mathbf{x}-\mathbf{x}'| + \nabla\psi(\mathbf{x}')\,.\,\nabla_{\mathbf{x}'}\ln|\mathbf{x}-\mathbf{x}'|)\,d\mathbf{x}'.$$

By equation (W.12), we have

$$\phi(\mathbf{x}) = p(\mathbf{x})\psi(\mathbf{x}) + \int_\Omega \nabla\psi(\mathbf{x}')\,.\,\nabla_{\mathbf{x}'}\ln|\mathbf{x}-\mathbf{x}'|\,d\mathbf{x}'. \tag{W.24}$$

In this formula, it can be shown using some elementary estimates that the integral term is continuous as $\mathbf{x}$ crosses the boundary S. It follows that $\phi(\mathbf{x})$ is discontinuous at S, so to solve Laplace's equation (W.22) using ϕ, we should use the limiting value at the boundary. Namely, for $\mathbf{x}_0$ in S and $\mathbf{x}$ in Ω but not in S, we have

$$\lim_{\mathbf{x}\to\mathbf{x}_0}\phi(\mathbf{x}) = 2\pi\psi(\mathbf{x}_0) + \int_\Omega \nabla\psi(\mathbf{x}')\,.\,\nabla_{\mathbf{x}'}\ln|\mathbf{x}_0-\mathbf{x}'|\,d\mathbf{x}',$$

whereas except at the corners, the value of ϕ on S is given by

$$\phi(\mathbf{x}_0) = \pi\psi(\mathbf{x}_0) + \int_\Omega \nabla\psi(\mathbf{x}')\,.\,\nabla_{\mathbf{x}'}\ln|\mathbf{x}_0-\mathbf{x}'|\,d\mathbf{x}'.$$

So we have

$$\lim_{\mathbf{x}\to\mathbf{x}_0}\phi(\mathbf{x}) = \phi(\mathbf{x}_0) + \pi\psi(\mathbf{x}_0).$$

In order to satisfy the boundary condition we want

$$\lim_{\mathbf{x}\to\mathbf{x}_0}\phi(\mathbf{x}) = f(\mathbf{x}_0).$$

So we must solve the equation

$$\phi(\mathbf{x}) + \pi\psi(\mathbf{x}) = f(\mathbf{x}) \tag{W.25}$$

on S. Notice that the value of ψ at corners is irrelevant to the integral (W.23), so we just ignore the anomalous values of ϕ at corners and solve (W.25) for all $\mathbf{x}$ in S.

We rewrite equation (W.25) as

$$\psi(\mathbf{x}) + \frac{1}{\pi}\int_S \psi(\mathbf{x}')\nabla_{\mathbf{x}'} \ln|\mathbf{x}-\mathbf{x}'|.\mathbf{n}'\,d\sigma' = \frac{1}{\pi}f(\mathbf{x}). \qquad \text{(W.26)}$$

Setting

$$K(\mathbf{x},\mathbf{x}') = -\frac{1}{\pi}\nabla_{\mathbf{x}'}\ln|\mathbf{x}-\mathbf{x}'|.\mathbf{n}' = \frac{(\mathbf{x}-\mathbf{x}').\mathbf{n}'}{\pi|\mathbf{x}-\mathbf{x}'|^2}$$

and $D = S$, we use equation (W.20) to obtain an operator $\mathbf{K}$ on $L^2(S)$ given by

$$\mathbf{K}\psi(\mathbf{x}) = -\frac{1}{\pi}\int_S \psi(\mathbf{x}')\nabla_{\mathbf{x}'}\ln|\mathbf{x}-\mathbf{x}'|.\mathbf{n}'\,d\sigma'.$$

Equation (W.26) then becomes

$$\psi - \mathbf{K}\psi = \frac{1}{\pi}f.$$

Applying Fredholm theory (Theorem W.7), we see that this equation always has a solution provided we can prove that the only solution of the equation

$$\psi - \mathbf{K}\psi = 0$$

is the zero function. So assume that ψ satisfies this equation, and define $\phi(\mathbf{x})$ by equation (W.23). Then $\nabla^2\phi = 0$, and $\phi(\mathbf{x}) \to 0$ as $\mathbf{x}$ approaches the boundary from inside Ω. So by Theorem W.8 (i), we have $\phi(\mathbf{x}) = 0$ for $\mathbf{x}$ in Ω. Similarly, we define $\phi(\mathbf{x})$ by equation (W.23) on Ω'. Then using equation (W.6) we find that $|\phi\,\nabla\phi\,.\,\mathbf{x}| \to 0$ as $R \to \infty$. So by Theorem W.8 (ii), we have $\phi(\mathbf{x}) = 0$ in Ω'. Now it follows from equation (W.24) that for a point $\mathbf{x}_0$ on S which is not a corner,

$$\lim_{\substack{\mathbf{x}\to\mathbf{x}_0 \\ \text{in } \Omega}} \phi(\mathbf{x}) - \lim_{\substack{\mathbf{x}\to\mathbf{x}_0 \\ \text{in } \Omega'}} \phi(\mathbf{x}) = 2\pi\psi(\mathbf{x}_0).$$

It follows that $\psi(\mathbf{x}_0) = 0$. Since we were only interested in ψ at points which are not corners, this completes the proof that the only solution of $\psi - \mathbf{K}\psi = 0$ is $\psi = 0$. Applying Fredholm theory as mentioned above, this completes the proof of existence of solutions of Laplace's equation.

Conservation of energy

We are now ready to begin proving existence and uniqueness for solutions of the wave equation (W.1). The basic tool for proving uniqueness of solutions is the conservation of energy. We define the *energy* $E(t)$ of a continuously differentiable function z of $\mathbf{x}$ and t to be the quantity

$$E(t) = \rho\int_\Omega \left(\left(\frac{\partial z}{\partial t}\right)^2 + c^2\nabla z.\nabla z\right)d\mathbf{x}. \qquad \text{(W.27)}$$

The two terms in this integral correspond to kinetic and potential energy respectively. Since $E(t)$ is obtained by integrating a sum of squares, it satisfies $E(t) \geq 0$. Furthermore, $E(t) = 0$ can only occur if the integrand is zero; namely if $\frac{\partial z}{\partial t}$ and ∇z are zero.

Suppose that z satisfies the wave equation (W.1). Differentiating, and using the divergence theorem (W.2), we get

$$\begin{aligned}\frac{dE}{dt} &= \int_\Omega \rho\left(2\frac{\partial z}{\partial t}\frac{\partial^2 z}{\partial t^2} + 2c^2\nabla z\,.\frac{\partial \nabla z}{\partial t}\right)d\mathbf{x}\\ &= \int_\Omega \rho\left(2\frac{\partial z}{\partial t}c^2\nabla^2 z + 2c^2\nabla z\,.\,\nabla\frac{\partial z}{\partial t}\right)d\mathbf{x}\\ &= \int_\Omega 2\rho c^2\nabla.\left(\frac{\partial z}{\partial t}\nabla z\right)d\mathbf{x}\\ &= \int_S 2\rho c^2\left(\frac{\partial z}{\partial t}\nabla z\right).\,\mathbf{n}\,d\sigma.\end{aligned}$$

Since $\frac{\partial z}{\partial t} = 0$ on S, we obtain

$$\frac{dE}{dt} = 0$$

so that E is a constant, independent of t. This is the statement of the conservation of energy for solutions of the wave equation.

Uniqueness of solutions

We now prove the uniqueness theorem for solutions to the wave equation. Suppose that z_1 and z_2 are solutions to the wave equation (W.1) on Ω, with the same initial conditions (i.e., the same values of z and $\frac{\partial z}{\partial t}$ for $t = 0$), and both vanishing on S. Then $z = z_1 - z_2$ satisfies the initial conditions $z = 0$ and $\frac{\partial z}{\partial t} = 0$ at $t = 0$. Equation (W.27) then shows that $E(0) = 0$. Conservation of energy implies that $E(t) = 0$ for all t. So $\frac{\partial z}{\partial t} = 0$ for all t, which implies that z is independent of t. Since it is zero at $t = 0$, we deduce that $z = 0$ for all values of t. Thus z_1 and z_2 are equal. It follows that there is at most one solution to the wave equation (W.1) for a given set of initial conditions for z and $\frac{\partial z}{\partial t}$.

It is less easy to prove existence of solutions. For this, we use the eigenvalue method. This will occupy the rest of the appendix.

Eigenvalues are nonnegative and real

We now prove that the eigenvalues of the Laplace operator ∇^2 are nonnegative and real—even if we allow f to take complex values (for real valued functions, ignore the bars in the proof of the lemma).

Lemma W.9. *Let Ω be a closed bounded region. If f is a nonzero (complex valued) twice differentiable function satisfying $\nabla^2 f = -\lambda f$ in Ω and $f = 0$ on the boundary S of Ω, then λ is a nonnegative real number.*

Proof. Let $\bar{f}$ be the complex conjugate of f. Then using Green's first identity (W.4), we have

$$\int_S (\bar{f}\,\nabla f)\,.\,\mathbf{n}\,d\sigma = \int_\Omega \nabla\bar{f}\,.\,\nabla f\,d\mathbf{x} + \int_\Omega \bar{f}(\nabla^2 f)\,d\mathbf{x}$$

$$= \int_\Omega |\nabla f|^2 \, d\mathbf{x} - \lambda \int_\Omega |f|^2 \, d\mathbf{x},$$

Since f is zero on S, the left hand side is zero. Since $\int_\Omega |f|^2 \, d\mathbf{x} > 0$ and $\int_\Omega |\nabla f|^2 \, d\mathbf{x} \geq 0$, this means that

$$\lambda = \frac{\int_\Omega |\nabla f|^2 \, d\mathbf{x}}{\int_\Omega |f|^2 \, d\mathbf{x}} \geq 0$$

so that λ is a nonnegative real number. This expression for λ is called *Rayleigh's quotient.* □

Orthogonality

The relationship between ∇^2 and the inner product for functions on Ω is expressed in the following lemma, which says that ∇^2 is *self-adjoint* with respect to the inner product, for functions vanishing on the boundary.

LEMMA W.10. *For twice continuously differentiable functions f and g on Ω vanishing on the boundary S, we have*

$$\langle f, \nabla^2 g \rangle = \langle \nabla^2 f, g \rangle.$$

PROOF. This follows from Green's second identity (W.5) (replacing f by $\bar{f}$) and the fact that $f(\mathbf{x})$ and $g(\mathbf{x})$ vanish on the boundary S. The left hand side of equation (W.5) is zero, while the right hand side is equal to $\langle f, \nabla^2 g \rangle - \langle \nabla^2 f, g \rangle$. □

This allows us to see easily why the eigenvalues of ∇^2 are real numbers (Lemma W.9). Namely if $\nabla^2 f = -\lambda f$, and $f(\mathbf{x}) = 0$ on the boundary S, then we have

$$\bar{\lambda}\langle f, f \rangle = \langle \lambda f, f \rangle = -\langle \nabla^2 f, f \rangle = -\langle f, \nabla^2 f \rangle = \langle f, \lambda f \rangle = \lambda \langle f, f \rangle.$$

Since $\langle f, f \rangle \neq 0$, we have $\lambda = \bar{\lambda}$. However, positivity is less easy to see from this point of view.

A similar argument shows that eigenfunctions with distinct eigenvalues are orthogonal, as in the following lemma.

LEMMA W.11. *Let f and g be Dirichlet eigenfunctions on Ω with eigenvalues λ and μ respectively. If $\lambda \neq \mu$ Then*

$$\langle f, g \rangle = 0.$$

PROOF. Using the fact that ∇^2 is self-adjoint (see Lemma W.10), we have

$$\lambda \langle f, g \rangle = \langle \nabla^2 f, g \rangle = \langle f, \nabla^2 g \rangle = \mu \langle f, g \rangle,$$

and so $(\lambda - \mu)\langle f, g \rangle = 0$. If $\lambda \neq \mu$, it follows that $\langle f, g \rangle = 0$. □

Inverting ∇^2

The key to understanding the eigenvalues and eigenfunctions of ∇^2 is to find an inverse $\mathbf{K}$ for the operator ∇^2 using Green's functions. The inverse is an integral operator with a wider domain of definition, and whose eigenvalues are the reciprocals of those for ∇^2. The operator $\mathbf{K}$ is an example of a *compact operator*, which is what makes the eigenvalue theory easier.

The construction of the inverse goes as follows. If $f(\mathbf{x})$ satisfies

$$\nabla^2 f(\mathbf{x}) = -\lambda f(\mathbf{x}) \tag{W.28}$$

on Ω and $f(\mathbf{x}) = 0$ on S, then we have

$$\begin{aligned} f(\mathbf{x}) &= \int_\Omega f(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}')\,d\mathbf{x}' = \int_\Omega f(\mathbf{x}')\nabla^2 G(\mathbf{x},\mathbf{x}')\,d\mathbf{x}' \\ &= \int_\Omega G(\mathbf{x},\mathbf{x}')\nabla^2 f(\mathbf{x}')\,d\mathbf{x}' = -\lambda\int_\Omega f(\mathbf{x}')G(\mathbf{x},\mathbf{x}')\,d\mathbf{x}'. \end{aligned}$$

In particular, $f(\mathbf{x}) \neq 0$ implies $\lambda \neq 0$, so zero is not an eigenvalue of ∇^2.

We write $\mathbf{K}$ for the operator defined by

$$\mathbf{K}f(\mathbf{x}) = -\int_\Omega f(\mathbf{x}')G(\mathbf{x},\mathbf{x}')\,d\mathbf{x}'.$$

Then the above calculation shows that if $f(\mathbf{x})$ satisfies (W.28) then

$$\mathbf{K}f(\mathbf{x}) = \frac{1}{\lambda}f(\mathbf{x}).$$

So $f(\mathbf{x})$ is an eigenfunction of $\mathbf{K}$ with eigenvalue $1/\lambda$. Conversely, if $f(\mathbf{x})$ is an eigenfunction of $\mathbf{K}$ with *nonzero* eigenvalue μ, and f is twice continuously differentiable, then $f(\mathbf{x})$ is also an eigenfunction of ∇^2 with eigenvalue $\lambda = 1/\mu$.

Compact operators

Let V be a Hilbert space. We say that a sequence of elements $x_1, x_2, \dots$ of elements of V is *bounded* if there is some positive constant M such that all the x_i satisfy $|x_i| \leq M$. A continuous operator $\mathbf{K}$ on V is said to be *compact* if, given any bounded sequence $x_1, x_2, \dots$, the images $\mathbf{K}x_1, \mathbf{K}x_2, \dots$ has a convergent subsequence.

Example. If the image of $\mathbf{K}$ is finite dimensional then the Bolzano–Weierstrass theorem implies that $\mathbf{K}$ is compact. More generally, the Fredholm alternative can be expressed in terms of compact operators.

If $\mathbf{K}$ is compact and self-adjoint then there is an upper bound to the values of $\langle \mathbf{K}x, x\rangle$ as x runs over the elements of V satisfying $|x| = 1$. This is because otherwise, there would be a sequence $x_1, x_2, \dots$ such that $\langle \mathbf{K}x_i, x_i\rangle > i$, and then by Schwartz' lemma, $\langle \mathbf{K}x_i, \mathbf{K}x_i\rangle > i^2$, so that there could not exist a convergent subsequence; this would contradict the fact that $\mathbf{K}$ is compact. Writing U for the least upper bound of the values for $\langle \mathbf{K}x, x\rangle$ for $|x| = 1$, we can find a sequence $x_1, x_2, \dots$ of elements with $|x_i| = 1$, such that $\langle \mathbf{K}x_1, x_1\rangle, \langle \mathbf{K}x_2, x_2\rangle, \dots$ converges to U. Using Schwartz' lemma again, we have

$$\begin{aligned}\langle \mathbf{K}x_i - Ux_i, \mathbf{K}x_i - Ux_i\rangle &= \langle \mathbf{K}x_i, \mathbf{K}x_i\rangle - 2U\langle \mathbf{K}x_i, x_i\rangle + U^2\\ &\le \langle \mathbf{K}x_i, x_i\rangle^2 - 2U\langle \mathbf{K}x_i, x_i\rangle + U^2\\ &\le 2U^2 - 2U\langle \mathbf{K}x_i, x_i\rangle\\ &= 2U(U - \langle \mathbf{K}x_i, x_i\rangle) \to 0 \quad \text{as} \quad i \to \infty,\end{aligned}$$

and so $\mathbf{K}x_i - Ux_i \to 0$ as $i \to \infty$.

Since $\mathbf{K}$ is compact, we can replace $x_1, x_2, \dots$ by a subsequence with the property that $\mathbf{K}x_1, \mathbf{K}x_2, \dots$ converges. So $Ux_1, Ux_2, \dots$ converges, and provided $U \neq 0$, this implies that $x_1, x_2, \dots$ also converges. Setting $x = \lim_{i\to\infty} x_i$, the continuity of $\mathbf{K}$ implies that $\mathbf{K}x = \lim_{i\to\infty} \mathbf{K}x_i$, so we have

$$\mathbf{K}x = Ux.$$

In other words, x is an eigenvector of $\mathbf{K}$ with eigenvalue U. So if $U \neq 0$ then U is an eigenvalue of $\mathbf{K}$.

Eigenvalue stripping

In the last section, we saw a method for finding an eigenvalue and eigenvector for $\mathbf{K}$. Suppose that we have already found some eigenvalues $\mu_1, \dots, \mu_n$ and corresponding eigenvectors $\psi_1, \dots, \psi_n$ of $\mathbf{K}$, and we wish to find some more. The most convenient method is to form a new operator $\mathbf{K}_n$ whose eigenvalues and eigenvectors are the same as $\mathbf{K}$ except for the removal of the ones we have found. As a preliminary step, we make sure that if there are repeated eigenvalues, then the corresponding eigenvectors are orthogonal. This can be done using the Gram–Schmidt process of linear algebra. Then we define

$$K_n(\mathbf{x}, \mathbf{x}') = K(\mathbf{x}, \mathbf{x}') - \sum_{i=1}^{n} \frac{\psi_i(\mathbf{x})\overline{\psi_i(\mathbf{x}')}}{\mu_i}.$$

Then we define $\mathbf{K}_n$ by

$$\mathbf{K}_n\psi = \int_\Omega K_n(\mathbf{x}, \mathbf{x}')\psi(\mathbf{x}')\, d\mathbf{x}',$$

so that $\mathbf{K}_n$ takes value zero on $\psi_1, \dots, \psi_n$, and takes the same value as $\mathbf{K}$ on any function orthogonal to $\psi_1, \dots, \psi_n$.

To be continued...

Bibliography

1. Pierre-Yves Asselin, *Musique et tempérament*, Éditions Costallat Paris, 1985. 236 pages, out of print. ISBN 2905335009.

   This French book starts with a few pages of explanation of harmonics, intervals and beats. The rest of the book describes various scales and temperaments, and gives instructions for how to tune them. The last chapter gives historical examples of pieces intended for various temperaments.

2. John Backus, *The acoustical foundations of music*, W. W. Norton & Co., 1969. Reprinted 1977. 384 pages, in print. ISBN 0393090965.

   This book gives a non-technical discussion of the physical basis for acoustics, the ear, and the production of sound in musical instruments. Very readable.

3. Denis Baggi, *Readings in computer generated music*, IEEE Computer Society Press Tutorial, 1992. 248 pages, in print. ISBN 0818627476.

4. J. Murray Barbour, *Tuning and temperament, a historical survey*, Michigan State College Press, E. Lansing, 1951. Reprinted by Da Capo Press, New York, 1972. 228 pages, out of print. ISBN 0306704226.

   This book is a standard source on scales and temeraments, and their history. It compares and contrasts Pythagorean tuning, just intonation, meantone, irregular temperaments, and finally equal temperament. Barbour displays a strong predisposition towards twelve tone equal temperament in this work, and interprets the history of scales and temperaments as an inexorable march towards equal temperament.

5. Georg von Békésy, *Experiments in hearing*, McGraw-Hill, New York/Toronto/London, 1960. 745 pages, in print.

   Von Békésy is responsible for the classical experiments in the functioning of the cochlea, described in this book.

6. Arthur H. Benade, *Fundamentals of musical acoustics*, Oxford Univ. Press, 1976. Reprinted by Dover, 1990. 596 pages, in print. ISBN 048626484X.

   Arthur Benade (1925–1987) made numerous contributions to the physics of music. This classic book mostly concerns the physics of musical instruments as well as the human voice.

7. Richard E. Berg and David G. Stork, *The physics of sound*, Prentice-Hall, 1982. Second edition, 1995. 416 pages, in print. ISBN 0131830473.

   A nicely presented textbook on elementary acoustics, musical instruments, and the human ear and voice.

8. Easley Blackwood, *The structure of recognizable diatonic tunings*, Princeton University Press, 1985. 318 pages, out of print. ISBN 0691091293.

   This book discusses various just, meantone and equal temperaments, and tries a little too hard to be mathematical about it. Example: THEOREM 16. *The number of greater* $(j+1)$*ths occurring in the diatonic scale is* h *where* $5j \equiv h \pmod 7$ *and* $0 \le h \le 6$.

9. Richard Charles Boulanger (ed.), *The CSound book: perspectives in software synthesis, sound design, signal procesing, and programming*, MIT Press, 2000. 782 pages, in

print. ISBN 0262522616.

CSound is a multiplatform *free* software synthesis program. It's hard to use at first, but is the most powerful thing around. Almost every synthesis technique you've ever heard of is implemented in a very flexible fashion. The first version came out in 1985, and it has been developing steadily since. This book contains separate articles by many authors, so there is something of a lack of overall coherence to the work. It comes with 2 CD-ROMs containing software for Mac, Linux and PC, hundreds of musical compositions, more than 3000 working instruments, and much more. There is a third CD-ROM available separately, called "The Csound Catalog with Audio," available from http://www.csounds.com. This CD-ROM contains over 2000 orchestra and score text files, and the corresponding audio files in mp3 format. It is also possible to order separately an updated version of the 2 CD-ROMs that came with the book, from the same web site, whether or not you own the book.

10. Pierre Buser and Michel Imbert, *Audition*, MIT Press, 1992. 406 pages, in print. ISBN 0262023318.

11. Murray Campbell and Clive Greated, *The musician's guide to acoustics*, OUP, 1986, reprinted 1998. 613 pages, in print. ISBN 0198165056.

    A well written account of acoustics for the musician, requiring essentially no mathematical background. This book is in print in the UK but not the USA, so try for example www.amazon.co.uk.

12. Peter Castine, *Set theory objects: abstractions for computer-aided analysis and composition of serial and atonal music*, European University Studies, vol. 36, Peter Lang Publishing, 1994. In print. ISBN 3631478976.

13. John Chowning and David Bristow, *FM Theory and applications*, Yamaha Music Foundation, 1986. 195 pages, out of print. ISBN 4636174828.

    This short book came out a couple of years after the Yamaha DX7 became available. It describes FM synthesis using the DX7 for the details of the examples. Note that the graphs for the Bessel functions J_{10} and J_{11} on page 176 have apparently been accidentally interchanged.

14. David Colton, *Partial differential equations, an introduction*, Random House, 1988. 308 pages. ISBN 0394358279.

    This book contains a good treatment of the solution of the wave equation, complete with the background from functional analysis necessary for the proof. The existence of a complete set of eigenfunctions can be found on page 233. A C^2 boundary is assumed, but only in order to solve Laplace's equation with logarithmic boundary conditions, for the construction of Green's functions.

15. Perry R. Cook (ed.), *Music, cognition, and computerized sound. An introduction to psychoacoustics*, MIT Press, 1999. 392 pages, in print. ISBN 0262032562.

    This is an excellent collection of essays on various aspects of psychoacoustics, written by some of the leading figures in the area of computer music. It comes with a CD full of sound examples.

    Chapter headings: 1. Max Mathews, *The ear and how it works.* 2. Max Mathews, *The auditory brain.* 3. Roger Shepard, *Cognitive psychology and music.* 4. John Pierce, *Sound waves and sine waves.* 5. John Pierce, *Introduction to pitch perception.* 6. Max Mathews, *What is loudness?* 7. Max Mathews, *Introduction to timbre.* 8. John Pierce, *Hearing in time and space.* 9. Perry R. Cook, *Voice physics and neurology.* 10. Roger Shepard, *Stream segregation and ambiguity in audition.* 11. Perry R. Cook, *Formant peaks and spectral valleys.* 12. Perry R. Cook, *Articulation in speech and sound.* 13. Roger Shepard, *Pitch perception and measurement.* 14. John Pierce, *Consonance and scales.* 15. Roger Shepard, *Tonal structure and scales.* 16. Perry R. Cook, *Pitch, periodicity, and noise in the voice.* 17. Daniel J. Levitin, *Memory for musical attributes.*

18. Brent Gillespie, *Haptics*. 19. Brent Gillespie, *Haptics in manipulation*. 20. John Chowning, *Perceptual fusion and auditory perspective*. 21. John Pierce, *Passive non-linearities in acoustics*. 22. John Pierce, *Storage and reproduction of music*. 23. Daniel J. Levitin, *Experimental design in psychoacoustic research*.

16. Deryck Cooke, *The language of music*, Oxford Univ. Press, 1959, reprinted in paperback, 1990. 289 pages, in print. ISBN 0198161808

    This wonderful little book explains how the basic elements of musical expression communicate emotional content, both locally and on a larger scale. Highly recommended to anyone trying to understand how music works. Deryck Cooke is the person who orchestrated Mahler's tenth symphony, starting with Mahler's original draft. Take a listen to the excellent Bournemouth Symphony/Simon Rattle recording.

17. David H. Cope, *New directions in music*, Wm. C. Brown Publishers, Dubuque, Iowa, Fifth edition, 1989. Sixth edition, Waveland Press, 1998. 439 pages, in print. ISBN 0697033422.

    An introduction to computers and the avant-garde in twentieth century music. Reads a bit like a scrapbook of ideas, pictures and music.

18. ______, *Computers and musical style*, Oxford University Press, 1991. 246 pages, in print. ISBN 019816274X.

    David Cope is well known for his attempts to induce computers to compose music in the style of various famous composers such as Bach and Mozart. Unsurprisingly, the compositions are not an unqualified success, but the account of the process presented in this book is interesting.

19. ______, *Experiments in musical intelligence*, Computer Music and Digital Audio, vol. 12, A-R Editions, Madison, Wisconsin, 1996. 263 pages, in print. ISBN 0895793148/0895793377.

    This book is a continuation of the project described in Cope's 1991 book, and comes with a CD-ROM full of examples for the Macintosh platform. I have not seen a copy, but from the review in Computer Music Journal 21 (3) (1997), it seems that the subject has progressed a good deal since [**18**] appeared in 1991. Artificial intelligence is still in a very primitive stage of development, and it will probably take another generation to produce a computational model which convincingly simulates one of the great composers. And then another generation after that, to compose with real originality. I think the real core of the problem is that when a human being composes, a hugely complex world view is invoked, which has taken a lifetime to accumulate. We'll end up teaching a baby computer how to talk before it grows up to be a real composer! But I'm glad that someone of the calibre of Cope is battling with these problems.

20. Lothar Cremer, *The physics of the violin*, MIT Press, 1984. 450 pages, in print. ISBN 0262031027.

    Translation of *Physik der Geige*, S. Hirzel Verlag, Stuttgart, 1981. This book is the standard reference on the physics of the violin. The technical standard is high and the writing is clear. Strongly recommended.

21. Alain Daniélou, *Music and the power of sound*, Inner Traditions, Rochester, Vermont, 1995, revised from 1943 publication under different title. 172 pages, in print. ISBN 0892813369.

    This is a book about tuning and scales in different cultures, especially Chinese, Indian and Greek, and their effect on the emotional content of music.

22. ______, *Introduction to the study of musical scales*, Munshiram Manoharlal Publishers Pvt. Ltd., New Delhi, 1999. Originally published by the India Society, London in 1943. 279 pages, in print. ISBN 8121509203.

    A companion to the above book by the same author.

23. Peter Desain and Henkjan Honig, *Music, mind and machine: Studies in computer music, music cognition, and artificial intelligence (Kennistechnologie)*, Thesis Publishers, 1992. 330 pages, in print. ISBN 9051701497.

24. Diana Deutsch (ed.), *The psychology of music*, Academic Press, 1982; 2nd ed., 1999. 807 pages, in print. ISBN 0122135652 (pbk), 0122135644 (hbk).

    This is an excellent collection of essays on various aspects of the psychology of music, by some of the leading figures in the field. The second edition has been completely revised to reflect recent progress in the subject. It is interesting to compare this collection of essays with Perry Cook's [**15**], which have a slightly different purpose.

    Chapter headings: 1. John R. Pierce, *The nature of musical sound.* 2. Manfred R. Schroeder, *Concert halls: from magic to number theory.* 3. Norman M. Weinberger, *Music and the auditory system.* 4. Rudolf Rasch and Reinier Plomp, *The perception of musical tones.* 5. Jean-Claude Risset and David L. Wessel, *Exploration of timbre by analysis and synthesis.* 6. Johan Sundberg, *The perception of singing.* 7. Edward M. Burns, *Intervals, scales and tuning.* 8. W. Dixon Ward, *Absolute pitch.* 9. Diana Deutsch, *Grouping mechanisms in music.* 10. Diana Deutsch, *The processing of pitch combinations.* 11. Jamshed J. Bharucha, *Neural nets, temporal composites, and tonality.* 12. Eugene Narmour, *Hierarchical expectation and musical style.* 13. Eric F. Clarke, *Rhythm and timing in music.* 14. Alf Gabrielson, *The performance of music.* 15. W. Jay Dowling, *The development of music perception and cognition.* 16. Rosamund Shuter-Dyson, *Musical ability.* 17. Oscar S. M. Marin and David W. Perry, *Neurological aspects of music perception and performance.* 18. Edward C. Carterette and Roger A. Kendall, *Comparative music perception and cognition.*

25. B. Chaitanya Deva, *The music of India: A scientific study*, Munshiram Manoharlal Publishers Pvt. Ltd., 1981. 278 pages, out of print.

26. Dominique Devie, *Le tempérament musical: philosophie, histoire, théorie et practique*, Société de musicologie du Languedoc Béziers, 1990. 540 pages, out of print. ISBN 2905400528.

    This French book is an extensive discussion of scales and temperaments, with a great deal of historical information and philosophical discussion.

27. Charles Dodge and Thomas A. Jerse, *Computer music: synthesis, composition, and performance*, Simon & Schuster, Second ed., 1997. 453 pages, in print. ISBN 0028646827 (pbk), 002873100X (hbk).

28. William C. Elmore and Mark A. Heald, *Physics of waves*, McGraw-Hill, 1969. Reprinted by Dover, 1985. 477 pages, in print. ISBN 0486649261.

    This book contains a useful discussion of waves on strings, rods and membranes.

29. Laurent Fichet, *Les théories scientifiques de la musique aux XIX^{e} et XX^{e} siècles*, Librairie J. Vrin, 1996. 382 pages, in print. ISBN 2711642844.

    This French book may be obtained from www.amazon.fr.

30. Neville H. Fletcher and Thomas D. Rossing, *The physics of musical instruments*, Springer-Verlag, Berlin/New York, 1991. ISBN 3540941517 (pbk), 3540969470 (hbk).

    This book is at a high technical level, and contains a wealth of interesting material. A difficult read, but worth the effort.

31. Allen Forte, *The structure of atonal music*, Yale Univ. Press, 1973. ISBN 0300021208.

    This book is about 12-tone music, and goes into a great deal of technical detail about the theory of pitch class sets, relations and complexes.

32. Steve De Furia and Joe Scacciaferro, *MIDI programmer's handbook*, M & T Publishing, Inc., 1989.

33. Trudi Hammel Garland and Charity Vaughan Kahn, *Math and music: harmonious connections*, Dale Seymore Publications, 1995. ISBN 0866518290.

This book is aimed at high school level, and avoids technical material. It looks as though it would make good classroom material at the intended level, and it seems to be the only book on the market with this aim.

34. H. Genevois and Y. Orlarey, *Musique & mathématiques*, Aléas–Grame, 1997. 194 pages, in print. ISBN 2908016834.

A collection of essays in French on various aspects of the connections between music and mathematics, coming out of the Rencontres Musicales Pluridisciplinaires at Lyons, 1996. This book can be ordered from www.amazon.fr.

35. Ben Gold and Nelson Morgan, *Speech and audio signal processing: processing and perception of speech and music*, Wiley & Sons, 2000. 537 pages, in print. ISBN 0471351547.

The basic purpose of this book is to understand sound well enough to be able to perform speech recognition, but it contains a lot of material relevant to music recognition and synthesis. By some quirk of international pricing, the price of this book in the UK is about half what it is in the USA, so it may be worth your while checking out UK online bookstores such as amazon.co.uk or the UK branch of bol.com for this one.

36. Heinz Götze and Rudolf Wille (eds.), *Musik und Mathematik. Salzburger Musikgespräch 1984 unter Vorsitz von Herbert von Karajan*, Springer-Verlag, Berlin/New York, 1995. ISBN 3540154078.

37. Karl F. Graff, *Wave motion in elastic solids*, Oxford University Press, 1975. Reprinted by Dover, 1991. ISBN 0486667456.

This book contains a lot of information about wave motion in strings, bars and plates, relevant to Chapter 3.

38. Niall Griffith and Peter M. Todd (eds.), *Musical networks: parallel distributed perception and performance*, MIT Press, 1999. 350 pages, in print. ISBN 0262071819.

39. Donald E. Hall, *Musical acoustics*, Wadsworth Publishing Company, Belmont, California, 1980. ISBN 0534007589.

This book has some good chapters on the physics of musical instruments, as well as briefer acounts of room acoustics and of tuning and temperament.

40. R. W. Hamming, *Digital filters*, Prentice Hall, 1989. Reprinted by Dover Publications. 296 pages, in print. ISBN 048665088X

Hamming is one of the pioneers of twentieth century communications and coding theory. This book on digital filters is a classic.

41. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, Oxford University Press, Fifth edition, 1980. 426 pages, in print. ISBN 0198531710.

This classic contains a good section on the theory of continued fractions, which may be used as a reference for the material presented in §6.2.

42. W. M. Hartmann, *Signals, sound and sensation*, Springer-Verlag, Berlin/New York, 1998. 647 pages, in print. ISBN 1563962837

This book contains a very nice discussion of psychoacoustics, Fourier theory and digital signal processing, and the relationships between these subjects.

43. Hermann Helmholtz, *Die Lehre von den Tonempfindungen*, Longmans & Co., Fourth German edition, 1877. Translated by Alexander Ellis as *On the sensations of tone*, Dover, 1954 (and reprinted many times). 576 pages, in print. ISBN 0486607534.

For anyone interested in scales and temperaments, or the origins acoustics and psychoacoustics, this book is an absolute gold mine. The appendices by the translator are also full of fascinating material. Strongly recommended.

44. Michael Hewitt, *The tonal Phoenix; a study of tonal progression through the prime numbers three, five and seven*, Verlag für systematische Musikwissenschaft GmbH, Bonn, 2000. 495 pages, in print. ISBN 3922626963.

    This German book should be available from www.amazon.de, but it doesn't yet seem to be listed.

45. Douglas R. Hofstadter, *Gödel, Escher, Bach*, Harvester Press, 1979. Reprinted by Basic Books, 1999. 777 pages, in print. ISBN 0465026567.

    A nice popularized account of the connections between mathematical logic, cognitive science, Escher's art and the music of J. S. Bach. A bit too longwinded to make a particularly good read.

46. David M. Howard and James Angus, *Acoustics and psychoacoustics*, Focal Press, 1996. 365 pages, in print. ISBN 0240514289.

47. Hua, *Introduction to number theory*, Springer-Verlag, Berlin/New York, 1982. ISBN 3540108181.

    This book contains a good section on continued fractions, which may be used as a supplement to §6.2. Be warned that the continued fraction for π given on page 252 of Hua is erronious. The correct continued fraction can be found here on page 152.

48. Stuart M. Isacoff, *Temperament: The idea that solved music's greatest riddle*, Knopf, 2001. 288 pages, expected to be in print on Nov 13, 2001. ISBN 0375403558.

49. Sir James Jeans, *Science & music*, Cambridge Univ. Press, 1937. Reprinted by Dover, 1968. 273 pages, in print. ISBN 0486619648.

50. Tom Johnson, *Self-similar melodies*, Editions 75, 75 rue de la Roquette, 75011 Paris, 1996. 291 pages, ring-bound, in print. ISBN 2907200011.

    Tom Johnson is a minimalist composer, whose work uses mathematical techniques such as the theory of automata to assist in the compositional process. Copies of this book may be obtained by writing to: Two Eighteen Press, PO Box 218, Village Station, New York, NY 10014, USA.

51. Ian Johnston, *Measured tones: The interplay of physics and music*, Institute of Physics Publishing, Bristol and Philadelphia, 1989. Reprinted 1997. 408 pages, in print. ISBN 0852742363.

    This very readable book is about acoustics and the physics of musical instruments, from a historical perspective, and with essentially no equations.

52. Owen H. Jorgensen, *Tuning*, Michigan State University Press, 1991. 798 pages, large format, out of print. ISBN 0870132903.

    This enormous book is subtitled: "Containing The Perfection of Eighteenth-Century Temperament, The Lost Art of Nineteenth-Century Temperament, and The Science of Equal Temperament, Complete With Instructions for Aural and Electronic Tuning." It is a mixture of history of tunings and temperaments, and explicit tuning instructions for various temperaments. An interesting thread running through the book is a detailed argument to the effect that equal temperament was not commonplace until the twentieth century.

53. Lawrence E. Kinsler, Austin R. Frey, Alan B. Coppens, and James V. Sanders, *Fundamentals of acoustics*, John Wiley & Sons, Fourth edition, 2000. 548 pages, in print. ISBN 0471847895.

    This is an excellent book on acoustics, and deservedly popular. The two original authors of the first (1950) edition were Kinsler and Frey, both now deceased. The book has gone through many print runs and editions. Coppens and Sanders have updated the book and added new material for the fourth edition. This is another book whose price in the UK is about half what it is in the USA, so it may be worth your while checking out UK online bookstores for this one.

54. T. W. Körner, *Fourier analysis*, Cambridge Univ. Press, 1988, reprinted 1990. 591 pages, in print. ISBN 0521389917.

This book makes great reading. There is a fair amount of high level mathematics, but also a number of sections of a more historical or narrative nature, and a wonderful sense of humor pervades the work. The account of the laying of the transatlantic cable in the nineteenth century and the technical problems associated with it is priceless. Several sections are devoted to the life of Fourier. There is also a companion volume entitled *Exercises for Fourier analysis*, ISBN 0521438497, in print.

55. Patricia Kruth and Henry Stobart (eds.), *Sound*, Cambridge Univ. Press, 2000. 235 pages, in print. ISBN 0521572096.

A nice collection of nontechnical essays on the nature of sound. I particularly like Jonathan Ashmore's contribution. Contents: 1. Philip Peek, *Re-sounding Silences.* 2. Charles Taylor, *The Physics of Sound.* 3. Jonathan Ashmore, *Hearing.* 4. Peter Slater, *Sounds Natural: The Song of Birds.* 5. Peter Ladefoged, *The Sounds of Speech.* 6. Christopher Page, *Ancestral Voices.* 7. Brian Ferneyhough, *Shaping Sound.* 8. Steven Feld, *Sound Worlds.* 9. Michel Chion, *Audio-Vision and Sound.*

56. J. Lattard, *Gammes et tempéraments musicaux*, Masson, Paris, 1988. 130 pages, in print. ISBN 2225812187.

Scales and musical temperaments, in French. This book can be obtained from alapage.com.

57. Marc Leman, *Music and schema theory: cognitive foundations of systematic musicology*, Springer Series on Information Science, vol. 31, Springer-Verlag, Berlin/New York, 1995. In print. ISBN 3540600213.

58. ———, *Music, Gestalt, and computing; studies in cognitive and systematic musicology*, Lecture Notes in Computer Science, vol. 1317, Springer-Verlag, Berlin/New York, 1997. 524 pages, in print. ISBN 3540635262.

59. E. Lendvai, *Symmetries of music*, Kodály Institute, Kecskemét, 1993. 155 pages, in print. ISBN 9637295100.

This book seems to be quite hard to get hold of. I suggest going to the Kodály Institute web site at www.kodaly-inst.hu and emailing them.

60. David Lewin, *Generalized musical intervals and transformations*, Yale University Press, New Haven/London, 1987. ISBN 0300034938.

This book discusses twelve tone music from a mathematical point of view, using some elementary group theory.

61. Carl E. Linderholm, *Mathematics made difficult*, Wolfe Publishing, Ltd., London, 1971. 207 pages, out of print.

This book isn't relevant to the subject of the text, but is well worth digging out to pass a happy evening.

62. Mark Lindley and Ronald Turner-Smith, *Mathematical models of musical scales*, Verlag für systematische Musikwissenschaft GmbH, Bonn, 1993. 308 pages, out of print. ISBN 3922626661.

63. Llewelyn S. Lloyd and Hugh Boyle, *Intervals, scales and temperaments*, Macdonald, London, 1963. 246 pages, out of print.

An extensive discussion of just intonation, meantone and equal temperament.

64. Charles Madden, *Fractals in music—Introductory mathematics for musical analysis*, High Art Press, 1999. ISBN 0967172756.

This book has a promising title, but both the mathematics and the musical examples could do with some improvement. There is certainly an interesting area here to be investigated, and maybe the real point of the book will be to make us more aware of the possibilities.

65. Max V. Mathews, *The technology of computer music*, MIT Press, 1969. 188 pages, out of print. ISBN 0262130505.

    This book appeared early in the game, and was at one stage a standard reference. Although much of the material is now outdated, it is still worth looking at for its description of the Music V language.

66. Max V. Mathews and John R. Pierce, *Current directions in computer music research*, MIT Press, 1989. Reprinted 1991. 432 pages, in print. ISBN 0262132419.

    A nice collection of articles on computer music, including an article by Pierce describing the Bohlen–Pierce scale. There is a companion CD, see Appendix R.

67. W. A. Mathieu, *Harmonic experience*, Inner Traditions International, Rochester, Vermont, 1997. 563 pages, large format, in print. ISBN 0892815604.

    You would not guess it from the title, but this book is about the conceptual transition from just intonation to equal temperament, and the parallel development of harmonic vocabulary. The writing is down to earth and easy to understand.

68. Guerino Mazzola, *Gruppen und Kategorien in der Musik*, Heldermann-Verlag, Berlin, 1985. Out of print.

69. ______, *Geometrie der Töne: Elemente der Mathematischen Musiktheorie*, Birkhäuser, 1990. ISBN 3764323531. 364 pages, in print.

    Geometry of tones: elements of mathematical music theory. This is a book in German about music and mathematics, almost completely disjoint in content from these course notes. The author was a graduate student under the direction of the mathematician Peter Gabriel in Zürich, and the influence is clear. I was rather surprised, for example, to see the appearance of Yoneda's lemma from category theory. This book can be ordered from www.amazon.de.

70. Ernest G. McClain, *The myth of invariance: The origin of the gods, mathematics and music from the Ṛg Veda to Plato*, Nicolas-Hays, Inc., York Beach, Maine, 1976. Paperback edition, 1984. 216 pages, in print. ISBN 0892540125.

    A strange mixture of mysticism and theory of scales and temperaments.

71. Brian C. J. Moore, *Psychology of hearing*, Academic Press, 1997. ISBN 0125056273.

    A standard work on psychoacoustics. Highly recommended.

72. F. Richard Moore, *Elements of computer music*, Prentice Hall, 1990. 560 pages, out of print. ISBN 0132525526.

    A very readable work by an expert in the field. The book is written in terms of the computer music language CMusic, which was a precursor of CSound.

73. Joseph Morgan, *The physical basis of musical sounds*, Robert E. Krieger Publishing Company, Huntington, New York, 1980. 145 pages, in print. ISBN 0882756567.

74. Philip M. Morse and K. Uno Ingard, *Theoretical acoustics*, McGraw Hill, 1968. Reprinted with corrections by Princeton University Press, 1986, ISBN 0691084254 (hbk), 0691024014 (pbk).

    This book is the best textbook on acoustics that I have found, for an audience with a good mathematical background.

75. Bernard Mulgrew, Peter Grant, and John Thompson, *Digital signal processing*, Macmillan Press, 1999. 356 pages, in print. ISBN 0333745310.

    A number of books have recently appeared on the subject of digital signal processing. This is a good readable one.

76. Cornelius Johannes Nederveen, *Acoustical aspects of woodwind instruments*, Northern Illinois Press, 1998. ISBN 0875805779.

77. Erich Neuwirth, *Musical temperaments*, Springer-Verlag, Berlin/New York, 1997. 70 pages, in print. ISBN 3211830405.

This very slim, overpriced volume explains the basics of scales and temperaments. It comes with a CD-ROM full of examples to go with the text.

78. Harry F. Olson, *Musical engineering*, McGraw Hill, 1952. Revised and enlarged version, Dover, 1967, with new title: *Music, physics and engineering*. ISBN 0486217698.

This work was a classic in its time, although it is now somewhat outdated.

79. Charles A. Padgham, *The well-tempered organ*, Positif Press, Oxford, 1986. ISBN 0906894131.

This book is hard to get hold of, but has a wealth of information about the usage of temperaments in organs.

80. Harry Partch, *Genesis of a music*, Second edition, enlarged. Da Capo Press, New York, 1974 (hbk), 1979 (pbk). 518 pages, in print. ISBN 030680106X.

Harry Partch is one of the twentieth century's most innovative experimental composers. This well written book explains the origins of his 43 tone scale, and its applications in his compositions, and puts it into historical context with some unusual insights. The book also contains descriptions and photos of many musical instruments invented and constructed by Partch using this scale.

81. George Perle, *Twelve-tone tonality*, University of California Press, 1977. Second edition, 1996. 256 pages, in print. ISBN 0520033876.

82. Hermann Pfrogner, *Lebendige Tonwelt*, Langen Müller, 1976. 680 pages, out of print. ISBN 3784415776.

Contains a discussion of musical scales in India, China, Greece and Arabia, followed by a discussion of the development of western tonality, and then a third section on the music of Arnold Schönberg.

83. James O. Pickles, *An introduction to the physiology of hearing*, Academic Press, London/San Diego, second edition, 1988. Out of print. ISBN 0125547544 (pbk).

84. John Robinson Pierce, *The science of musical sound*, Scientific American Books, 1983; 2nd ed., W. H. Freeman & Co, 1992. 270 pages, in print. ISBN 0716760053.

A classic by an expert in the field. Well worth reading. The second edition has been updated and expanded.

85. Ken C. Pohlmann, *Principals of digital audio*, McGraw-Hill, fourth edition, 2000. 736 pages, in print. ISBN 0071348190.

This is a standard work on digital audio. The fourth edition has been brought completely up to date, with sections on the newest technologies.

86. Stephen Travis Pope (ed.), *The well-tempered object: Musical applications of object-oriented software technology*, MIT Press, 1991. ISBN 0262161265.

An edited collection of articles from the Computer Music Journal on applications of object oriented programming to music technology.

87. Daniel R. Raichel, *The science and applications of acoustics*, Amer. Inst. of Physics, 2000. 598 pages, in print. ISBN 0387989072.

A general interdisciplinary textbook on modern acoustics, containing a discussion of musical instruments, as well as music and voice synthesis, and psychoacoustics.

88. Jean-Philippe Rameau, *Traité de l'harmonie*, Ballard, Paris, 1722. Reprinted as "Treatise on Harmony" in English translation by Dover, 1971. 444 pages, in print. ISBN 0486224619.

89. J. W. S. Rayleigh, *The theory of sound (2 vols)*, Second edition, Macmillan, 1896. Dover, 1945. 480/504 pages, in print. ISBN 0486602923/0486602931.

This book revolutionized the field when it came out. It is now mostly of historical interest, because the subject has advanced a great deal during the twentieth century.

90. Geza Révész, *Einführung in die Musikpsychologie*, Amsterdam, 1946. Translated by G. I. C. de Courcy as *Introduction to the psychology of music*, University of Oklahoma Press, 1954, and reprinted by Dover, 2001. 265 pages, in print. ISBN 048641678X.

    This book contains an interesting discussion (pages 160–167) of the question of whether mathematicians are more musically gifted than exponents of other special branches and professions. The author gives evidence for a negative answer to this question, in sharp contrast with widely held views on the subject.

91. John S. Rigden, *Physics and the sound of music*, Wiley & Sons, 1977. 286 pages. ISBN 0471024333. Second edition, 1985. 368 pages, in print. ISBN 0471874124.

92. Curtis Roads, *The music machine. Selected readings from Computer Music Journal*, MIT Press, 1989. 725 pages. ISBN 0262680785.

93. ———, *The computer music tutorial*, MIT Press, 1996. 1234 pages, large format, in print. ISBN 0262181584 (hbk), 0262680823 (pbk).

    This is a huge work by a renowned expert. It contains an excellent section on various methods of synthesis, but surprisingly, doesn't go far enough with technical aspects of the subject.

94. Curtis Roads, Stephen Travis Pope, Aldo Piccialli, and Giovanni De Poli (eds.), *Musical signal processing*, Swets & Zeitlinger Publishers, 1997. 477 pages, in print. ISBN 9026514824 (hbk), 9026514832 (pbk).

    A collection of articles by various authors, in four sections: I, Foundations of musical signal processing. II, Innovations in musical signal processing. III, Musical signal macrostructures. IV, Composition and musical signal processing.

95. Curtis Roads and John Strawn (eds.), *Foundations of computer music. Selected readings from Computer Music Journal*, MIT Press, 1985. ISBN 0262181142 (hbk), 0262680513 (pbk).

96. Juan G. Roederer, *The physics and psychophysics of music*, Springer-Verlag, Berlin/New York, 1995. 219 pages, in print. ISBN 3540943668.

97. Thomas D. Rossing, *The science of sound*, Addison-Wesley, Reading, Mass., Second edition, 1990. 686 pages, in print. ISBN 0201157276.

98. ———, *Science of percussion instruments*, World Scientific, 2000. 208 pages, in print. ISBN 9810241585 (Hbk), 9810241593 (Pbk).

99. Thomas D. Rossing and Neville H. Fletcher (contributor), *Principles of vibration and sound*, Springer-Verlag, Berlin/New York, 1995. 247 pages, in print. ISBN 0387943048.

100. Joseph Rothstein, *MIDI, A comprehensive introduction*, Oxford Univ. Press, 1992. 226 pages, in print. ISBN 0198162936

    Rothstein is one of the editors of the Computer Music Journal.

101. Heiner Ruland, *Expanding tonal awareness*, Rudolf Steiner Press, London, 1992. 187 pages, out of print. ISBN 1855841703.

    A somewhat ideosynchratic account of the history of scales and temperaments.

102. Joseph Schillinger, *The Schillinger system of musical composition, I*, Carl Fischer, Inc, 1941. 878 pages, out of print. ISBN 0306775522.

    Schillinger uses many mathematical concepts in describing his theory of musical composition.

103. Albrecht Schneider, *Tonhöhe, Skala, Klang: Akustiche, tonometrische und psychoakustische Studien auf vergleichender Grundlage*, Verlag für systematische Musikwissenschaft, Bonn, 1997. 597 pages, in print. ISBN 3922626890.

Pitch, scale, timbre: Acoustic, tonometric and psychoacoustic studies on comparative foundations. This German book can be ordered from www.amazon.de.

104. Günter Schnitzler, *Musik und Zahl*, Verlag für systematische Musikwissenschaft, 1976. 297 pages, out of print.

A collection of essays on music and mathematics, in German, by a number of different authors.

105. William A. Sethares, *Tuning, timbre, spectrum, scale*, Springer-Verlag, Berlin/New York, 1998. ISBN 354076173X.

The basic thesis of this book is the idea, first put forward by John Pierce, that the harmonic spectrum or timbre of an instrument determines the most appropriate scales and temperaments to be used, and therefore we could start with a prescribed scale and design a harmonic spectrum for which it would be appropriate. The book comes with a CD of examples.

106. Ken Steiglitz, *A digital signal processing primer: with applications to digital audio and computer music*, Addison-Wesley, 1996. 314 pages, in print. ISBN 0805316841.

107. Reinhard Steinberg (ed.), *Music and the mind machine. The psychophysiology and psychopathology of the sense of music*, Springer-Verlag, Berlin/New York, 1995. Out of print. ISBN 3540585281.

108. Charles Taylor, *Exploring music: The science and technology of tones and tunes*, Institute of Physics Publishing, Bristol and Philadelphia, 1992. Reprinted 1994. 255 pages, in print. ISBN 0750302135.

109. Stan Tempelaars, *Signal processing, speech and music*, Swets & Zeitlinger Publishers, 1996. 360 pages, in print. ISBN 9026514816.

110. Martin Vogel, *Die Lehre von den Tonbeziehungen*, Verlag für systematische Musikwissenschaft, 1975. 480 pages, in print.

This German book is one of the standard works on scales and temperaments. It can be ordered from www.amazon.de.

111. G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge Univ. Press, 1922. Reprinted, 1996. 804 pages, in print. ISBN 0521483913.

This encyclopedic tome contains more than you will ever want to know about Bessel functions. After all these years, it is still the standard work on the subject.

112. Scott R. Wilkinson, *Tuning in: Microtonality in electronic music*, Hal Leonard Books, Milwaukee, 1988. ISBN 0881886335.

This is a nice short popularized account of the theory of scales and temperaments, designed for synthesists. Beware, though, that the tables have some inaccuracies. This applies especially to the values in cents given in the tables for the Werckmeister III and Vallotti-Young temperaments.

113. Fritz Winckel, *Music, sound and sensation, a modern exposition*, Dover, 1967. 189 pages, in print. ISBN 0486217647.

114. Iannis Xenakis, *Formalized music: Thought and mathematics in composition*, Indiana University Press, Bloomington/London, 1971. ISBN 0253323789. Pendragon revised edition (four new chapters and a new appendix), Pendragon Press, New York, 1992. 387 pages, in print. ISBN 0945193246.

Xenakis is one of the twentieth century's leading composers and theorists of music using aleatoric, or stochastic processes. This book is about the theory behind these processes. It is hard to read, and the mathematics is suspect in places. Nonetheless, the book is full of interesting ideas.

115. Joseph Yasser, *A theory of evolving tonality*, American Library of Musicology, Inc., 1932. 381 pages, out of print

This book is a classic. Somebody should reprint it.

116. William A. Yost, *Fundamentals of hearing. An introduction*, Academic Press, San Diego, 1977. 326 pages, in print. ISBN 012772690X.

117. Eberhard Zwicker and H. Fastl, *Psychoacoustics: facts and models*, Springer-Verlag, Berlin/New York, Second edition, 1999. 380 pages, in print. ISBN 3540650636.

Eberhard Zwicker was one of the great names in psychoacoustics. This book, written with his student Hugo Fastl, is an excellent introduction to psychoacoustics, and presents a modern account with a great deal of qualitative and quantitative information. The second edition has been updated by Fastl.

Mobile instrument, Arthur Frick

Index