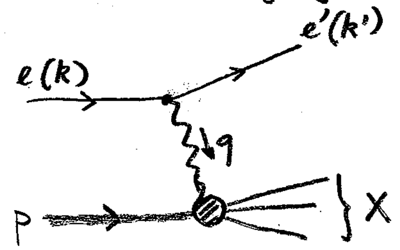
DIS $e^\pm p \to e^\pm X$

<u>idea</u>: probe the proton structure

1) kinematics: exchange of a virtual photon



$$\begin{cases} s = (p+k)^2 & \text{energy}^2 \text{ of the } ep \text{ system} \\ W^2 = (p+q)^2 & \text{" " " } \gamma^* p \text{ "} \\ Q^2 = -q^2 > 0 & \text{photon virtuality} \end{cases}$$

<u>Other useful variables</u>:
- $\nu = p.q$ neglecting the p. mass, $W^2 = 2\nu - Q^2$
- $y = \frac{p.q}{p.k} = \frac{W^2+Q^2}{s} = \frac{2\nu}{s}$
- $x = \frac{Q^2}{2\nu} \Rightarrow Q^2 = sxy$

<u>Experimentally</u>: 2 degrees of freedom (integrating over the electron azimutal angle)
↳ the energy & scattering angle of the outgoing electron

$$\begin{cases} p = (0,0,P,P) \\ k = (0,0,-E,E) \\ k' = (E'\sin(\theta), 0, -E'\cos(\theta), E') \end{cases} \Rightarrow \begin{cases} s = 4PE \\ Q^2 = 4EE'\cos^2(\theta/2) \\ \nu = 2P\,(E - E'\sin^2(\theta/2)) \\ x = \dfrac{EE'\cos^2(\theta/2)}{P(E-E'\sin^2(\theta/2))} \quad y = 1 - \dfrac{E'}{E}\sin^2(\theta/2) \end{cases}$$

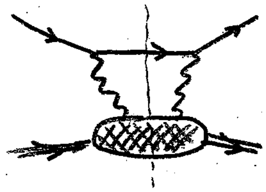
$$\Rightarrow q = (-E's_\theta, 0, -E + E'c_\theta, E - E')$$

<u>Usual choice</u>: x, Q^2.

At large Q^2, the photon acts like a "microscope" probing the proton at small distances $\sim 1/Q$

⇝ Bjorken limit: large Q^2, fixed x

2) proton structure functions



$$|\mathcal{M}|^2 = L_{\mu\nu} \cdot W^{\mu\nu}$$

$$L_{\mu\nu} = e^2 \mathrm{tr}\left(\not k \gamma_\mu \not k' \gamma_\nu\right) = 4e^2\left(k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu}\, k.k'\right)$$

$$W_{\mu\nu} = \frac{1}{4\pi}\sum_X \langle P|j_\nu^\dagger(0)|X\rangle\langle X|j_\mu(0)|P\rangle\,(2\pi)^4\delta^{(4)}(p+q-p_X)$$

$$= \frac{1}{4\pi}\sum_X \int d^4z\; e^{i(p+q-p_X).z}\langle P|j_\nu^\dagger(0)|X\rangle\langle X|j_\mu(0)|P\rangle$$

$$= \frac{1}{4\pi}\int d^4z\; e^{iq.z}\langle P|j_\nu^\dagger(z)\,j_\mu(0)|P\rangle$$

$$= \frac{1}{4\pi}\int d^4z\; e^{iq.z}\langle P|[j_\nu^\dagger(z), j_\mu(0)]|P\rangle$$

In general, $W^{\mu\nu}$ can be parametrised as $a g^{\mu\nu} + b p^\mu p^\nu + c p^\mu q^\nu + d p^\nu q^\mu + e q^\mu q^\nu$ with $a, \dots, e$ depending on x and Q^2

From gauge invariance, we need $q_\mu W^{\mu\nu} = q_\nu W^{\mu\nu} = 0$. This allows to write

$$W^{\mu\nu} = -\left(g^{\mu\nu} + \frac{q^\mu q^\nu}{Q^2}\right)W_1(x,Q^2) + \left(p^\mu + \frac{q^\mu}{2x}\right)\left(p^\nu + \frac{q^\nu}{2x}\right)W_2(x,Q^2)$$

Instead of $W_{1,2}$, one usually introduces

$$F_1 = W_1 \qquad \text{(all dimensionless)}$$
$$F_2 = \nu W_2$$
$$F_L = F_2 - 2x F_1$$

which are known as the <u>proton structure functions</u>

<u>Notes</u>:
- we can also have Z exchange
- We can do DIS with (anti-)neutrino: $\overset{(-)}{\nu} p \to e^\pm X$ with a $W^\pm$ exchange or having $e^\pm p \to \overset{(-)}{\nu} X$

  This introduces a 3rd struct. function (F_3) with opposite sign for ν & $\bar\nu$.

$\Rightarrow$ Neutral currents: F_1^{ep}, F_2^{ep}

Charged currents: $F_i^{\nu}, F_i^{\bar\nu}$ $\quad i = 1, 2, 3$

3) Parton model

* Go in a frame where the proton is highly boosted ($P \gg$)

$$p^\mu \equiv (0, 0, P, P)$$

$$n^\mu \equiv \left(0, 0, \frac{-1}{2P}, \frac{1}{2P}\right) \qquad (n \cdot p = 1)$$

and

$$q^\mu \equiv q_\perp^\mu + \nu n^\mu \qquad (n \cdot q = 0)$$

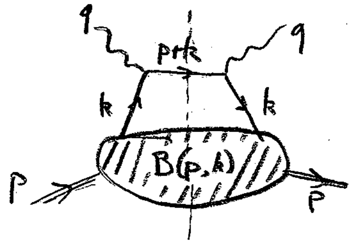
with $(q_\perp^\mu)^2 = -\vec{q}_\perp^{\,2} = -Q^2$

Then: $$p^\mu p^\nu W_{\mu\nu} = -\frac{\nu^2}{Q^2} W_1 + \frac{\nu^2}{4x^2} W_2 = \frac{\nu}{4x^2}\left(F_2 - \frac{4x^2\nu}{Q^2} F_1\right) = \frac{\nu}{4x^2} F_L$$

$$n^\mu n^\nu W_{\mu\nu} = W_2 = \frac{1}{\nu} F_2$$

$$\Rightarrow \begin{cases} F_2 = \nu\, n^\mu n^\nu W_{\mu\nu} \\ F_L = \frac{4x^2}{\nu} p^\mu p^\nu W_{\mu\nu} \end{cases}$$

* The photon scatters on a quark from the proton (large $Q^2 \Rightarrow$ pt-like object)



$$W^{\mu\nu} = e_q^2 \int \frac{d^4k}{(2\pi)^4}\, \mathrm{tr}\left(\gamma^\mu (\not{k} + \not{q}) \gamma^\nu B(p,k)\right) \delta\left((k+q)^2\right)$$

$$k^\mu \equiv \xi p^\mu + \frac{k^2 + k_\perp^2}{2\xi} n^\mu + k_\perp^\mu \quad \Rightarrow \quad \delta((k+q)^2) = \delta\left(k^2 - Q^2 + 2\xi\nu - 2\vec{k}_\perp \cdot \vec{q}_\perp\right)$$

$$\overset{Q^2 \gg}{\simeq} \delta\left(2\nu\xi - Q^2\right)$$

$$\simeq \frac{1}{2\nu}\, \delta(\xi - x)$$

i.e. the photon scatters on a quark carrying a longitudinal fraction x of the proton momentum.

$$F_2 = \nu n^\mu n^\nu W_{\mu\nu} = \frac{1}{2} e_q^2 \int \frac{d^4k}{(2\pi)^4}\, \underbrace{\mathrm{tr}(\not{n} \overbrace{\not{k} \not{n}}^{-\not{n}\not{k} + 2n\cdot k} B(p,k))}_{2\xi\, \mathrm{tr}(\not{n} B)}\, \delta(\xi - x)$$

Introducing

$$q(x) = \int \frac{d^4k}{(2\pi)^4} \, tr(\not{x}\, B(p,k))\, \delta(\xi - x)$$

we have

$$F_2(x) = e_q^2\, x\, q(x)$$

or, summing over flavours,

$$F_2(x) = \sum_q e_q^2\, x\, (q(x) + \bar{q}(x))$$

Notes:
- $q(x)$ can be interpreted as the density of quarks q in the proton : $q(\xi)d\xi$ quarks with x in $[\xi, \xi + d\xi]$
  $\Rightarrow$ Parton Distribution Functions (PDF)
- $F_2(x, Q^2) = F_2(x)$: only dependent on x $\rightarrow$ Bjorken scaling
- $F_L = \frac{4x^2}{\nu} p^\mu p^\nu W_{\mu\nu}$

  $$= \frac{4x^2}{2\nu^2} \int \frac{d^4k}{(2\pi)^4} \, tr(\not{p}\, (\not{k} + \not{q})\, \not{p}\, B(p,k))\, \delta(\xi - x)$$

  $$\underbrace{\quad}_{-\not{p}(\not{k}+\not{q}) + 2p\cdot(k+q)} \qquad \underbrace{\quad}_{\frac{k^2 + k_\perp^2}{\xi} + 2\nu}$$

  $$\simeq \frac{4x^2}{\nu} \int \frac{d^4k}{(2\pi)^4} \, tr(\not{p}\, B(p,k))\, \delta(\xi - x)$$

  is suppressed at large Q^2 $\Rightarrow$ $\boxed{F_2 = 2x F_1}$

  Callan-Gross relation ; tells that quarks are spin-$\frac{1}{2}$ fermions.
- $F_2^{em}(x) = \sum_q e_q^2\, x\, (q(x) + \bar{q}(x))$

  $F_2^{\nu}(x) = 2x\,(d + s + \bar{u} + \bar{c})$ $\qquad (\nu \to e^- W^+)$

  $F_2^{\bar{\nu}}(x) = 2x\,(u + c + \bar{d} + \bar{s})$

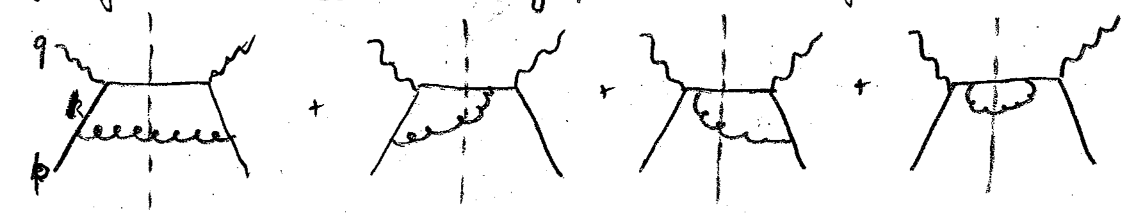
  $x F_3^{\nu}(x) = 2x\,(d + s - \bar{u} - \bar{c})$

  $x F_3^{\bar{\nu}}(x) = 2x\,(u + c - \bar{d} - \bar{s})$

4) Scaling violations

In QCD, the quark probed by the photon can radiate gluons

So, at first order in α_s, there are 4 graphs to consider (for the real emission)



We'll consider a parton of momentum $\xi(0,0,P,P) = p$

Useful: $n.p = 1$

$p.k = \frac{k_\perp^2 - |k^2|}{2\xi}$

$p.q = \nu$

NB at 0^{th} order in α_s.

$$\hat{F}_2 = \frac{\nu\, n^\mu n^\nu}{4\pi} \overline{\sum} |M|^2 \; (2\pi)\, \underbrace{\delta((p+q)^2)}_{\frac{1}{2\nu}\delta(\xi - x)}$$

$$\frac{\nu}{4\pi} \; \frac{1}{2}\left[n^\mu n^\nu \; e_q^2 \,\mathrm{tr}(\gamma^\mu \not{p} \gamma^\nu (\not{p}+\not{q})) = e_q^2 \mathrm{tr}(\not{n} \not{p} \not{n} (\not{p}+\not{q})) = 8 e_q^2 \right]$$

$$= e_q^2\, \delta(1-\xi)$$

Now let's consider the $O(\alpha_s)$ corrections. (Recall $q^\mu = \nu n^\mu + q_\perp^\mu$)

Phase space:

$$\int d\Phi_2 = \int \frac{d^4 k}{(2\pi)^3}\, (2\pi)^2 \delta^+((p-k)^2)\, \delta^+((k+q)^2) = \frac{1}{4\pi^2} \int d^4 k \; \delta^+((p-k)^2)\, \delta^+((k+q)^2)$$

$$k^\mu = \xi p^\mu + \frac{k_\perp^2 - |k^2|}{2\xi} n^\mu + k_\perp^\mu$$

$$k^\mu = \left(k_\perp, \; \xi P - \frac{k_\perp^2 - |k^2|}{4\xi P}, \; \xi P + \frac{k_\perp^2 - |k^2|}{4\xi P} \right)$$

$$\Rightarrow (p-k)^2 = k^2 - 2p.k = -|k^2| - \frac{k_\perp^2 - |k^2|}{\xi} = \frac{1-\xi}{\xi}|k^2| - \frac{k_\perp^2}{\xi}$$

$$(k+q)^2 = -|k^2| - Q^2 - 2\vec{k}_\perp \cdot \vec{q}_\perp + 2\xi\nu$$

$$d^4 k = d^2 k_\perp \, d\xi \, d|k^2| \left| \begin{matrix} \frac{1}{4\xi P} & P + O(\frac{1}{P^2}) \\ \frac{-1}{4\xi P} & P + O(\frac{1}{P^2}) \end{matrix} \right| = \frac{d\xi}{2\xi} d|k^2| \, d^2 k_\perp$$

$$\Rightarrow \int d\Phi_2 = \int \frac{d\xi}{2\xi} d|k^2|\, d^2 k_\perp \; \frac{1}{4\pi^2} \, \delta^+\!\left(\frac{(1-\xi)}{\xi}|k^2| - k_\perp^2 \right) \delta^+\!\left(2\xi\nu - Q^2 - |k^2| - 2\vec{k}_\perp \cdot \vec{q}_\perp\right)$$

$$= \frac{1}{16\nu\pi^2} \int d\xi \, d|k^2| \, d\hat{k}_\perp \, d\theta \; \delta^+\!\left((1-\xi)|k^2| - k_\perp^2\right) \delta^+\!\left(\xi - x - \frac{|k^2| + 2\vec{k}_\perp \cdot \vec{q}_\perp}{2\nu}\right)$$

NB $p.k = \frac{k_\perp^2 - |k^2|}{2\xi} = \frac{-1}{2}|k^2|$

amplitude

$$\overline{\sum}|\mathcal{M}|^2 = \frac{1}{2N_c} e_q^2 \sum_{a,B,A} \left(t^a_{AB} t^a_{BA}\right) g^2\, \mathrm{tr}\left(\gamma^\nu (\not k+\not q)\gamma^\mu \not k \gamma^\alpha \not p \gamma^\beta \not k\right)\frac{1}{k^4}\sum \varepsilon_\alpha \varepsilon^*_\beta(p-k)$$

we'll use the light-cone gauge $n\cdot A=0$ $\Rightarrow$ $\sum \varepsilon_\alpha\varepsilon^*_\beta = -g_{\alpha\beta} + \frac{n_\alpha(p-k)_\beta + n_\beta(p-k)_\alpha}{n\cdot(p-k)}$

$$\Rightarrow n^\mu n^\nu \overline{\sum}|\mathcal{M}^2| = \frac{1}{8} e_q^2 C_F \cdot g^2 \frac{1}{k^4} \mathrm{tr}\left(\not n (\not k+\not q)\not n \not k\gamma^\alpha \not p\gamma^\beta \not k\right)\left[-g_{\alpha\beta} + \frac{n_\alpha(p-k)_\beta + (p-k)_\alpha n_\beta}{1-\xi}\right]$$

$$-\mathrm{tr}\left(\not n(\not k+\not q)\not n \not k\gamma^\alpha\not p\gamma_\alpha \not k\right) + \frac{1}{1-\xi}\mathrm{tr}\left(\not n(\not k+\not q)\not n\not k\not n\not p(\not p-\not k)\not k\right) + \frac{1}{1-\xi}\mathrm{tr}\left(\not n(\not k+\not q)\not n\not k(\not p-\not k)\not p\not n\not k\right)$$

$+2n\cdot(k+q)\not n = +2\xi\not n$; $-2\not p$; $-\not p\not k\not k = -\not p k^2 = \not p|k^2|$; $-\not k\not k\not p = -k^2\not p$

$$=+2\xi\left[2\,\mathrm{tr}(\not n\not k\not p\not k) + \frac{2|k^2|}{1-\xi}\mathrm{tr}(\not n\not k\not n\not p)\right]$$

$$=4\xi\left[\mathrm{tr}(\not n\not k\not p\not k) + \frac{|k^2|}{1-\xi}\mathrm{tr}(\not n\not k\not n\not p)\right]$$

$$=4\xi\left(8\, n\cdot k\, p\cdot k - 4\, n\cdot p\, k^2 + \frac{8|k^2|}{1-\xi}\, n\cdot k\, n\cdot p\right)$$

$2n\cdot k\, p\cdot k = 2\xi\,\frac{k_\perp^2-|k^2|}{2\xi} = k_\perp^2 - |k^2| = -\xi|k^2|$

$$=16\xi\left(-\xi|k^2|+|k^2|+\frac{2\xi}{1-\xi}|k^2|\right)$$

$$=16\xi|k^2|\,\frac{1+\xi^2}{1-\xi}$$

$$\Rightarrow n^\mu n^\nu\overline{\sum}|\mathcal{M}|^2 = 8e_q^2 g^2\frac{1}{|k^2|}\,C_F\frac{1+\xi^2}{1-\xi}\,\xi = 32\pi e_q^2\alpha_s\,\xi P(\xi)\frac{1}{|k^2|}$$

$$\boxed{P(\xi)=C_F\,\frac{1+\xi^2}{1-\xi}}$$

$$\Rightarrow \hat F_2 = \frac{1}{4\pi}\cdot\nu\, n^\mu n^\nu \sum|\mathcal M|^2 d\Phi_2 = 8e_q^2\alpha_s\frac{1}{16\pi^2}\int d\xi\,\xi P(\xi)\int\frac{d|k^2|}{|k^2|}\underbrace{k_\perp dk_\perp}_{\frac12 dk_\perp^2} d\theta\,\underbrace{\delta^+\left((1-\xi)|k^2|-k_\perp^2\right)}_{k_\perp^2=(1-\xi)|k^2|}\delta^+\left(\xi-x-\frac{|k^2|+2k_\perp\cdot q}{2\nu}\right)$$

$$= \frac{e_q^2\alpha_s}{4\pi^2}\int\frac{d|k^2|}{|k^2|}\int d\xi\,\xi P(\xi)\int d\theta\,\delta\left(\xi-x-\frac{|k^2|}{2\nu}+\frac{\sqrt{(1-\xi)|k^2|}\,Q\cos(\theta)}{\nu}\right)$$

$$2\int_0^\pi d\theta.\ \delta\left(\xi-x-z+2\sqrt{(1-\xi)zx}\cos(\theta)\right) \qquad z=\frac{|k^2|}{2\nu}$$

$\Rightarrow \cos(\theta)=\frac{z+x-\xi}{2\sqrt{(1-\xi)zx}}$ $\quad\cos^2(\theta)\le1 \Leftrightarrow (\xi-z-x)^2\le 4(1-\xi)zx \le 4(1-x-z)zx - 4(\xi-x-z)zx$

$\Leftrightarrow (\xi-x-z-2xz)^2 \le 4(1-x-z)xz+4x^2z^2 \le 4xz(1-x)(1-z)$

$\Leftrightarrow \xi_-\le\xi\le\xi_+$ $\qquad \xi_\pm = x+z+2xz\pm\sqrt{z(1-z)x(1-z)}$

$$\int_0^\pi d\theta\,\delta\left(\xi-x-z+2\sqrt{(1-\xi)xz}\cos(\theta)\right) = \frac{1}{2\sqrt{(1-\xi)zx}}\,\frac{1}{\sqrt{1-\cos^2(\theta)}} = \frac{1}{\sqrt{4(1-\xi)zx-(\xi-x-z)^2}} = \frac{1}{\sqrt{(\xi-\xi_-)(\xi_+-\xi)}}$$

$\xi_\pm$ real $\Rightarrow$ $0\le z\le1$ $\Rightarrow$ $0\le|k^2|\le2\nu$

$$\Rightarrow \hat{F}_2 = \frac{e_q^2 \alpha_s}{2\pi^2} \cdot \int_{\mu^2}^{2\nu} \frac{d|k^2|}{|k^2|} \int_{\xi_-}^{\xi_+} d\xi \frac{\xi P(\xi)}{\sqrt{(\xi_+ - \xi)(\xi - \xi_-)}}$$

Where we have introduced μ^2 to regularise the (otherwise divergent) $|k^2|$ integration.
If we are only interested in the dominant contribution ($\propto \log(\nu/\mu^2) \equiv \log(Q^2/\mu^2)$) i.e. those which are logarithmically enhanced compare to the constant terms, we have that $\xi \simeq x$ $(\xi = x + O(\sqrt{z}))$

This has 2 consequences

1) the other diagrams have no log enhancement (see next page)

2) $\int_{\xi_-}^{\xi_+} d\xi \frac{\xi P(\xi)}{\sqrt{(\xi_+ - \xi)(\xi - \xi_-)}} \rightarrow x P(x) \cdot \pi$

$$\Rightarrow \boxed{\hat{F}_2 = e_q^2 \frac{\alpha_s}{2\pi} x P(x) \log(Q^2/\mu^2)}$$ keeping only the leading Q^2 behaviour.

We still need to compute the virtual corrections which also appear at that order. Those will be proportional to $\delta((p+q)^2)$ i.e. to $\delta(1-x)$. This means that

$$P(x) \rightarrow P(x) + K\,\delta(1-x)$$

Because of baryon number conservation, $\int dx\, q(x)$ must be constant in Q^2

$$\Rightarrow \int_0^1 dx\; P(x) + K\delta(1-x) = 0 \qquad \Rightarrow \quad K = -\int_0^1 dx\, P(x) = -C_F \int_0^1 dx \frac{1+x^2}{1-x}$$

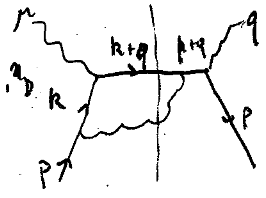
$$\Rightarrow P(x) = C_F \cdot \left(\frac{1+x^2}{1-x}\right)_+ \qquad \text{with} \quad \left[\frac{f(x)}{1-x}\right]_+ = \frac{f(x)}{1-x} - \int_0^1 dx \frac{f(x)}{1-x}\,\delta(1-x)$$

$$= C_F \left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)\right] \qquad \text{with} \quad \frac{1}{(1-x)_+} = \frac{1}{1-x} \text{ and } \int_0^1 dx \frac{f(x)}{(1-x)_+} = \int_0^1 dx \frac{f(x) - f(1)}{1-x}$$

the "+-distribution"

At the end, reinserting the LO contribution

$$\boxed{\hat{F}_2(x, Q^2) = e_q^2\, x \left[\delta(1-x) + \frac{\alpha_s}{2\pi} P(x) \log(Q^2/\mu^2)\right]}$$



$$= \frac{1}{2} e_q^2 C_F g^2 \frac{1}{k^2 (p+q)^2} \cdot \mathrm{tr}\left(\gamma^\beta (\not p+\not q)\gamma^\mu (\not k+\not q)\not\epsilon \not k \gamma^\alpha \not p\right)\left(-g_{\alpha\beta} + \frac{n_\alpha (p-k)_\beta + n_\beta (p-k)_\alpha}{1-\xi}\right)$$

$$= 2\,\mathrm{tr}\left(\gamma(\not p+\not q)\not k \gamma (\not k+\not q)\not p\right) + \frac{1}{1-\xi}\mathrm{tr}\left(\not n(\not p+\not q)\gamma(\not k+\not q)\gamma\not k(\not p-\not k)\not p\right) + \frac{1}{1-\xi}\mathrm{tr}\left(\not n(\not p+\not q)(\not p-\not k)(\not k+\not q)\gamma\not k\gamma\not p\right)$$

$$= 4\,\mathrm{tr}\left((\not p+\not q)\not k \not n(\not q+\not k)\right) - 2\,\mathrm{tr}\left(\not q \not n \not k \not n(\not k+\not q)\not p\right) + \frac{4}{1-\xi}\,n\cdot(p+q)\,n\cdot(k+q)\,\mathrm{tr}\left(\not n\not k(\not p-\not k)\not p\right) + \frac{4}{1-\xi}\,n\cdot k\,n\cdot p\,\mathrm{tr}\left(\not n(\not p+\not q)(\not p-\not k)(\not k+\not q)\right)$$

$(k+q)+(p-k)$; $2\xi\not n$; $-4k^2\, n\cdot p$; $(p+q)\cdot(k+q)\cdots$; $4(p+q)^2\, n\cdot(k+q)$

$$= \mathrm{tr}\left((\not p-\not k)\not k\not n(\not q+\not k)\right)$$

$\hookrightarrow k\cdot p \not p$

$$= \mathrm{tr}\left((\not p-\not k)\not p\not n(\not q+\not k)\right)$$

$$= -\mathrm{tr}\left(\not k\not p\not n(\not q+\not k)\right)$$

$$= -4\,k\cdot(q+k) - 4k\cdot p\, n\cdot(q+k) + 4k\cdot n\, p\cdot(q+k)$$

$$-4\xi\,\mathrm{tr}\left(\not q\not n(\not k+\not q)\not p\right)$$

$$= -4\xi\left[4\,p\cdot q\, n\cdot(k+q) - 4\,q\cdot(k+q)\, n\cdot p\right]$$

$$(p+q)^2 = 2\nu - Q^2 = Q^2\frac{1-x}{x}$$

$$k\cdot(k+q) = -|k^2| + k\cdot q = -|k^2| + \frac{|k^2|+Q^2}{2} = \frac{Q^2-|k^2|}{2}$$

$$= 16\cdot\left[-k\cdot(q+k) - \xi k\cdot p + \xi p\cdot q + \xi p\cdot k - \xi^2 p\cdot q + \xi q(k+q) + \frac{\xi}{1-\xi}|k^2| + \frac{\xi^2 (p+q)^2}{1-\xi}\right]$$

$$= 16\cdot\left[+\xi(1-\xi)\,p\cdot q + \frac{\xi^2}{1-\xi}(p+q)^2 + \frac{\xi}{1-\xi}|k^2| + (-k+\xi q)\cdot(k+q)\right]$$

$$-(1+\xi)\,k\cdot(k+q) = -\frac{1+\xi}{2}Q^2 + \frac{1+\xi}{2}|k^2|$$

$$= 16 Q^2\left[+\frac{\xi(1-\xi)}{2x} + \frac{\xi^2(1-x)}{x(1-\xi)} - \frac{1+\xi}{2}\right] + O(|k^2|)$$

for $\xi = x + O(\sqrt{z})$,

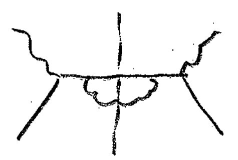
$$\frac{\xi(1-\xi)}{2x} + \frac{\xi^2(1-x)}{x(1-\xi)} - \frac{1+\xi}{2} = \frac{x(1-x)}{2x} + \frac{x^2(1-x)}{x(1-x)} - \frac{1+x}{2} + O(\sqrt{z})$$

$$= \frac{1-x}{2} + x - \frac{1+x}{2} + O(\sqrt{z})$$

$$= O(\sqrt{z})$$

Thus, the interference term has no log enhancement

<u>NOTE</u>: This is mostly due to our gauge choice $n\cdot A = 0$!

*  has no $\frac{1}{k^2}$ propagator

$\Rightarrow$ no log enhancement.

Factorisation & PDF evolution

We have seen that, the $\mathcal{O}(\alpha_s)$ corrections to F_2 show a divergence when $|k^2| \to 0$, that we have so far regularised by introducing a small cut-off $\mathcal{K}^2$.

This divergence when $|k^2| \to 0$ corresponds to $k_\perp \to 0$ i.e. the emitted gluon is <u>collinear</u> with the incoming parton. Note that this $k_\perp \to 0$, $0<x<1$ divergence is reminiscent of the $\theta \to 0$, $0<z<1$ (collinear) divergence in the $e^+e^- \to q\bar{q}g$ case with the difference that, there, it was regularised by the virtual corrections.

In this case, we will show that this collinear divergence can be reabsorbed into a redefinition of the parton density. Indeed, going from the parton-γ^* case to the proton-γ^* one, we can write F_2 as a convolution of $\hat{F}_2$ with the PDF $q_{bare}(x)$

$$F_2(x,Q^2) = \sum_{q,\bar{q}} x e_q^2 \int_x^1 \frac{d\xi}{\xi}\, q_{bare}(\xi) \left[\delta(1-\tfrac{x}{\xi}) + \frac{\alpha_s}{2\pi} \log\left(\frac{Q^2}{\mu_0^2}\right) P(x/\xi)\right]$$

The divergence can then be absorbed into the PDF by defining

$$q(x,\mu_F^2) = q_{bare}(x) + \int_x^1 \frac{d\xi}{\xi} \frac{\alpha_s}{2\pi} q_{bare}(\xi)\, P(x/\xi) \log(\mu_F^2/\mu_0^2)$$

Leading to

$$F_2(x,Q^2) = \sum_{q,\bar{q}} x e_q^2 \int_x^1 \frac{d\xi}{\xi}\, q(\xi,\mu_F^2) \left[\delta(1-\tfrac{x}{\xi}) + \frac{\alpha_s}{2\pi} P(x/\xi) \log(Q^2/\mu_F^2)\right]$$

We have thus reabsorbed the soft ($k_\perp \to 0$), long-range divergence into the "non-perturbative" PDF, keeping the perturbative $\frac{\alpha_s}{2\pi}\log(Q^2/\mu_F^2)$ part only in $\hat{F}_2$.

As a first consequence, F_2 now gets a Q^2 dependence i.e. QCD causes Bjorken scaling violations. Then, we see that when Q^2 is large, $\frac{\alpha_s}{2\pi}\log(Q^2/\mu_F^2)$ can get large also.(*) Anticipating that the higher order corrections will go like $\left[\frac{\alpha_s}{2\pi}\log(Q^2/\mu_F^2)\right]^n$, this means a need for resumming those terms at large Q^2. We also need to make sure that F_2 is not depending on the scale μ_F^2 introduced by hand.

(*) with $\alpha_s \equiv \alpha_s(Q^2)$, $\frac{\alpha_s}{2\pi}\log(Q^2/\mu_F^2) \simeq$ constant ~ 1

This requirement that F_2 is independent of the factorisation scale μ_F actually implies

$$\partial_{\mu_F^2} \int_x^1 \frac{d\xi}{\xi}\, q(\xi,\mu_F^2)\left[\delta\left(1-\frac{x}{\xi}\right) + \frac{\alpha_s}{2\pi} P(x/\xi)\log(Q^2/\mu_F^2)\right] = 0$$

i.e.

$$\int_x^1 \frac{d\xi}{\xi}\,\partial_{\mu_F^2} q(\xi,\mu_F^2)\left[\delta\left(1-\frac{x}{\xi}\right) + \frac{\alpha_s}{2\pi} P(x/\xi)\log(Q^2/\mu_F^2)\right] = \frac{1}{\mu_F^2}\int_x^1 \frac{d\xi}{\xi}\, q(\xi,\mu_F^2)\,\frac{\alpha_s}{2\pi} P(x/\xi)$$

Hence, setting $Q^2=\mu_F^2$

$$\boxed{Q^2\partial_{Q^2} q(x,Q^2) = \frac{\alpha_s}{2\pi}\int_x^1 \frac{d\xi}{\xi} P(x/\xi)\, q(\xi,Q^2)} \qquad (*)$$

known as the DGLAP equation. (Dokshitzer, Gribov, Lipatov, Altarelli, Parisi)

Taking $\mu_F^2=Q^2$ in the expression for F_2, we get

$$F_2(x,Q^2) = \sum_{q,\bar q} x\, e_q^2\, q(x,Q^2)$$

which has the same form as in the parton model except that now q depends on Q^2.

The DGLAP equation also resums $\log(Q^2/\mu_F^2)$ to all orders. To see that, let $t=\log(Q^2/\mu_F^2)$.

The solution of $(*)$ can be written

$$q(x,t) = q(x,t_0) + \int_{t_0}^{t} dt_1\, \frac{\alpha_s}{2\pi}\int_x^1 \frac{d\xi_1}{\xi_1} P(x/\xi_1)\, q(\xi_1,t_0) + \int_{t_0}^{t} dt_2\int_{t_0}^{t_2} dt_1 \left(\frac{\alpha_s}{2\pi}\right)^2 \int_x^1 \frac{d\xi_2}{\xi_2} P\left(\frac{x}{\xi_2}\right)\int_{\xi_2}^1 \frac{d\xi_1}{\xi_1} P\left(\frac{\xi_2}{\xi_1}\right) q(\xi_1,t_0) + \dots$$

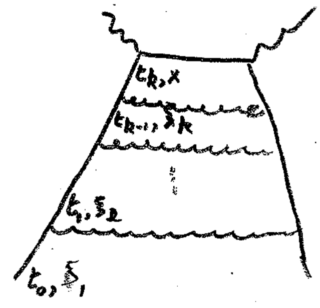
$$= \sum_{k=0}^{\infty} \int_{t_0}^{t} dt_k \int_{t_0}^{t_k} dt_{k-1} \cdots \int_{t_0}^{t_2} dt_1 \left(\frac{\alpha_s}{2\pi}\right)^k \int_x^1 \frac{d\xi_k}{\xi_k} P(x/\xi_k)\int_{\xi_k}^1 \frac{d\xi_{k-1}}{\xi_{k-1}} P(\xi_k/\xi_{k-1})\cdots\int_{\xi_2}^1 \frac{d\xi_1}{\xi_1} P(\xi_2/\xi_1)\, q(\xi_1,t_0)$$

In that sum, the nested t_i integration give a contribution $\frac{1}{k!}(t-t_0)^k = \frac{1}{k!}\log^k(Q^2/Q_0^2)$ hence, eq $(*)$ resums terms of the form $\frac{1}{k!}\left(\frac{\alpha_s}{2\pi}\right)^k \log^k(Q^2/Q_0^2)$.

This solution can also be written

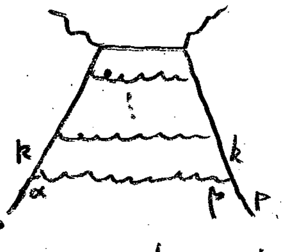
$$q(x,Q^2) = q(x,\mu_F^2) + \frac{\alpha_s}{2\pi}\int_{\mu_F^2}^{Q^2} \frac{dk^2}{k^2}\int_x^1 \frac{d\xi}{\xi} P(x/\xi)\, q(\xi,k^2)$$

Under the "Z" form, it resums gluon ladders of the form



It is interesting to see that this actually resums the relevant leading logs to all order (in $\alpha_s^k \log^k(Q^2/\mu_F^2)$)

To see that, let us consider the ladder



and concentrate on the bottom rung i.e. on

$$R_k = (\not k \gamma_\alpha \not p \gamma_\beta \not k)\left(-g^{\alpha\beta} + \frac{(p-k)^\alpha n^\beta + (p-k)^\beta n^\alpha}{1-\xi}\right)$$

Since $(p-k)^2 = 0$, $2p.k = k^2 = -|k^2|$. The log divergence will come from the $|k^2|$ part of R_k $\left(\to \frac{d|k^2|}{|k^4|} \cdot |k^2|\right)$

$$R_k = 2\not k \not p \not k + \frac{1}{1-\xi}\left(\not k \not n \not p (\not p - \not k)\not k + \not k (\not p - \not k)\not p \not n \not k\right)$$

$$= +4p.k\,\not k - 2k^2 \not p + \frac{|k^2|}{1-\xi}\left(\not k \not n \not p + \not p \not n \not k\right)$$

$$= |k^2| \cdot \left[-2\not k + 2\not p + \frac{1}{1-\xi}\left(\not k \not n \not p + \not p \not n \not k\right)\right]$$

The important point to realise here is that, since $p.k \propto |k^2|$ and $n.k = \xi = \xi(n.p)$, only the p component of k will NOT give an extra $|k^2|$ factor. Hence, we can replace k by ξp in $[\ldots]$, i.e.

$$R_k \overset{LL}{=} |k^2|\left(-2\xi \not p + 2\not p + \frac{2\xi}{1-\xi} \not p \not n \not p\right) \quad \text{(NB we neglected a } |k^2|^2 \not n \text{ contrib, not } \Delta \text{ at this order)}$$

($\not p \not n \not p \to 2\not p$)

$$= 2|k^2| \not p \underbrace{\left(\frac{(1-\xi)^2 + 2\xi}{1-\xi}\right)}_{\frac{1+\xi^2}{1-\xi}}$$

With the k phase space and the colour factor, we get

$$\frac{1}{(2\pi)^4}\int \frac{d\xi}{2\xi}\int \frac{dk_\perp^2}{2}\int d\theta \int \frac{d|k^2|}{|k^2|^2}\,(2\pi)\,\delta(k_\perp^2 - c.|k^2|)\, C_F . g^2 . 2|k^2|\, \frac{1+\xi^2}{1-\xi}\not p$$

$$= \frac{\alpha_s}{2\pi}\int_{\xi_{next}}^{1} \frac{d\xi}{\xi}\int_{\mu^2}^{k^2_{next}} \frac{d|k^2|}{|k^2|} \,.\, P(\xi)$$

which is exactly what gets resummed by the DGLAP eq.

+ We also see from that computat° that F_L has no $\alpha_s^k \log^k(Q^2/\mu_F^2)$. Indeed, this term would multiply $\mathrm{Tr}(\not p(\not k + \not q)\not p \not p) = 0$.

Instead, we should keep the $|k^2|\not n$ term and so, F_L has one less log i.e. starts at $\alpha_s^n \log^{n-1}(Q^2/\mu_F^2)$

* Strictly speaking, we have only shown that the collinear singularity can be reabsorbed into a redefinition of the PDF at first order in α_s.

(From the argument in the previous pages, it should be clear that it holds for the resummation of the $\alpha_s^k \log(Q^2/\mu_0^2) q_0 \to \alpha_s^k \log(Q^2/\mu_F^2) q(\mu_F^2)$ terms.)

Actually, this factorisation of the collinear singularity into

1) a non-perturbative / long-range / divergent part that we can reabsorb into the PDF at a scale μ_F
2) a perturbative part that one can complete at scales larger than μ_F^2

is true to any order of the perturbation theory, provided we keep only the leading-twist contributions (i.e. those that are not suppressed by powers of $1/Q$).

The independence on the choice of the factorisation scale μ_F leads to the DGLAP eq where the splitting function $P(x)$ can be computed order-by-order in perturbation theory:

$$P(x) = \frac{\alpha_s}{2\pi} P_0(x) + \left(\frac{\alpha_s}{2\pi}\right)^2 P_1(x) + \dots$$

resumming the $\alpha_s^k \log^k(Q^2/\mu_F^2)$, $\alpha_s^k \log^{k-1}(Q^2/\mu_F^2)$, ... terms. At higher orders, the relation to F_2 also has corrections (the <u>Coefficient functions</u> or <u>Wilson coefficients</u>) that can also be computed in perturbation theory.

<u>NB</u> at higher order there is a scheme dependence.

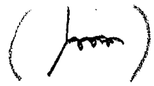
When limited to the 1st order, we speak about the LO DGLAP eqn
2nd NLO
3rd NNLO
⋮ ⋮

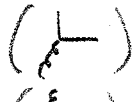
To account for the running of the coupling, we should set

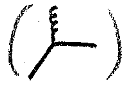
$$\alpha_s \to \alpha_s(Q^2)$$

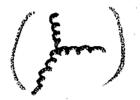
in the DGLAP eqn.

* The evolution equation we've considered so far is actually not complete.
If gluons can be emitted from quarks, we should also take into account an initial gluon splitting into a $q\bar{q}$ pair (). Also, in the resummed ladders, we can have  and  . The splitting function we have computed is thus one among a set of 4. We introduce P_{ab} as the probability to have a parton of type a out of a parton of type b. At LO, we have 4 choices

$$P_{qq} = C_F \left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)\right]$$ 

$$P_{qg} = T_R \left[x^2 + (1-x)^2\right]$$ 

$$P_{gq} = C_F \frac{1+(1-x)^2}{x}$$ 

$$P_{gg} = 2C_A \left[\frac{x}{(1-x)_+} + \frac{1-x}{x} + x(1-x)\right] + \frac{11 C_A - 4 n_f T_R}{6}\delta(1+x)$$ 

And the evolut° eqns becomes

$$Q^2 \partial_{Q^2} q(x,Q^2) = \frac{\alpha_s}{2\pi}\int_x^1 \frac{d\xi}{\xi}\left[P_{qq}\left(\frac{x}{\xi}\right) q(\xi,Q^2) + P_{qg}\left(\frac{x}{\xi}\right) g(\xi,Q^2)\right] \qquad \text{(id. for } \bar{q}\text{)}$$

$$Q^2 \partial_{Q^2} g(x,Q^2) = \frac{\alpha_s}{2\pi}\int_x^1 \frac{d\xi}{\xi}\left[\sum_{q,\bar{q}} P_{gq}\left(\frac{x}{\xi}\right) q(\xi,Q^2) + P_{gg}(x/\xi)\, g(\xi,Q^2)\right]$$

It is often useful to consider flavour singlet/non-singlet linear combinations of q & $\bar{q}$
e.g. • the valence distributions $V_i = q_i - \bar{q}_i$

- $T_3 = u^+ - d^+$
  $T_8 = u^+ + d^+ - 2s^+$ with $q_i^+ = q_i + \bar{q}_i$
  $T_{15} = u^+ + d^+ + s^+ - 3c^+$
  ...
- the sea: $\Sigma = \sum_{q,\bar{q}} q$

Then

$$Q^2 \partial_{Q^2} V_i(x,Q^2) = \frac{\alpha_s}{2\pi}\int_x^1 \frac{d\xi}{\xi} P_{qq}(x/\xi)\, V_i(\xi,Q^2)$$

$$Q^2 \partial_{Q^2} T_i(x,Q^2) = \frac{\alpha_s}{2\pi}\int_x^1 \frac{d\xi}{\xi} P_{qq}(x/\xi)\, T_i(\xi,Q^2)$$

$$Q^2 \partial_{Q^2} \begin{pmatrix}\Sigma(x,Q^2)\\ g(x,Q^2)\end{pmatrix} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{d\xi}{\xi} \begin{pmatrix} P_{qq}(x/\xi) & 2n_f P_{qg}(x/\xi) \\ P_{gq}(x/\xi) & P_{gg}(x/\xi)\end{pmatrix} \begin{pmatrix}\Sigma(\xi,Q^2)\\ g(\xi,Q^2)\end{pmatrix}$$

Note that at higher orders, the "P_{qq}" entering the evolution of V_i, T_i & (Σ, g) are different.

* Exercise: compute P_{qg}, P_{gq} and P_{gg}

Hint: for gluons the projector will be $(-g^{\alpha\beta})$

* Probabilities & sum rules

$q(x,Q^2)$ & $g(x,Q^2)$ can be considered as the probabilities to extract from the proton a quark/gluon of longitudinal momentum xp and virtuality less than Q^2.

This implies sum rules for the splitting functions:

quark number conservation: $\int_0^1 dx\, P_{qq}(x) = 0$

momentum conservation: $\int_0^1 dx\, x\left[P_{qq}(x) + P_{gq}(x)\right] = 0$

$$\int_0^1 dx\, x\left[2n_f P_{qg}(x) + P_{gg}(x)\right] = 0$$

* Solutions of the DGLAP equation(s)

The DGLAP eqn is linear, so we should be able to solve it analytically.
The idea is to get rid of the convolution with the splitting function by going to Mellin space:

$$f(x) \to \tilde{f}(j) = \int_0^1 dx\, x^{j-1} f(x)$$

which can be inverted a

$$f(x) = \int_{c-i\infty}^{c+i\infty} \frac{dj}{2i\pi}\, x^{-j}\, \tilde{f}(j)$$

With that transformation,

$$\int_0^1 dx\, x^{j-1} \int_x^1 \frac{d\xi}{\xi} P(x/\xi)\, q(\xi) = \int_0^1 \frac{d\xi}{\xi} \int_0^{\xi} dx\, x^{j-1} P(x/\xi)\, q(\xi) \qquad u = x/\xi$$

$$= \int_0^1 \frac{d\xi}{\xi}\, q(\xi) \int_0^1 \xi\, du\, u^{j-1} \xi^{j-1} P(u)$$

$$= \int_0^1 d\xi\, \xi^{j-1} q(\xi) \int_0^1 du\, u^{j-1} P(u)$$

$$= \tilde{q}(j)\, \tilde{P}(j)$$

a convolution becomes a product in j-space.
The Mellin transform of the splitting function $P(x)$ is called the anomalous dimension and usually denoted $\gamma(j)$.

We have (at LO)

$$\gamma_{qq}(j) = \int_0^1 dx\, x^{j-1} P_{qq}(x)$$

$$= C_F \cdot \left[\frac{3}{2} + \int_0^1 dx\, \frac{x^{j-1}(1+x^2)}{(1-x)_+}\right]$$

$$= C_F \left[\frac{3}{2} + \int_0^1 dx\, \frac{x^{j-1}+x^{j+1}-2}{1-x}\right] \qquad \int_0^1 dx\, \frac{x^k-1}{x-1} = \gamma_E + \psi(k+1)$$

$$= C_F \cdot \left[\frac{3}{2} - 2\gamma_E - \psi(j) - \psi(j+2)\right]$$

$$\gamma_{qg}(j) = T_R \int_0^1 dx\, x^{j-1}\left(1-2x+2x^2\right)$$

$$= T_R \cdot \left(\frac{1}{j} - \frac{2}{j+1} + \frac{2}{j+2}\right)$$

$$\gamma_{gq}(j) = C_F \int_0^1 dx\, x^{j-2}\left(2-2x+x^2\right)$$

$$= C_F \cdot \left(\frac{2}{j-1} - \frac{2}{j} + \frac{1}{j+1}\right)$$

$$\gamma_{gg}(j) = \frac{11C_A - 4n_f T_R}{6} + 2C_A \int_0^1 dx\, \frac{x^j-1}{1-x} + x^{j-2} - x^{j-1} + x^j - x^{j+1}$$

$$= \frac{11C_A - 4n_f T_R}{6} + 2C_A \left[\frac{1}{j-1} - \frac{1}{j} + \frac{1}{j+1} - \frac{1}{j+2} - \gamma_E - \psi(j+1)\right]$$

We can now turn to the solution of the DGLAP eqn. Let us start with the non-singlet case:

$$Q^2 \partial_{Q^2} \tilde{q}(j, Q^2) = \frac{\alpha_s}{2\pi} \gamma_{qq}(j) \tilde{q}(j, Q^2)$$

At fixed coupling, this gives

$$\tilde{q}(j, Q^2) = \tilde{q}(j, \mu_F^2) \exp\left[\frac{\alpha_s}{2\pi} \gamma_{qq}(j) \cdot \log(Q^2/\mu_F^2)\right] = \tilde{q}(j, \mu_F^2)\left(\frac{Q^2}{\mu_F^2}\right)^{\frac{\alpha_s}{2\pi}\gamma_{qq}(j)},$$

which justifies the term "anomalous dimension" for $\gamma(j)$.

For running coupling, $\alpha_s \to \alpha_s(Q^2) = \frac{1}{b_0 \log(Q^2/\Lambda_{QCD}^2)}$ $\left(b_0 = \frac{11C_A - 2n_f}{12\pi}\right)$ and the evol eqn is

$$\log(Q^2/\Lambda^2)\, \partial_{\log(Q^2/\Lambda^2)} \tilde{q}(j, Q^2) = \frac{1}{2\pi b_0} \cdot \gamma_{qq}(j)\, \tilde{q}(j, Q^2)$$

Hence

$$\tilde{q}(j, Q^2) = \tilde{q}(j, \mu_F^2) \exp\left\{\frac{\gamma_{qq}(j)}{2\pi b_0}\left[\log\left(\log(Q^2/\Lambda^2)\right) - \log\left(\log(\mu_F^2/\Lambda^2)\right)\right]\right\}$$

or

$$\tilde{q}(j, Q^2) = \tilde{q}(j, \mu_F^2) \cdot \left[\frac{\alpha_s(\mu_F^2)}{\alpha_s(Q^2)}\right]^{\frac{1}{2\pi b_0}\gamma_{qq}(j)}$$

The case of the singlet PDF is a bit more complex:

$$\partial_{\log(Q^2)} \begin{pmatrix} \tilde{\Sigma}(j,Q^2) \\ \tilde{g}(j,Q^2) \end{pmatrix} = \frac{\alpha_s}{2\pi} \begin{pmatrix} \gamma_{qq}(j) & 2n_f \gamma_{qg}(j) \\ \gamma_{gq}(j) & \gamma_{gg}(j) \end{pmatrix} \begin{pmatrix} \tilde{\Sigma}(j,Q^2) \\ \tilde{g}(j,Q^2) \end{pmatrix}$$

The anomalous dimension matrix first have to be diagonalised. The eigenvalues are given by

$$\begin{vmatrix} \gamma_{qq}-\gamma & 2n_f\gamma_{qg} \\ \gamma_{gq} & \gamma_{gg}-\gamma \end{vmatrix} = 0 \iff \gamma^2 - (\gamma_{qq}+\gamma_{gg})\gamma + \gamma_{gg}\gamma_{qq} - 2n_f\gamma_{qg}\gamma_{gq} = 0$$

$$\iff \left(\gamma - \frac{\gamma_{qq}+\gamma_{gg}}{2}\right)^2 = \frac{1}{4}\left((\gamma_{qq}+\gamma_{gg})^2 - 4\gamma_{qq}\gamma_{gg} + 8n_f\gamma_{gq}\gamma_{qg}\right)$$

$$= \frac{1}{4}\left[(\gamma_{qq}-\gamma_{gg})^2 + 8n_f\gamma_{qg}\gamma_{gq}\right]$$

$$\iff \gamma_\pm = \frac{1}{2}\left[\gamma_{qq}+\gamma_{gg} \pm \sqrt{(\gamma_{qq}-\gamma_{gg})^2 + 8n_f\gamma_{qg}\gamma_{gq}}\right]$$

with eigenvectors

$$V_\pm = \begin{pmatrix} 2n_f\gamma_{qg} \\ \gamma_\pm - \gamma_{qq} \end{pmatrix}$$

Thus

$$\begin{pmatrix} \tilde{\Sigma}(j,Q^2) \\ \tilde{g}(j,Q^2) \end{pmatrix} = \sum_\pm a_\pm(j,\mu_F^2) \begin{pmatrix} 2n_f\gamma_{qg} \\ \gamma_\pm - \gamma_{qq} \end{pmatrix} \left[\frac{\alpha_s(\mu_F^2)}{\alpha_s(Q^2)}\right]^{\frac{1}{2\pi b_0}\gamma_\pm(j)}$$

where $a_\pm(j,\mu_F^2)$ are fixed from $\tilde{\Sigma}(j,\mu_F^2)$ and $\tilde{g}(j,\mu_F^2)$

Instead of considering the general solution, it is more interesting to look at asymptotic limits. In general, QCD radiation will generate a Q^2 dependence going like

$$\left(\frac{Q^2}{\mu_F^2}\right)^{\frac{\alpha_s}{2\pi}\gamma(j)} \quad \text{or} \quad \left[\frac{\alpha_s(\mu_F^2)}{\alpha_s(Q^2)}\right]^{\frac{1}{2\pi b_0}\gamma(j)}$$

at fixed or running coupling. This is an increase (resp. decrease) when $\gamma(j)>0$ (resp. $\gamma(j)<0$)

In the next paragraphs, we'll consider the behaviour at small and large x.

- large-x. For simplicity, we deal with the non-singlet case.

  The large-x behaviour is dictated by the large-j behaviour of the anomalous dimension $\gamma_{qq}(j) \simeq -2C_F\log(j)$ and thus can be obtained from our j-space solution (note that $\gamma_{qq}(j)<0$ implies that $\tilde{q}(x)$ will decrease at large x).

  Let us take the initial condition

$$q(x,\mu_F^2) = a(\mu_F^2)\,(1-x)^{b(\mu_F^2)}$$

  NB we can recover the same result working directly in x space

$$\int_0^1 dx\ x^{j-1}(1-x)^b = B(j, b+1) \simeq \Gamma(b+1)\ j^{-b-1}$$

$$\Rightarrow \tilde{q}(j,Q^2) \overset{j\gg 1}{\simeq} a\Gamma(b+1)\cdot j^{-b-1}\cdot \exp\left[\frac{-2C_F}{3\pi b_0}\log\left(\frac{\alpha_s(\mu_F^2)}{\alpha_s(Q^2)}\right)\log(j)\right] \qquad (a\equiv a(\mu_F^2), b\equiv b(\mu_F^2))$$

$$\simeq a\Gamma(b+1)\cdot j^{-b-1-\frac{C_F}{b_0\pi}\log\left(\frac{\alpha_s(\mu_F^2)}{\alpha_s(Q^2)}\right)}$$

$$\Rightarrow q(x,Q^2) \overset{x\simeq 1}{\simeq} a\cdot\frac{\Gamma(b+1)}{\Gamma\left(b+1+\frac{C_F}{b_0\pi}\log\left(\frac{\alpha_s(\mu_F^2)}{\alpha_s(Q^2)}\right)\right)}(1-x)^{b+\frac{C_F}{b_0\pi}\log\left(\frac{\alpha_s(\mu_F^2)}{\alpha_s(Q^2)}\right)}$$

Which can be recast as

$$q(x,Q^2) = a(Q^2)\,(1-x)^{b(Q^2)}$$

with

$$a(Q^2) = \frac{\Gamma[b(\mu_F^2)+1]}{\Gamma[b(Q^2)+1]}\,a(\mu_F^2)$$

$$b(Q^2) = b(\mu_F^2) + \frac{C_F}{b_0\pi}\log\left(\frac{\alpha_s(\mu_F^2)}{\alpha_s(Q^2)}\right)$$

When Q^2 increases, $b(Q^2)$ increases also. Thus $q(x,Q^2)$ decreases as expected.

-) <u>small x</u>:

The small-x behaviour is dictated by the rightmost singularity in the complex j-plane. (note $\int_0^1 dx\, x^{j-1}(1/x)^\alpha = \frac{1}{j-\alpha}$). This is

$\gamma_{qq}(j) \simeq \frac{C_F}{j}$ corresponding to $P_{qq}(x) \to C_F$ when $x\to 0$

$\gamma_{qg}(j) \simeq \frac{T_R}{j}$ — $P_{qg}(x) \to T_R$

$\gamma_{gq}(j) \simeq \frac{2C_F}{j-1} - \frac{2C_F}{j}$ — $P_{gq}(x) \to \frac{2C_F}{x}(1-x)$

$\gamma_{gg}(j) \simeq \frac{2C_A}{j-1} - \frac{2C_A}{j}$ — $P_{gg}(x) \to \frac{2C_A}{x}(1-x)$

In that case, the singlet case (with the leading $\frac{1}{j-1}$ behaviour) is the interesting one to study.

Since

$$\gamma_\pm = \frac{1}{2}\left[\gamma_{qq}+\gamma_{gg} \pm \sqrt{(\gamma_{gg}-\gamma_{qq})^2 + 8n_f\gamma_{qg}\gamma_{gq}}\right] \simeq \frac{1}{2}\cdot\left[\frac{2C_A}{j-1} \pm \sqrt{\left(\frac{2C_A}{j-1}\right)^2}\right] \begin{cases} + & \frac{2C_A}{j-1}\ (\equiv \gamma_{gg}) \\ - & O\left(\frac{1}{j}\right) \end{cases}$$

only one of the 2 eigenvalue will contribute

This means

$$\begin{pmatrix} \tilde{\Sigma}(j,Q^2) \\ \tilde{g}(j,Q^2) \end{pmatrix} = \begin{pmatrix} \tilde{\Sigma}(j,\mu_F^2) \\ \tilde{g}(j,\mu_F^2) \end{pmatrix} \left[\frac{\alpha_s(\mu_F^2)}{\alpha_s(Q^2)}\right]^{\frac{C_A}{2b_0\pi}\frac{1}{j-1}}$$

v. To go back to x-space we have to compute

$$\int_{c-i\infty}^{c+i\infty} \frac{dj}{2i\pi}\, x^{-j} e^{a/j-1} = \int_{c-i\infty}^{c+i\infty} \frac{d\gamma}{2i\pi}\, e^{j\log(1/x)+\frac{a}{j-1}}$$

This is done in the saddle point approximation

$$\log\left(\frac{1}{x}\right) - \frac{a}{(j-1)^2} = 0 \Rightarrow j = 1 + \sqrt{\frac{a}{\log(1/x)}}$$

$$\Rightarrow j\log(1/x) + \frac{a}{j-1} = \log(1/x) + \sqrt{a\log(1/x)} + \sqrt{a\log(1/x)}$$

giving

$$\int_{c-i\infty}^{c+i\infty} \frac{dj}{2i\pi}\, x^{-j} e^{a/j-1} \propto \frac{1}{x} e^{2\sqrt{a\log(1/x)}}$$

In our case, this will give $\left(a = \frac{C_A}{2b_0\pi}\cdot\log\left(\frac{\alpha_s(\mu_F^2)}{\alpha_s(Q^2)}\right)\right)$

$$\boxed{g(x,Q^2) \sim g(x,\mu_F^2)\exp\left[\sqrt{\frac{2C_A}{b_0\pi}\log(1/x)\log\left(\frac{\alpha_s(\mu^2)}{\alpha_s(Q_0^2)}\right)}\right]}$$

in the small-x, large-Q^2 limit.

Note that this limit is also known as the double-log approximation (DLA)

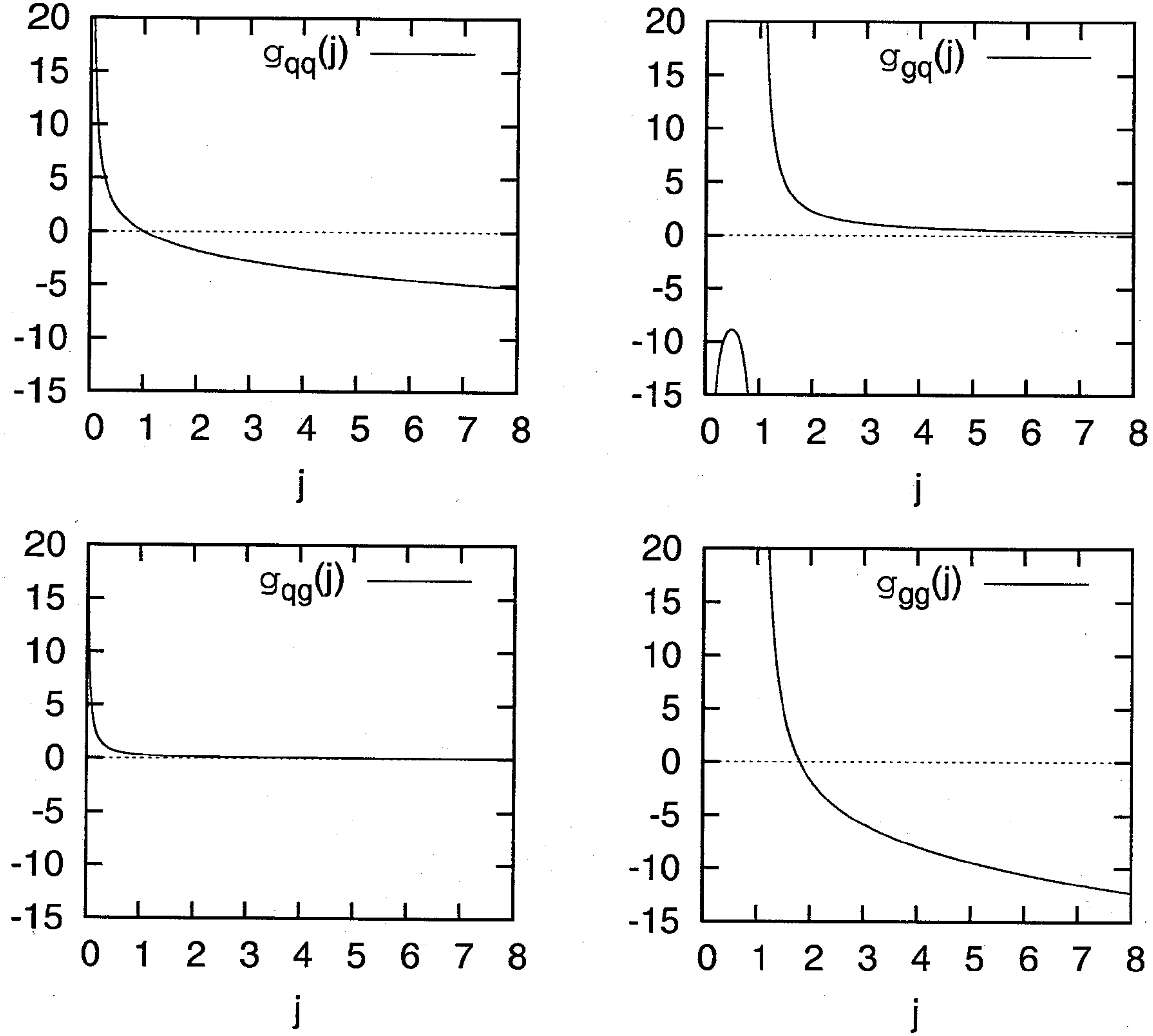
It is actually possible to rewrite the DGLAP solution in the small-x limit as (for fixed coupling)

$$xg(x,Q^2) \propto \sum_{k=0}^{\infty}\left(\frac{1}{k!}\right)^2\left(\frac{2C_A\alpha_s}{\pi}\right)^k \log^k(Q^2/\mu_F^2)\log^k(1/x)$$

This shows that at small x, we have terms enhanced by logarithms of $1/x$: $\alpha_s^k\log^k(1/x)$.

They should also be resummed. In the present formalism (DGLAP/large Q^2), we have resummed the dominant contribution at large Q^2 i.e. terms like $\alpha_s^k\log^k(Q^2)\log^k(1/x)$ (hence the name DLA).

In general, it is possible to resum the leading (and next-to-leading) $\log(1/x)$ without also requiring that Q^2 is large. This leads to the BFKL equation (Balitsky-Fadin-Kuraev-Lipatov).



$g_{qq}(j)$
$g_{gq}(j)$
$g_{qg}(j)$
$g_{gg}(j)$
j

* Higher order corrections:

At any order of the perturbation theory, we have FACTORISATION under the form

$$F_2(x,Q^2) = x\int_x^1 \frac{d\xi}{\xi}\left[C_q(x/\xi,\alpha_s)\, q(\xi,Q^2) + C_g(x/\xi,\alpha_s)\, g(\xi,Q^2)\right] + \mathcal{O}\left(\frac{\Lambda_{QCD}^2}{Q^2}\right)$$

with

$$Q^2\partial_{Q^2} q(x,Q^2) = \int_x^1 \frac{d\xi}{\xi}\, P_{qq}(x/\xi,\alpha_s)\, q(\xi,Q^2) + P_{qg}(x/\xi,\alpha_s)\, g(\xi,Q^2)$$

$$Q^2\partial_{Q^2} g(x,Q^2) = \int_x^1 \frac{d\xi}{\xi}\, P_{gq}(x/\xi,\alpha_s)\, q(\xi,Q^2) + P_{gg}(x/\xi,\alpha_s)\, g(\xi,Q^2)$$

the coefficient functions $C_{q,g}$ for specific processes ($F_2, F_L, F_2^d, F_3^\nu, F_2^\nu, \dots$) and the splitting functions P_{ab} can be computed order by order in perturbation theory

$$C_a(x,\alpha_s) = C_a^{(0)}(x) + \left(\frac{\alpha_s}{2\pi}\right) C_a^{(1)}(x) + \dots$$

$$P_{ab}(x,\alpha_s) = \left(\frac{\alpha_s}{2\pi}\right) P_{ab}^{(0)}(x) + \left(\frac{\alpha_s}{2\pi}\right)^2 P_{ab}^{(1)}(x) + \dots$$

The only "non-perturbative" input is then the PDF at the factorisation scale $q, g(x,\mu_F^2)$.

* Notes:
  * starting at NLO (i.e. $\mathcal{O}(\alpha_s)$ in C, $\mathcal{O}(\alpha_s^2)$ in P), there is a scheme dependence. $\overline{MS}$ is a common choice
  * at LO, $C_q^{(0)}(x) = e_q^2\,\delta(1-x)$, $C_g^{(0)}(x) = 0$ (for F_2)
    starting at NLO, both C_q and $C_g \neq 0$. Except for the DIS scheme which is (nearly) defined as
    $$C_q(x,\alpha_s) = e_q^2\,\delta(1-x)$$
    to any order
  * perturbative expansion:
    - $P_{ab}(x,\alpha_s)$ is known (since 2003) at NNLO i.e. $\mathcal{O}(\alpha_s^3)$
    - the Wilson coefficients are known at NLO or NNLO depending on the process under consideration (structure functions: NNLO, jets: NLO, ...)
  * Drell-Yan, jets: see later when we discuss pp collisions.
    ↳ $\ell^+\ell^-$ production in pp collision
    Remarkable fact: one can use THE SAME PDF as in DIS, a great example of their universality

* Global QCD fits:

- **General method**

  (i) at a fixed factorisation scale, parametrise the x dependence of the PDFs
  $q(x, \mu_F^2; \vec{a})$ $\quad g(x, \mu_F^2; \vec{a})$
  with free parameters $\vec{a}$ (remember sum rules constraints)

  (ii) use DGLAP evolution (at a given order) to get the PDFs at all Q^2

  (iii) adjust the parameters $\vec{a}$ to reproduce the data.

  typically:
  - F_2^p most important constraint, especially for the sea ($\Sigma q + \bar{q}$) and gluons
  - F_2^d $F_2^d = (F_2^n + F_2^p)/2$ with $F_2^p \xrightarrow{u \rightleftarrows d} F_2^n$ i.e. constraint on the sea asymmetry ($\bar{d} - \bar{u}$) (note: $\bar{d} > \bar{u}$ because of the Pauli principle and $u_V > d_V$)
  - $F_2^{\nu+\bar{\nu}}$ mostly a constraint on the strange quarks ($\neq$ coefs than F_2^p & F_2^d)
  - $xF_3^{\nu+\bar{\nu}}$ $\propto u_V + d_V \Rightarrow$ constraint on the valence quarks
  - DY quarks only at leading order.
  - jets constraint on the large-x gluon distrib (see later)

- **Many PDF available:**
  - Different teams: MRST, CTEQ, Alekhin, GRV
    H1, ZEUS (now being combined to get HERA PDF)
    NNPDF (Neural Network PDF: only DIS so far)
  - Many updates (due to new data, new improvements, higher orders)
    ex: LO, NLO, NNLO
    MRST01, 02, 04, 06, (08) CTEQ3M, 4M, 4HQ, 5M, 5HQ, 5HQ1, 5HJ, 5M1, 6M, 6HQ, 61M, 65M, 66.
  - include a study of the error on the PDF

- **BUT**: the "general method" is not sufficient!
  - pure DGLAP means large $Q^2 \Rightarrow$ massless quarks
    For massive quarks, we have $\log(Q^2/m^2) \Rightarrow$ need for a special treatment
    Various options are available:
    ZMFS (Zero Mass Flavour Scheme), VMFS (Variable Mass Flavour Scheme)
    Thorne-Roberts, ACOT, ACOT-χ, ...
    A "simple" option: - for the coefficient functions use exact results for $m \neq 0$
    - for evolution, (i) start at $Q^2 = m_q^2$
    (ii) ensure $W^2 \geq 4m_q^2$ ($\Leftrightarrow Q^2 \frac{1-x}{x} \geq 4m_q^2 \Leftrightarrow x\left(1 + \frac{4m_q^2}{Q^2}\right) \leq 1$)
    by using $x_{eff} = x\left(1 + \frac{4m_q^2}{Q^2}\right)$
  - Nuclear target corrections
  - Dependence on the parametric form used for the initial conditions
  - Dependence on the data included in the fit (dataset & cuts)
  - There is no valence strange quark in the proton. BUT, one could have
    $s(x) \neq \bar{s}(x)$ provided $\int_0 dx\, s(x) - \bar{s}(x) = 0$.