Feynman Rules & other bits and pieces

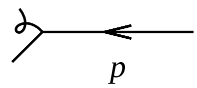
For the QCD course (G. Salam & M. Cacciari in the parcours théorique of the M2 Concepts Fondamentaux de la Physique). Information about the course, book recommendations, exact timetable, etc., can be accessed at `http://tinyurl.com/atktmk` (`http://www.lpthe.jussieu.fr/~salam/teaching/M2-CFP-QCD.html`).

The Feynman rules here are mostly taken from Peskin & Schroeder, second edition. The one difference is that *capital letters* are used to represent adjoint (gluon/ghost) colour indices, while fundamental representation indices are made explicit as small letters (following Ellis, Kunszt & Stirling). Note that P&S contains a whole bunch of other useful things in its appendix A.

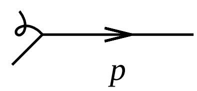
External particles

We'll start with the rules for external quarks and gluons:

External quarks:

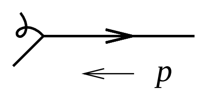


$$= u(p) \quad \text{(initial)} \tag{1}$$

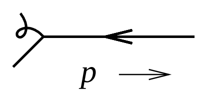


$$= \bar{u}(p) \quad \text{(final)} \tag{2}$$

External antiquarks:

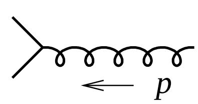


$$= \bar{v}(p) \quad \text{(initial)} \tag{3}$$

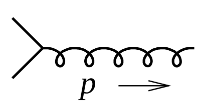


$$= v(p) \quad \text{(final)} \tag{4}$$

External gluons:



$$= \epsilon_\mu(p) \quad \text{(initial)} \tag{5}$$



$$= \epsilon^*_\mu(p) \quad \text{(final)} \tag{6}$$

For reference, recall certain basic spinor properties

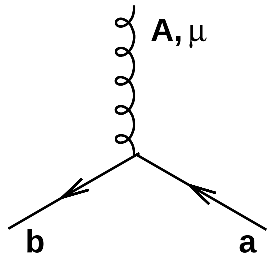
$$\begin{aligned} 0 &= (\not{p} - m)u(p) = \bar{u}(p)(\not{p} - m), \\ &= (\not{p} + m)v(p) = \bar{v}(p)(\not{p} + m), \end{aligned}$$

and if we assign a spin s to the spinors and sum over spins,

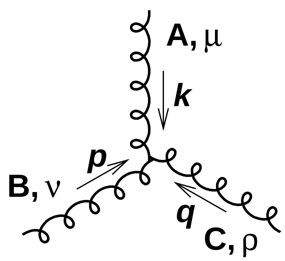
$$\sum_s u^s(p)\bar{u}^s(p) = \not{p} + m, \qquad \sum_s v^s(p)\bar{v}^s(p) = \not{p} - m \tag{7}$$

Remember also that there is a *symmetry factor* associated with each diagram (e.g. two final gluons $\to \frac{1}{2!}$).

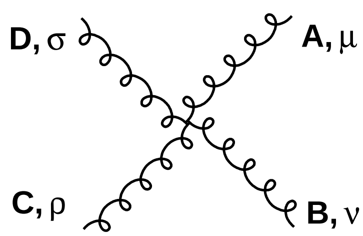
Internal components



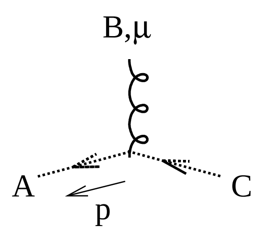
$$= \quad ig\gamma^\mu t^A \tag{8}$$



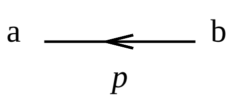
$$= \quad gf^{ABC}[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu] \tag{9}$$



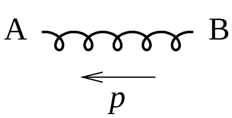
$$= \quad \begin{aligned} -ig^2[& f^{ABE}f^{CDE}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\ & + f^{ACE}f^{BDE}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\ & + f^{ADE}f^{BCE}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})] \end{aligned} \tag{10}$$



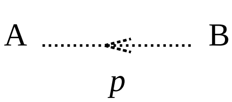
$$= \quad -gf^{ABC}p^\mu \tag{11}$$



$$= \quad \frac{i(\not{p}+m)\delta^{ab}}{p^2 - m^2 + i\epsilon} \tag{12}$$



$$= \quad \frac{-ig^{\mu\nu}\delta^{AB}}{p^2 + i\epsilon} \tag{13}$$



$$= \quad \frac{i\delta^{AB}}{p^2 + i\epsilon} \tag{14}$$

- Loops are associated with an integral over the loop momena $\int \frac{d^4\ell}{(2\pi)^4}$.

- Fermion and ghost loops are associated with an extra factor of -1.

Colour algebra

In SU(N), we have the matrices t^C_{ab} in fundamental representation, normalised so that

$$\mathrm{Tr}\,(t^A t^B) = T_R\,\delta^{AB} = \frac{1}{2}\delta^{AB}\,. \tag{15}$$

We've suppressed the $_{ab}$ indices here (and elsewhere) to aid readability.

The commutation relation of the group is

$$[t^A, t^B] = i f^{ABC} t^C \tag{16}$$

where the f^{ABC} are the (real, antisymmetric) structure constants of the group.

The casimirs of the group arise in the following relations:

$$(t^A t^A)_{ab} = C_F \delta_{ab}\,, \qquad C_F = \frac{N^2-1}{2N} \tag{17a}$$

$$f^{ACD} f^{BCD} = C_A \delta^{AB}\,, \qquad C_A = N \tag{17b}$$

Another useful identity is the Fierz identity:

$$t^A_{ab} t^A_{cd} = \frac{1}{2}\left(\delta_{ad}\delta_{cb} - \frac{1}{N}\delta_{ab}\delta_{cd}\right). \tag{18}$$

Finally, anticommutation relations:

$$\{t^A, t^B\} = \frac{1}{N} I + d^{ABC} t^C\,, \tag{19a}$$

$$\sum_{A,B} d^{ABC} d^{ABD} = \frac{N^2-4}{N}\delta^{CD}\,, \quad d^{AAC} \equiv 0\,. \tag{19b}$$

Specifics for SU(3)

$SU(3)$ local gauge symmetry $\leftrightarrow$ 8 ($= 3^2 - 1$) generators $t^1_{ab} \ldots t^8_{ab}$ corresponding to 8 gluons $\mathcal{A}^1_\mu \ldots \mathcal{A}^8_\mu$.

A representation is: $t^A = \frac{1}{2}\lambda^A$,

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},\ \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},\ \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix},\ \lambda^8 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{-2}{\sqrt{3}} \end{pmatrix},$$

Dirac algebra shortcuts

Start from the β and α (both Hermitian) of the Dirac equation;

$$\gamma_0 = \beta; \qquad \beta^2 = 1; \qquad \gamma_i = \beta\alpha_i \tag{20a}$$

$$\{\gamma_\mu, \gamma_\nu\} = 2g^{\mu\nu} \qquad \gamma_\mu^\dagger = \gamma_0\gamma_\mu\gamma_0 \tag{20b}$$

Check last one since $\gamma_i^\dagger = \alpha_i\beta = \gamma_0^2\alpha_i\gamma_0 = \gamma_0\gamma_i\gamma_0$.

Basic identities for traces:

$$\mathrm{Tr}\,(\mathbf{1}) = 4 \tag{21a}$$

$$\mathrm{Tr}\,(\text{odd number of } \gamma\text{'s}) = 0 \tag{21b}$$

$$\mathrm{Tr}\,(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu} \tag{21c}$$

$$\mathrm{Tr}\,(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} + g^{\nu\rho}g^{\sigma\mu} - g^{\mu\rho}g^{\nu\sigma}) \tag{21d}$$

$$\mathrm{Tr}\,(\gamma^5) = 0 \tag{21e}$$

$$\mathrm{Tr}\,(\gamma^\mu\gamma^\nu\gamma^5) = 0 \tag{21f}$$

$$\mathrm{Tr}\,(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) = -4i\epsilon^{\mu\nu\rho\sigma} \tag{21g}$$

Note: extension to $d \neq 4$ is non-trivial for expressions involving $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. Cyclic permutations and reversal of order of γ matrices leave traces unchanged. Some common manipulations of γ matrices in d dimensions are:

$$\gamma^\mu\gamma_\mu = d \tag{22a}$$

$$\gamma^\mu\gamma^\nu\gamma_\mu = -(d-2)\gamma^\nu \tag{22b}$$

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho} - (4-d)\gamma^\nu\gamma^\rho \tag{22c}$$

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu + (4-d)\gamma^\nu\gamma^\rho\gamma^\sigma \tag{22d}$$

Cross-sections, etc.

Cross sections are given by

$$d\sigma = \frac{1}{2E_A 2E_B|v_A - v_B|}\left(\prod_f \frac{d^3p_f}{(2\pi)^3\,2E_f}\right)|M(p_A, p_B \to \{p_f\})|^2(2\pi)^4\delta^4(p_A + p_B - \sum_f p_f)\,, \tag{23}$$

in terms of the matrix element M. Decay rates are given by

$$d\Gamma = \frac{1}{2m_A}\left(\prod_f \frac{d^3p_f}{(2\pi)^3\,2E_f}\right)|M(m_A \to \{p_f\})|^2(2\pi)^4\delta^4(p_A - \sum_f p_f)\,. \tag{24}$$

The two-body phase-space can be written as

$$\left(\prod_f \frac{d^3p_f}{(2\pi)^3\,2E_f}\right)(2\pi)^4\delta^4(\sum_i p_i - \sum_f p_f) = \int \frac{d\Omega_{cm}}{4\pi}\frac{1}{8\pi}\left(\frac{2|\vec{p}|}{E_{cm}}\right)\,, \tag{25}$$

where $\vec{p}$ is the 3-momentum of either of the outgoing particles in the centre-of-mass frame.

Loop integrals

Feynman parametrisation:

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[xA + (1-x)B]^2} \tag{26}$$

Inside symmetric integrals, one can replace

$$\ell^\mu \ell^\nu \to \frac{1}{d}\ell^2 g^{\mu\nu} \tag{27a}$$

$$\ell^\mu \ell^\nu \ell^\rho \ell^\sigma \to \frac{1}{d(d+2)}(\ell^2)^2 (g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma}g^{\mu\sigma}g^{\nu\rho}) \tag{27b}$$

Actual integrations are performed by Wick-rotating to Euclidean space, with the substitution $\ell^0 = i\ell_E^0$, $\ell^2 = -\ell_E^2$, but for the purposes of the course, the following table should be enough to get you going:

$$\int \frac{d^d\ell}{(2\pi)^d}\frac{1}{(\ell^2-\Delta)^n} = i\frac{(-1)^n}{(4\pi)^{d/2}}\frac{\Gamma(n-d/2)}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}} \tag{28a}$$

$$\int \frac{d^d\ell}{(2\pi)^d}\frac{\ell^2}{(\ell^2-\Delta)^n} = i\frac{(-1)^{n-1}}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma(n-d/2-1)}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1} \tag{28b}$$

$$\int \frac{d^d\ell}{(2\pi)^d}\frac{(\ell^2)^2}{(\ell^2-\Delta)^n} = i\frac{(-1)^n}{(4\pi)^{d/2}}\frac{d(d+2)}{4}\frac{\Gamma(n-d/2-2)}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-2} \tag{28c}$$

Relevant expansion coefficients:

$$\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} = 1 - (2-\frac{d}{2})\ln\Delta\,, \qquad \Gamma(x) = \frac{1}{x} - \gamma_E + \mathcal{O}(x) \tag{29}$$

with $\gamma_E \simeq 0.5772$. A common combination is:

$$\frac{\Gamma(2-d/2)}{(4\pi)^{d/2}}\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} = \frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon} - \ln\Delta - \gamma_E + \ln(4\pi) + \mathcal{O}(\epsilon)\right), \qquad \epsilon = 4-d \tag{30}$$