

LECTURES ON GEOMETRY, QUANTUM INTEGRABILITY AND SYMMETRIC FUNCTIONS

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SUMMARY OF INTRODUCTION

A simple observation, which was apparently made first in [9], is that both quantum integrable systems and exceptional cohomology theories (with certain technical assumptions) are classified by 1-dimensional formal groups. In quantum integrability, this is required by the fact that spectral parameters need to be subtracted in order to write the *Yang–Baxter equation* (with the so-called *difference property*), and therefore must belong to an abelian group. In generalized cohomology theories, the formal group can be thought of as the space of *equivariant parameters*. In the end, if we require the formal group to be an actual (smooth) curve, we find a classification into three classes of curves / cohomology theories / integrable systems:

- $\mathbb{G}_a := (\mathbb{K}, +)$ / Ordinary cohomology / Rational integrable systems
- $\mathbb{G}_m := (\mathbb{K}^\times, \times)$ / K -theory / Trigonometric integrable systems
- Elliptic curves E_τ / Elliptic cohomology / Elliptic integrable systems

In fact, it is not hard to see that given a simple algebraic group G and its Cartan subgroup T , acting on some algebraic variety X , and a cohomology theory $H_T^*(X)$, we can naturally extract from the Weyl group action a generalized R -matrix (in the sense of [3], i.e., which satisfies a generalized form of the Yang–Baxter equation), and therefore in some sense an integrable system. The purpose of these lectures is to make this construction explicit in the simplest possible setting, when the variety X is a (type A) Grassmannian, the gauge group G is $GL(n)$ (really, $PGL(n)$), and the cohomology theory is ordinary cohomology.

More introduction and references, e.g., [22, 8, 1, 2, 12, 14, 9, 6, 20, 11, 21, 10, 18, 23]. Comment on how quantum integrability gives “meaning” to a lot of constructions in geometric representation theory, which otherwise look like arbitrary engineering; and provides new tools.

Plan of what follows. In its current form, only a summary of the first two lectures is given (but this is classical material).

1. SUMMARY OF LECTURE 1: COHOMOLOGY

1.1. **Homology.** Singular homology. Grading by dimension. Homotopy invariance.

Example of torus: $H_*(\text{torus}) = \mathbb{Z} \oplus \mathbb{Z}_1^2 \oplus \mathbb{Z}_2$.

Pushforward (for proper maps). Covariant functor.

In all that follows: X is a smooth, complex projective algebraic variety.

I would like to thank V. Gorbunov and A. Knutson for discussions and ongoing collaboration, from which part of these lecture notes are derived.

Each (closed) subvariety $Y \subset X$ has a fundamental class $[Y] \in H_*(X)$. If $f : X \rightarrow X'$,

$$f_*([X]) = \begin{cases} 0 & \text{if } \dim f(X) < \dim X \\ m[f(X)] & \text{if } X \xrightarrow[\text{generically } m\text{-to-1}]{f} f(X) \end{cases}$$

We are more specifically interested in varieties X such that $H_*(X)$ is generated by fundamental classes of (complex, algebraic) subvarieties of X . (This implies in particular $H_{2i+1}(X) = 0$.)

Better example: $\mathbb{C}\mathbb{P}^1 = \mathbb{P}^1$ (topologically, sphere). $H_*(\mathbb{P}^1) = H_*(\text{sphere}) = \mathbb{Z} \oplus \mathbb{Z}$.

Fact. *If X admits a cell decomposition where each cell closure is a (complex) subvariety of X , then $H_*(X)$ is a free \mathbb{Z} -module with basis the classes of cell closures.*

Example of $\mathbb{P}^{n-1} = \bigsqcup_{k=0}^{n-1} \mathbb{C}^k \Rightarrow H_*(\mathbb{P}^{n-1}) = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$.

1.2. **Grassmannian.** Definition: given $0 \leq k \leq n$,

$$Gr(k, n) = \{V \text{ linear subspace of } \mathbb{C}^n, \dim_{\mathbb{C}} V = k\}$$

They are smooth projective (Plücker embedding) complex algebraic varieties of dimension $\dim_{\mathbb{C}} Gr(k, n) = k(n - k)$. Note $Gr(1, n) = \mathbb{P}^{n-1}$.

Description as

$$Gr(k, n) = GL(k) \backslash \text{Mat}^{max}(k, n)$$

where $\text{Mat}^{max}(k, n)$ is the space of $k \times n$ matrices of maximal rank (k).

Transitive action of $GL(n)$.

Definition of Schubert cells:

- Geometrically, given a full flag $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n$, take the fibers of the map $V \mapsto (d_i = \dim(V \cap V_i))_{0 \leq i \leq n}$.

These fibers are in bijection with Young diagrams in a $k \times (n - k)$ rectangle (reading (d_n, \dots, d_0) , move up one step in \mathbb{Z}^2 each time $d_{i-1} = d_i - 1$, move right one step each time $d_{i-1} = d_i$, starting from $(0, -k)$ and arriving at $(n - k, 0)$; then consider the area delimited by the two axes and that path) and with subsets of $\{1, \dots, n\}$ of cardinality k (subset of i such that $d_{n-i+1} - d_{n-i} = 1$).

Denote by S_I the corresponding fiber (Schubert cell), where I is such a subset (or equivalently, such a Young diagram).

(Example of $Gr(2, 4) = PGr(1, 3)$.)

- In coordinates, row echelon form: by $GL(k)$ action, any matrix in $\text{Mat}^{max}(k, n)$ can be reduced to a unique matrix of the form

$$\begin{pmatrix} 0 & 1 & \star & 0 & \star & 0 & \star & \cdots \\ 0 & 0 & 0 & 1 & \star & 0 & \star & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \star & \cdots \end{pmatrix}$$

where \star represents unconstrained entries. Denote by I the set of columns where 1's occur.

This shows that $S_I \cong \mathbb{C}^{k(n-k)-|I|}$ where $|I|$ is the number of boxes of (the Young diagram associated to) I .

- Consider the Borel subgroup $B \subset GL(n)$ of upper triangular matrices. Then the S_I are B -orbits. (more on that later)

The closures $\overline{S_I}$ (replace equalities with inequalities: $\dim(V \cap V_i) \geq d_i$) are called Schubert varieties. They are (closed) algebraic subvarieties of $Gr(k, n)$. Their own cell decomposition is

$$\overline{S_I} = \bigsqcup_{J \geq I} S_J$$

where the order relation is inclusion of Young diagrams.

Their classes $s_I = [\overline{S_I}]$ form a basis of $H_*(Gr(k, n))$ as a free graded \mathbb{Z} -module (the degree of s_I being $k(n - k) - |I|$).

1.3. Cohomology. Recall that X is a smooth projective complex algebraic variety. Let $d = \dim_{\mathbb{R}} X$. Define

$$H^i(X) = H_{d-i}(X)$$

New grading by codimension. (pushforward does not respect that new gradation)

Product \sim generic intersection. . . One sufficient condition for $[X][Y] = [X \cap Y]$: transversality of the intersection. One necessary condition: $\text{codim } X + \text{codim } Y = \text{codim}(X \cap Y)$.

$H^*(X)$ is a commutative graded ring.

Examples of \mathbb{P}^{n-1} : $H^*(\mathbb{P}^{n-1}) = \mathbb{Z}[x]/\langle x^n \rangle$, $\deg x = 2$.

Note $H^d(X) = H_0(X) \cong \mathbb{Z}$: generated by the class of a point. Define $\pi : X \rightarrow \{\cdot\}$ and $\pi_* : H_*(X) = H^*(X) \rightarrow \mathbb{Z}$ (extracts the coefficient of the class of a point).

Assume that $H^*(X)$ has no torsion. Then

$$\langle a | b \rangle = \pi_*(ab)$$

is a (symmetric) perfect pairing between $H^i(X)$ and $H^{d-i}(X)$ (Poincaré duality). (Combined with our definition $H^{d-i}(X) = H_i(X)$, we recover the ordinary duality between $H^i(X)$ and $H_i(X)$.)

This pairing is compatible with the product, in the sense that $\langle ab | c \rangle = \langle a | bc \rangle$.

Pullbacks: given $f : X \rightarrow Y$, define f^* to be the transposed (dual) map of f_* , i.e.,

$$f^* : H^i(Y) \cong (H_i(Y))^* \xrightarrow{f_*^T} (H_i(X))^* \cong H^i(X)$$

f^* is a graded ring morphism. contravariant functor.

More examples in $Gr(2, 4)$. “How many lines intersect four given lines (in general position) in 3-space?” (answer: 2).

Fact.

$$\langle s_I | s_J \rangle = \delta_{I, J^*}$$

where J^* is the 180 degree rotation of the complement of the Young diagram of J .

Presentation of $H^*(Gr(k, n))$:

Fact. $H^*(Gr(k, n)) = \mathbb{Z}[e_1, \dots, e_k]/\mathcal{I}_{k,n}$, with $\deg e_i = 2i$, and where $\mathcal{I}_{k,n}$ is the ideal generated by the relations

$$\left(1 + \sum_{i=1}^k e_i t^i \right)^{-1} = O(t^{n-k})$$

Sketch of proof: to e_i we assign the Schubert class of the one-column Young diagram with i boxes. Need to show that the kernel is $\mathcal{I}_{k,n}$ (the relations come from the exact sequence of vector bundles $0 \rightarrow V \rightarrow \mathbb{C}^n \rightarrow \mathbb{C}^n/V \rightarrow 0$, where the Chern classes of the tautological vector bundle V are identified with the e_i), and that the map is surjective.

1.4. Schur functions. Equivalently, one can reformulate the presentation above as follows. $\mathbb{Z}[e_1, \dots, e_k] \cong \mathbb{Z}[y_1, \dots, y_k]^{S_k}$ (with $\deg y_i = 2$) where $e_i(y_1, \dots, y_k) = \sum_{a_1 < \dots < a_i} y_{a_1} \dots y_{a_i}$ is the i^{th} elementary symmetric polynomial of the y_i . The ideal $\mathcal{I}_{k,n}$ is then generated by the relations

$$\prod_{i=1}^k (1 + ty_i)^{-1} = O(t^{n-k})$$

(the y_i are called Chern roots; in integrable systems, they will become Bethe roots.)

A symmetric function $P = (P_k)_{k \geq 0}$ is a sequence of symmetric polynomials such that for all $k \geq 1$, $P_k(y_1, \dots, y_{k-1}, 0) = P_{k-1}(y_1, \dots, y_{k-1})$.

Are there symmetric functions s_λ (for every unbounded Young diagram λ) such that s_Γ is the equivalence class of $s_\lambda(y_1, \dots, y_k)$ for all k, n ? (where λ is the Young diagram of I ignoring the bounding box) Yes, they are called Schur functions (and each polynomial in the sequence, Schur polynomial) and form a basis of the ring of symmetric functions.

2. SUMMARY OF LECTURE 2: EQUIVARIANT COHOMOLOGY

2.1. Generalities. Let X be a space, Γ a group acting on it. If Γ acts freely we define

$$H_\Gamma^*(X) = H^*(X/\Gamma)$$

In general, let $E\Gamma$ be a contractible space with a free Γ -action. Then we define

$$H_\Gamma^*(X) = H^*((X \times E\Gamma)/\Gamma)$$

In particular, $H_\Gamma^*(\cdot) = H^*(E\Gamma/\Gamma)$.

Γ -equivariant maps $f : X \rightarrow Y$ induce pullbacks and pushforwards (properness required for the latter).

The map $f : X \rightarrow \{\cdot\}$ induces a graded ring map from $H_\Gamma^*(\cdot)$ to $H_\Gamma^*(X)$, which endows the latter with a structure of $H_\Gamma^*(\cdot)$ -module.

Any fiber of $(X \times E\Gamma)/\Gamma \rightarrow E\Gamma/\Gamma$ is a copy of X , and the corresponding inclusion map $i : X \rightarrow (X \times E\Gamma)/\Gamma$ induces a graded ring map from $H_\Gamma^*(X) \rightarrow H^*(X)$. (In our case this map will be surjective)

As before, we assume in what follows that X is a smooth projective variety, and that Γ is an algebraic group.

From a homological viewpoint, any Γ -invariant subvariety Y of X possesses a fundamental class $[Y]_\Gamma$ in $H_\Gamma^*(X)$. However, the product structure becomes more subtle to define because the Γ -invariance constraint may prevent from moving freely class representatives to make them transverse.

2.2. Torus. In what follows we focus on the case where Γ is an algebraic torus, i.e.

$$\Gamma = T \cong (\mathbb{C}^\times)^n$$

Consider first $n = 1$. Then

$$ET = \{(z_i)_{i \in \mathbb{Z}_{>0}}, 0 < \#\{i : z_i \neq 0\} < \infty\}$$

and $ET/T = \mathbb{P}^\infty$. Then $H_{\mathbb{C}^\times}(\cdot) = H^*(ET/T) = \mathbb{Z}[x]$.

For general n , $H_{(\mathbb{C}^\times)^n}(\cdot) = \mathbb{Z}[x_1, \dots, x_n]$. More abstractly, $H_T(\cdot) = \text{Sym}(T^*)$, where $T^* = \{\chi : T \rightarrow \mathbb{C}^\times \text{ group morphism}\}$ is the lattice of weights of T (viewed as an additive group).

Note that if one has a sub-torus $T' \subset T$, then there is a map $H_T^*(X) \rightarrow H_{T'}^*(X)$ corresponding to imposing linear relations between the x_i (the lattice of weights of T' is a

quotient of that of T), and in particular the forgetful map $H_T^*(X) \rightarrow H^*(X)$ mentioned above corresponds to “setting all the x_i to zero”.

Now assume that we have a T -invariant cell decomposition of X (i.e., whose cells are T -invariant algebraic subvarieties). Then $H_T^*(X)$ is a free $H_T^*(\cdot)$ -module with basis, the classes of closures of cells.

Example. \mathbb{P}^1 with $T = \mathbb{C}^\times$ action $z \mapsto tz$, viewing \mathbb{P}^1 as $\mathbb{C} \cup \{\infty\}$. Three T -invariant subspaces: \mathbb{P}^1 , $\{0\}$, $\{\infty\}$. Pick one of $\{0\}$ or $\{\infty\}$ for the cell decomposition. We don’t know how to compute $[\{0\}]^2$. . . Admit for now: $H_{\mathbb{C}^\times}(\mathbb{P}^1) = \mathbb{Z}[y, x] / \langle y(y - x) \rangle$ where $y = [\{0\}]$, $y - x = [\{\infty\}]$.

2.3. Grassmannian. $Gr(k, n)$ has a $T \cong (\mathbb{C}^\times)^n$ -action, where T is a Cartan torus of $G := GL(n)$. Note that there is a circle \mathbb{C}^\times inside T acting trivially (scalar matrices), so really T/\mathbb{C}^\times (the Cartan torus of $PGL(n)$) acts.

Schubert cells are T -invariant iff the corresponding flag is T -invariant. Fixing an (ordered) diagonalization basis $(\epsilon_i)_{i=1, \dots, n}$ of T , T -invariant flags are exactly of the form

$$V_i = \text{span}(\epsilon_{w_{n-i+1}}, \dots, \epsilon_{w_n}), \quad i = 0, \dots, n$$

for w some permutation of \mathcal{S}_n (equivalently, think of these as permutations of the variables x_1, \dots, x_n .)

Let us denote $S_I^{(w)}$ and $s_I^{(w)}$ the corresponding Schubert cells and Schubert classes, and $s_I := s_I^{(1)}$.

In the same way as ordinary Schubert classes related to Schur polynomials, equivariant Schubert classes are related to *factorial* Schur polynomials [15, 17]. There is no natural analogue of going over to Schur functions, because of the presence of equivariant parameters (see however the notion of double Schur functions).

Presentation of $H_T^*(Gr(k, n))$:

Fact. $H_T^*(Gr(k, n)) = \mathbb{Z}[y_1, \dots, y_k, x_1, \dots, x_n]^{\mathcal{S}_k} / \mathcal{I}_{k,n}$ (where \mathcal{S}_k permutes the y_i), with $\deg y_i = \deg x_i = 2$, and where $\mathcal{I}_{k,n}$ is the ideal generated by the relations

$$\frac{\prod_{i=1}^n (1 + tx_i)}{\prod_{i=1}^k (1 + ty_i)} = O(t^{n-k})$$

Example. $Gr(1, n) = \mathbb{P}^{n-1}$. $H_T^*(\mathbb{P}^{n-1}) = \mathbb{Z}[y, x_1, \dots, x_n] / \langle \prod_{i=1}^n (y - x_i) \rangle$. (we don’t quite recover the case of \mathbb{P}^1 above because the torus is bigger, see remark on trivially acting circle; we can match the equation for y by setting $x = x_1 - x_2$ and shifting y by x_2)

Example. One box in $Gr(k, n)$:

$$s_{\square} = \sum_{i=1}^k y_i - \sum_{i=1}^k x_i$$

Note that it is *not* symmetric by permutation of the x_i . This can be traced back to the arbitrary choice made in selecting the flag among the $n!$ ones. . . (more on this later)

2.4. Localization. The idea is that we allow ourselves to invert any (nonzero) element of the base ring $H_T^*(\cdot)$, turning it into its fraction field F . We therefore denote for any $H_T^*(\cdot)$ -module M

$$\tilde{M} = M \otimes_{H_T^*(\cdot)} F$$

(this is overkill because we only need certain denominators. . .). In particular $\tilde{H}_T^*(\cdot) = F$. This way $\tilde{H}_T^*(X)$ is simply a *finite-dimensional algebra* over F .

In practice, for $T \cong (\mathbb{C}^\times)^n$, $F = \mathbb{Q}(x_1, \dots, x_n)$.

Before proceeding, let us discuss the pairing in the equivariant setting. $\pi : X \rightarrow \{\cdot\}$ still induces a map $\pi_* : \tilde{H}_T^*(X) \rightarrow F$ (which sends every fixed point to 1), and therefore a scalar product $\langle a|b \rangle = \pi_*(ab)$ on $\tilde{H}_T^*(X)$. This scalar product is compatible with the product as before, making $\tilde{H}_T^*(X)$ a Frobenius algebra.

As is obvious in the case of \mathbb{P}^1 , the basis of Schubert classes in $\tilde{H}_T^*(Gr(k, n))$ is no more globally self-dual. However, one has:

$$\left\langle s_I^{(w_0)} \middle| s_J \right\rangle = \delta_{I, J^*}$$

where w_0 is the longest permutation (the corresponding cells are usually called opposite Schubert cells; they intersect transversally the original Schubert cells).

2.5. Localization and fixed points. The most important consequence of localization involves T -fixed points. We assume in what follows that X has isolated fixed points (so that they are in finite number), and denote their set X^T .

We first formulate the statement using only the F -vector space structure of $\tilde{H}_T^*(X)$.

Fact (Equivariant localization, version 1). *The fixed points of X form a basis of $\tilde{H}_T^*(X)$ (as a vector space over F).*

(Note that the statement would make no sense without equivariance, since all points have the same nonequivariant class...)

In particular, this implies that their number is $\dim \tilde{H}_T^*(X)$.

Check: case of $Gr(k, n)$. Fixed points are coordinate subspaces $\mathbb{C}^I := \text{span}(\epsilon_i, i \in I)$, where I is as usual a subset of cardinality k in $\{1, \dots, n\}$. There is a natural bijection between Schubert cells S_I and fixed points \mathbb{C}^I , since each Schubert cell contains exactly one fixed point: $\mathbb{C}^I \in S_I$, and in fact, is its B -orbit: $S_I = \mathbb{C}^I B$ ($B \subset GL(n)$ acting by multiplication on the right if we think of $Gr(k, n)$ as $k \times n$ matrices). One must be careful however that if we use the other Schubert cells $S_I^{(w)}$, the bijection gets mixed up: $\mathbb{C}^{w^{-1}(I)} \in S_I^{(w)}$ (more on this later).

In practice, to understand how to decompose on this basis of fixed points, we introduce the embedding $i : X^T \rightarrow X$. It induces two maps $i^* : \tilde{H}_T^*(X) \rightarrow \tilde{H}_T^*(X^T)$ and $i_* : \tilde{H}_T^*(X^T) \rightarrow \tilde{H}_T^*(X)$, which are *both* (vector space) isomorphisms. However they cannot be inverses of each other, for grading reasons. $\tilde{H}_T^*(X^T)$ is nothing but $|X^T|$ copies of F . By definition, i_* simply sends the class of the fixed point p as a subspace of itself ($1_p \in \tilde{H}_T^*(p)$) to its class $[p]$ as a subspace of X . $i^*([p])$ on the other hand must be a class of degree $\dim X$ in $\tilde{H}_T^*(p) \cong H_T^*(\cdot)$: let us call it $i^*([p]) = E_p$.

Before trying to compute E_p , it is best to reinterpret the result above, but using the product structure. We've noticed on our favorite example that $\tilde{H}_T^*(X)$ is a (commutative) semi-simple algebra, contrary to its nonequivariant counterpart (which was nilpotent). So the action of the algebra on itself by multiplication should be diagonalizable; equivalently, it means that the algebra should possess primitive idempotents (which project onto the one-dimensional eigenspaces).

Fact (Equivariant localization, version 2). *The fixed points of X are (up to normalization) the primitive idempotents of the commutative algebra $\tilde{H}_T^*(X)$.*

It is obvious that the classes of two distinct points p and q satisfy $[p][q] = 0$. What about $[p]^2$? Using what precedes,

$$[p]^2 = [p]i_*(1_p) = i_*(i^*([p])1_p) = i_*(E_p 1_p) = E_p [p]$$

(where in the second step, we have used the fact that i_* is a $\tilde{H}_T^*(X)$ -module map)

Therefore, $[p][q] = \delta_{pq} E_p [p]$, i.e., $\frac{[p]}{E_p}$ are the idempotents we were looking for.

We finally provide a formula for E_p :

E_p is the product of weights of the action of T in the tangent space of X at p .

(justification: if we restricted T to a subgroup T' such that any weight of the tangent space became zero, we'd have a line of fixed points going through p , which means we'd be able to move one p away from the other, which means we'd have $[p]_{T'}^2 = 0$. This suggests that $[p]^2$ is a multiple of $E_p = \prod$ (weights); but by degree there can't be anything else)

To conclude, everything can be summarized by the formula

$$1 = \sum_{p \in X^T} \frac{[p]}{E_p}$$

which algebraically expresses the identity as the sum of primitive idempotents, and geometrically is the decomposition of the class of the whole space X as a linear combination of classes of fixed points. (by using denominators, we managed to express a degree zero class as a linear combination of degree $\dim X$ classes!)

2.6. Key formulae in the case of the Grassmannian.

- Restriction to the fixed point \mathbb{C}^I :

$$(1) \quad \begin{aligned} i_I^* : H_T^*(Gr(k, n)) &\rightarrow H_T^*(\cdot) \\ P(y_1, \dots, y_k; x_1, \dots, x_n) &\mapsto P(x_{I_1}, \dots, x_{I_k}; x_1, \dots, x_n) \end{aligned}$$

- Classes of fixed points:

$$(2) \quad [\mathbb{C}^I] = \prod_{i=1}^k \prod_{\substack{j=1 \\ j \notin I}}^n (y_i - x_j)$$

(Lagrange interpolation!)

- Normalization of the idempotents (norm of the Bethe state)

$$(3) \quad E_I = i_I^*([\mathbb{C}^I]) = \prod_{\substack{i=1 \\ i \in I}}^k \prod_{\substack{j=1 \\ j \notin I}}^n (x_i - x_j)$$

3. THE R -MATRIX

We want to define the R -matrix as the change of basis from one basis $(s_I^{(w)})$ to another one $(s_J^{(w')})$ in $\tilde{H}_T^*(Gr(k, n))$, where w is an element of the Weyl group \mathcal{S}_n . But first we need to modify a bit the setup.

3.1. The Hilbert space. $\tilde{H}_T^*(Gr(k, n))$ has dimension $\binom{n}{k}$. However in the context of integrable models (spin chains), we're more used to a Hilbert space of dimension 2^n . So the first thing we do is consider

$$Gr(\cdot, n) = \bigsqcup_{k=0}^n Gr(k, n), \quad \mathcal{H} := \tilde{H}_T^*(Gr(\cdot, n)) = \bigoplus_{k=0}^n \tilde{H}_T^*(Gr(k, n))$$

\mathcal{H} is a vector space of dimension 2^n over the field $F = \mathbb{Q}(x_1, \dots, x_n)$; in fact, we want to think about it as

$$\mathcal{H} \cong \bigotimes_{i=1}^n \mathbb{Z}[x_i]^2 = (\text{some completion of}) \bigotimes_{i=1}^n \mathbb{Q}(x_i)^2$$

where $\mathbb{Z}[x_i]^2 = \text{span}_{\mathbb{Z}[x_i]}(\bullet, \circ)$; i.e., each “site” has two states (occupied, empty) and one parameter x_i attached to it.

In this view of \mathcal{H} , it has a natural basis indexed by binary strings of (\bullet, \circ) of length n , i.e., of subsets $I \subset \{1, \dots, n\}$. We identify the basis element in question with s_I in $\tilde{H}_T^*(Gr(|I|, n))$.

The different $\tilde{H}_T^*(Gr(k, n))$ seem all mixed up, but it's of course very easy to distinguish them, namely they are weight spaces / eigenspaces of the operator σ_z :

$$\sigma_z := \sum_{i=1}^n 1 \otimes \dots \otimes 1 \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_i \otimes 1 \otimes \dots \otimes 1$$

$$\tilde{H}_T^*(Gr(k, n)) = \{v \in \mathcal{H} : \sigma_z v = (2k - n)v\}$$

The pairing will not be immediately obvious in the integrable setting, so we shall consider both \mathcal{H} and the dual space \mathcal{H}^* . In particular we shall also write \mathcal{H}^* as (some completion of) $\bigotimes_{i=1}^n \mathbb{Q}(x_i)^2$, but this time we shall identify the natural basis with the *dual* basis of the s_I , which was found to be the $s_{I^*}^{(w_0)}$.

Example. $Gr(\cdot, 2) = \{\cdot\} \sqcup \mathbb{P}^1 \sqcup \{\cdot\}$.

$$\begin{aligned} |\bullet\bullet\rangle &= [\cdot] \in \tilde{H}_T^*(Gr(2, 2)) \\ |\bullet\circ\rangle &= [\mathbb{P}^1] \in \tilde{H}_T^*(Gr(1, 2)) \\ |\circ\bullet\rangle &= [\infty] \in \tilde{H}_T^*(Gr(1, 2)) \\ |\circ\circ\rangle &= [\cdot] \in \tilde{H}_T^*(Gr(0, 2)) \end{aligned}$$

and bras:

$$\begin{aligned} \langle\bullet\bullet| &= [\cdot] \in \tilde{H}_T^*(Gr(2, 2)) \\ \langle\bullet\circ| &= [0] \in \tilde{H}_T^*(Gr(1, 2)) \\ \langle\circ\bullet| &= [\mathbb{P}^1] \in \tilde{H}_T^*(Gr(1, 2)) \\ \langle\circ\circ| &= [\cdot] \in \tilde{H}_T^*(Gr(0, 2)) \end{aligned}$$

3.2. Definition of the R -matrix. Given $w \in \mathcal{S}_n$, we define the R -matrix $\check{R}_{I,J}^{(w)}$ to be the matrix of change of basis between the $s_I^{(w)}$ and the s_J in \mathcal{H} :

$$(4) \quad s_J = \sum_I \check{R}_{I,J}^{(w)} s_I^{(w)}$$

or equivalently, in bracket notation, $\check{R}_{I,J}^{(w)} = \langle I^{(w)} | J \rangle$.

Obviously, $\check{R}^{(1)} = Id$.

Example. $Gr(\cdot, 2)$. There is only one nontrivial permutation, (21). The only nontrivial column of the matrix, $J = \circ\bullet$, is given by

$$[\infty] = [0] + (x_2 - x_1)[\mathbb{P}^1]$$

We then find:

$$\check{R}^{(21)} = \begin{matrix} & \bullet\bullet & \bullet\circ & \circ\bullet & \circ\circ \\ \bullet\bullet & \left(\begin{array}{cccc} 1 & & & \\ & 1 & x_2 - x_1 & \\ & 0 & 1 & \\ \circ\circ & & & 1 \end{array} \right) \\ \bullet\circ & & & & \\ \circ\bullet & & & & \\ \circ\circ & & & & \end{matrix}$$

Alternatively, it would make more sense to compare Schubert varieties containing the same fixed point. Therefore, we could consider instead the change of basis $R_{I,J}^{(w)}$ from the $s_{w(I)}^{(w)}$ to the s_I . We then have

$$R^{(21)} = \begin{matrix} & \bullet\bullet & \bullet\circ & \circ\bullet & \circ\circ \\ \bullet\bullet & \left(\begin{array}{cccc} 1 & & & \\ & 0 & 1 & \\ \bullet\circ & & 1 & x_2 - x_1 & \\ \circ\circ & & & & 1 \end{array} \right) \\ \circ\bullet & & & & \\ \bullet\circ & & & & \\ \circ\circ & & & & \end{matrix}$$

In what follows we shall use the $\check{R}^{(w)}$ matrices only.

3.3. First properties of the R -matrix. Note that $\check{R}^{(w)}(x_i = 0) = 1$ (nonequivariantly, all Schubert bases are the same).

Next, by composing carefully two such changes of basis, and using the fact that $S_I^{(w)} = S_I w$ (where the right action of w is the one on $k \times n$ matrices), we find the following identity:

$$\check{R}^{(vw)} = \tau^{(w^{-1})}(\check{R}^{(v)})\check{R}^{(w)}$$

where $\tau^{(w^{-1})}$ is the permutation of the variables x_i , i.e., the \mathbb{Q} -linear automorphism of F that sends x_i to $x_{w^{-1}(i)}$.

This has two important consequences:

- (1) This allows to express any matrix $\check{R}^{(w)}$ in terms of only elementary transpositions. Write \check{R}_i and τ_i for the elementary transposition $(i, i + 1)$.
- (2) Consider the (\mathbb{Q} -linear) operators $\tau^{(w)}\check{R}^{(w)}$. Then according to the above, they form a representation of the symmetric group \mathcal{S}_n . This is simply the counterpart of the natural geometric action of the Weyl group of $GL(n)$ on $Gr(k, n)$ (i.e., if one expands a class $[X]$ in the s_I , and then applies $\tau^{(w)}\check{R}^{(w)}$ to the vector of its entries, one obtains the class $[X\omega^{-1}]$, where ω is any representative in the normalizer $N(T)$ of the class $w \in N(T)/T \cong \mathcal{S}_n$).

Putting these facts together, we can write the Coxeter relations:

$$\tau_i(\check{R}_i)\check{R}_i = 1$$

(a form of the unitarity equation), and

$$\tau_{i+1}\tau_i(\check{R}_i)\tau_i(\check{R}_{i+1})\check{R}_i = \tau_i\tau_{i+1}(\check{R}_{i+1})\tau_{i+1}(\check{R}_i)\check{R}_{i+1}$$

(a form of the Yang–Baxter equation). We shall rewrite more simply and reinterpret graphically these relations below.

Example. Consider $Gr(1, 3) = \mathbb{P}^2$. Here are the Schubert classes s_I with the usual presentation of $\check{H}_T^*(Gr(1, 3))$:

$$\begin{aligned} |\bullet \circ \circ\rangle &= s_{\{1\}} = 1 \\ |\circ \bullet \circ\rangle &= s_{\{2\}} = y - x_1 \\ |\circ \circ \bullet\rangle &= s_{\{3\}} = (y - x_1)(y - x_2) \end{aligned}$$

In this presentation, one obtains $s_I^{(w)}$ by substituting $x_i \mapsto x_{w^{-1}(i)}$. For instance, for $w = (231)$, one gets $s_{\{1\}}^{(w)} = 1$, $s_{\{2\}}^{(w)} = y - x_3$, $s_{\{3\}}^{(w)} = (y - x_3)(y - x_1)$. By expanding the s_I in terms of the $s_I^{(w)}$, we find the following 3×3 block of the R -matrix:

$$\check{R}^{(231)} = \begin{pmatrix} 1 & x_3 - x_1 & (x_3 - x_1)(x_3 - x_2) \\ 0 & 1 & x_3 - x_2 \\ 0 & 0 & 1 \end{pmatrix} = \tau^{(132)} \begin{pmatrix} 1 & x_2 - x_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x_3 - x_2 \\ 0 & 0 & 1 \end{pmatrix}$$

where the decomposition matches $(231) = (213)(132)$ (careful that $\tau^{(132)}$ only acts on $R^{(213)}$).

It is also not hard to verify that $\tau^{(231)}\check{R}^{(231)}$ implements the natural geometric action of (231) , namely, right multiplication by its inverse (312) (check on fixed points!).

3.4. General form.

Fact. For general n , the R -matrix \check{R}_i associated to the elementary transposition $(i, i+1)$ is of the form

$$\check{R}_i = 1 \otimes \cdots \otimes 1 \otimes \begin{pmatrix} 1 & & & & \\ & 1 & x_{i+1} - x_i & & \\ & 0 & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}_{i, i+1} \otimes 1 \otimes \cdots \otimes 1$$

Sketch of proof. We shall use the definition of Schubert cells/varieties in terms of B -orbits, namely $S_I = \mathbb{C}^I B$.

We are trying to compare Schubert cells/varieties before/after action of the transposition $w = (i, i+1)$. Note that if we simply act with w , we break B -invariance of Schubert varieties. We therefore introduce the following (minimal parabolic) subgroup of $GL(n)$:

$$P_i = \left\{ i+1 \begin{pmatrix} \star & \cdots & i & \cdots & \star \\ 0 & \ddots & & & \vdots \\ 0 & 0 & \star & & \\ 0 & \cdots & \star & \star & \\ 0 & \cdots & & 0 & \ddots & \vdots \\ 0 & & \cdots & 0 & \star \end{pmatrix} \right\}$$

which is generated by B and w .

The natural action of P_i by right multiplication on $Gr(k, n)$, or any B -invariant subspace X of it, factors through the space $Gr(k, n) \times_B P_i$ (quotient of $Gr(k, n) \times P_i$ by right-action of B on $Gr(k, n)$ + left-action on P_i). The latter naturally projects to $B \backslash P_i$:

$$\begin{array}{ccc}
& \hookrightarrow^{Gr(k,n)} & \\
& X \times_B P_i & \\
f \swarrow & & \searrow g \\
Gr(k,n) & & B \setminus P_i \cong \mathbb{P}^1
\end{array}$$

By definition one has

$$\begin{aligned}
f_*g^*[1] &= [X] \\
f_*g^*[w] &= [Xw] \\
f_*g^*[\mathbb{P}^1] &= [XP_i] \delta_{\dim(XP_i), \dim X+1}
\end{aligned}$$

Note $XP_i = XwP_i$.

Now write the decomposition of the identity

$$[\mathbb{P}^1] = \frac{[1]}{x_{i+1} - x_i} + \frac{[w]}{x_i - x_{i+1}}$$

This implies

$$[X] = [Xw] + (x_{i+1} - x_i)[XP_i] \delta_{\dim(XP_i), \dim X+1}$$

Now take $X = \overline{S_I}$. There are 4 cases depending on whether $i, i+1 \in I$. In three cases out of 4, $\overline{S_I}$ is actually P_i -invariant. Only if $i \notin I$ and $i+1 \in I$ do we find $\overline{S_I P_i} = \overline{S_{w(I)}}$, hence the 3rd column of the R -matrix. \square

Remark: equivalently, this proof gives an inductive definition of Schubert classes in terms of *divided difference operators*...

Note that the R -matrix is “local” in the sense that it affects only factors i and $i+1$ of the tensor product in \mathcal{H} , and also only depends on variables x_i and x_{i+1} .

In particular, the Coxeter relations can be expressed entirely in terms of the 4×4 matrix $\check{R}(x) = \begin{pmatrix} 1 & & & \\ & x & & \\ & 0 & 1 & \\ & & & 1 \end{pmatrix}$; they are respectively the unitarity relation

$$\check{R}(-x)\check{R}(x) = 1$$

and the Yang–Baxter equation

$$(\check{R}(x) \otimes 1)(1 \otimes \check{R}(x+y))(\check{R}(y) \otimes 1) = (1 \otimes \check{R}(y))(\check{R}(x+y) \otimes 1)(1 \otimes \check{R}(x))$$

(1 being the 2×2 identity matrix so that the equation involves 8×8 matrices).

Example. Here’s the list of Schubert classes with the usual presentation of $\tilde{H}_T^*(Gr(2,4))$, for more nontrivial checks:

$$\begin{aligned}
s_{\{1,2\}} &= 1 \\
s_{\{1,3\}} &= y_1 + y_2 - x_1 - x_2 \\
s_{\{1,4\}} &= y_1^2 + y_1 y_2 - y_1 x_1 - y_1 x_2 - y_1 x_3 + y_2^2 - y_2 x_1 - y_2 x_2 - y_2 x_3 + x_1 x_2 + x_1 x_3 + x_2 x_3 \\
s_{\{2,3\}} &= (y_1 - x_1)(y_2 - x_1) \\
s_{\{2,4\}} &= (y_1 - x_1)(y_2 - x_1)(y_1 + y_2 - x_2 - x_3) \\
s_{\{3,4\}} &= (y_1 - x_1)(y_2 - x_1)(y_1 - x_2)(y_2 - x_2)
\end{aligned}$$

3.5. Graphical representation. We now try to formalize what we've learnt so far.

First, from what precedes we conclude that the R -matrix $\check{R}^{(w)}$ should be thought of as an operator

$$\check{R}^{(w)} : \bigotimes_{i=1}^N \mathbb{Z}[x_i]^2 \rightarrow \bigotimes_{i=1}^N \mathbb{Z}[x_{w^{-1}(i)}]^2$$

(In particular this explains why the $\check{R}^{(w)}$ can't be composed – however, the $\tau^{(w)}\check{R}^{(w)}$ can.)

Each space $\mathbb{Z}[x_i]^2$ is represented graphically by an oriented line (with the label “ x_i ”). The line can have two states, occupied or empty. Tensor product is implemented by juxtaposing lines next to each other, reading the tensor product from right to left if one looks in the direction of the orientation. Reading an equation (i.e., a product of operators) from right to left corresponds to following the orientation of lines.

The 4×4 \check{R} -matrix is now given by

$$\check{R}^{(21)} = \begin{array}{c} x_1 \quad x_2 \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \hline \end{array} = \begin{array}{ccccc} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} & \begin{array}{c} \circ \quad \bullet \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \bullet \quad \circ \end{array} & \begin{array}{c} \bullet \quad \circ \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \bullet \quad \circ \end{array} & \begin{array}{c} \circ \quad \bullet \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \circ \quad \bullet \end{array} & \begin{array}{c} \circ \quad \circ \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \circ \quad \circ \end{array} \\ 1 & x_2 - x_1 & 1 & 1 & 1 \end{array}$$

so that each line preserves its label and orientation across the crossing, but may change states.

The distinction between R and \check{R} becomes irrelevant, simply corresponding to a different way of reading the same picture.

More generally, say in size $n = 3$,

$$\check{R}_1 = \check{R}^{(213)} = \begin{array}{c} x_1 \quad x_2 \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \hline \end{array} \quad \begin{array}{c} x_3 \\ \downarrow \\ \hline \end{array}, \quad \check{R}^{(231)} = \begin{array}{c} x_1 \quad x_2 \quad x_3 \\ \diagdown \quad \diagup \quad \diagdown \\ \diagup \quad \diagdown \quad \diagup \\ \hline \end{array}$$

so that the decomposition of $R^{(w)}$ in terms of the \check{R}_i becomes graphically evident.

With these conventions, YBE and unitarity become:

$$\begin{array}{c} x_1 \quad x_2 \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \hline \end{array} = \begin{array}{c} x_1 \\ \downarrow \\ \hline \end{array} \quad \begin{array}{c} x_2 \\ \downarrow \\ \hline \end{array}, \quad \begin{array}{c} x_1 \quad x_2 \quad x_3 \\ \diagdown \quad \diagup \quad \diagdown \\ \diagup \quad \diagdown \quad \diagup \\ \hline \end{array} = \begin{array}{c} x_1 \quad x_2 \quad x_3 \\ \diagdown \quad \diagup \quad \diagdown \\ \diagup \quad \diagdown \quad \diagup \\ \hline \end{array}$$

(In all such pictures, summation over the states of internal edges, which corresponds to operator product, is implied; whereas the states of external edges are arbitrary but fixed.)

Also, $\check{R}(0) = 1$ becomes

$$\begin{array}{c} x \quad x \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \hline \end{array} = \begin{array}{c} x \quad x \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ \hline \end{array}$$

4. TRANSFER MATRICES, ALGEBRAIC BETHE ANSATZ

The R -matrix defined above has 5 nonzero entries, and is the building block of the so-called (rational) *five-vertex model*, a statistical lattice model which we describe now.

4.1. Monodromy matrix. The five-vertex model is typically defined on a finite domain of the square lattice, say a $m \times n$ rectangle. Edges of the lattice can be in two states, empty or occupied, and the partition function is the sum over all configurations of their Boltzmann weight, which is defined as the product over all vertices of the R -matrix weight $R(u_i - x_j)$ at row i and column j . In the context of integrable models the parameters u_i and x_j are called *spectral* parameters. This definition should be complemented with some choice of boundary conditions, e.g., the external edges of the domain should be fixed in some way.

In the formalism of the previous section, the partition function becomes a certain matrix element (depending on bottom and top boundary conditions) of the product of m matrices (transfer matrices) which describe one single row of the domain. These matrices can be defined as follows.

First introduce the monodromy matrix $T(u)$ as the following graphical object:

$$T(u) = \begin{array}{c} x_1 \quad x_2 \quad \dots \quad x_n \\ \leftarrow u \end{array}$$

It is an operator from $\mathcal{H} \otimes \mathbb{Z}[u]^2$ to $\mathbb{Z}[u]^2 \otimes \mathcal{H}$; we usually emphasize dependence on the formal parameter u , hence the notation $T(u)$.

For convenience, we redraw the R -matrix rotated 45 degrees:

$$\begin{array}{c} x_i \\ \leftarrow u \end{array} = \begin{array}{ccccc} \bullet & \circ & \bullet & \circ & \circ \\ \bullet & \bullet & \circ & \bullet & \circ \\ \circ & \circ & \bullet & \circ & \bullet \\ \circ & \bullet & \circ & \circ & \circ \end{array}$$

1 $u - x_i$ 1 1 1

More explicitly, it can be thought of as a 2×2 matrix of operators on \mathcal{H} , depending on the boundary conditions at the left and at the right:

$$T(u) = \begin{array}{cc} \text{left} \backslash \text{right} & \begin{array}{c} \bullet \\ \circ \end{array} \\ \begin{array}{c} \bullet \\ \circ \end{array} & \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \end{array}$$

These operators are transfer matrices with fixed (horizontal) boundary conditions.

As a consequence of the Yang–Baxter equation, $T(u)$ satisfies the so-called *RTT* relations:

Explicitly, they are quadratic relations satisfied by the operators $A(u), B(u), C(u), D(u)$:

$$\begin{aligned} A(u)A(v) &= A(v)A(u) \\ B(u)A(v) &= B(v)A(u) \\ A(u)B(v) &= (u - v)B(v)A(u) + A(v)B(u) \\ B(u)B(v) &= B(v)B(u) \\ A(u)C(v) &= A(v)C(u) \\ B(u)C(v) &= B(v)C(u) \\ A(u)D(v) &= A(v)D(u) + (u - v)B(v)C(u) \end{aligned}$$

$$\begin{aligned}
B(u)D(v) &= B(v)D(u) \\
C(u)A(v) + (u-v)A(u)C(v) &= C(v)A(u) \\
D(u)A(v) + (u-v)B(u)C(v) &= D(v)A(u) \\
(u-v)A(u)D(v) + C(u)B(v) &= (u-v)D(v)A(u) + C(v)B(u) \\
D(u)B(v) + (u-v)B(u)D(v) &= D(v)B(u) \\
C(u)C(v) &= C(v)C(u) \\
D(u)C(v) &= D(v)C(u) \\
C(u)D(v) &= (u-v)D(v)C(u) + C(v)D(u) \\
D(u)D(v) &= D(v)D(u)
\end{aligned}$$

Such a quadratic algebra is called a *Yang–Baxter algebra*.

[Small digression: if we were working with the *six-vertex model*, a slightly more general model from which the five-vertex model is obtained by a degeneration process, then this Yang–Baxter algebra would be nothing but the *Yangian* of $\mathfrak{sl}(2)$ in disguise. Here we obtain a degenerate limit of this Yangian]

In particular, as a consequence of the *RTT* relations, we find the following commutation relations for $\mathcal{T}(u) := A(u) + \kappa D(u)$:

$$[\mathcal{T}(u), \mathcal{T}(v)] = 0$$

where κ is another arbitrary parameter. $\mathcal{T}(u)$ is nothing but the *transfer matrix with twisted periodic boundary conditions* of the five-vertex model.

Let us now briefly discuss the geometric meaning of these operators.

We shall show in what follows that $A(u)$, restricted to each weight space of \mathcal{H} , is a multiplication operator (by a certain u -dependent class) in the equivariant cohomology of $Gr(k, n)$. Collectively, the $A(u)$ generate the cohomology algebra viewed as acting on itself by multiplication.

On the other hand, note that $B(u)$ and $C(u)$ change the value of σ^z by ± 1 (i.e., jump from one Grassmannian to another); they can be defined in terms of certain *correspondences*, i.e., are part of a *convolution* algebra. $D(u)$ can be expressed in terms of $B(u)$, $C(u)$ and $A(u)^{-1}$. All of these interpretations will be discussed in Sect. 4.5.

More generally, $\mathcal{T}(u) = A(u) + \kappa D(u)$ is a multiplication operator in the (small) *quantum* cohomology ring of $Gr(k, n)$.

4.2. Algebraic Bethe Ansatz. There is a standard method to diagonalize the transfer matrix $\mathcal{T}(u) = A(u) + \kappa D(u)$, called Algebraic Bethe Ansatz. The “Ansatz” is to assume that (right) eigenvectors of $\mathcal{T}(u)$ are of the form

$$C(y_1) \dots C(y_k) |\emptyset\rangle$$

where $|\emptyset\rangle$ is the all empty state, and the y_i are parameters to be determined. Similarly, left eigenvectors are taken to be of the form

$$\langle \emptyset | B(y_1) \dots B(y_k)$$

Fact. $C(y_1) \dots C(y_k) |\emptyset\rangle$ (resp. $\langle \emptyset | B(y_1) \dots B(y_k)$) is a right (resp. left) eigenvector of $\mathcal{T}(u) = A(u) + \kappa D(u)$ iff the y_i are distinct and satisfy the equations:

$$(5) \quad \prod_{j=1}^n (y_i - x_j) = (-1)^{k-1} \kappa$$

The corresponding eigenvalue is

$$(6) \quad \frac{(-1)^k \kappa + \prod_{j=1}^n (u - x_j)}{\prod_{i=1}^k (u - y_i)}$$

Sketch of proof. Apply repeatedly the commutation relations between $A(u)$ and $D(u)$ on the one hand, and $B(y_i)$ on the other, among the relations of the Yang–Baxter algebra. Then separate the “good” terms (the ones which we want for the eigenvector equation) from the “bad” terms (the rest). The Bethe equations are exactly the coefficient in front of the bad terms. \square

Remark: in general, Bethe equations are *coupled* equations between the y_i . The fact that here each equation depends on only one y_i is a sign that this model is not just integrable: it’s actually a *free* (fermionic) theory.

4.3. Fixed points as Bethe states. From now on we specialize to $\kappa = 0$, i.e., look at eigenvectors of $A(u)$.

First, note that it is trivial to solve the Bethe equations, namely, the y_i have to satisfy $\prod_{j=1}^n (y_i - x_j) = 0$ and be distinct, therefore

$$y_i = x_{I_i}, \quad i = 1, \dots, k$$

for some subset $I \subset \{1, \dots, n\}$. (the operators $B(y_i)$ commute so the ordering of the I_i is unimportant). We denote these left and right eigenvectors, with a bit of foresight, $\langle \mathbb{C}^I |$ and $|\mathbb{C}^I \rangle$. The corresponding eigenvalue is:

$$(7) \quad A(u) |\mathbb{C}^I \rangle = \prod_{\substack{j=1 \\ j \notin I}}^n (u - x_j) |\mathbb{C}^I \rangle$$

and the same for $\langle \mathbb{C}^I |$. In fact, these eigenvalues can easily be found without Bethe Ansatz since $A(u)$ is a triangular matrix.

Fact. $|\mathbb{C}^I \rangle$ and $\langle \mathbb{C}^I |$ are the class of the fixed point \mathbb{C}^I .

Proof. The proof consists in showing that the $|\mathbb{C}^I \rangle$, as well as the classes of the fixed points, satisfy two properties which fix them entirely:

- (1) The R -matrix permutes them:

$$\tau^{(w)} \check{R}^{(w)} |\mathbb{C}^I \rangle = |\mathbb{C}^{w(I)} \rangle$$

(obvious graphical proof for w elementary transposition), which matches with the natural geometric action of w , $\mathbb{C}^I w^{-1} = \mathbb{C}^{w(I)}$.

- (2) They agree in a special case: if $I = \{n - k + 1, \dots, n\}$, then the fixed point \mathbb{C}^I is nothing but the Schubert variety/cell S_I , and the eigenvector is nothing but $|\circ \cdots \circ \bullet \cdots \bullet \rangle$ by triangularity of $A(u)$.

A similar proof works for $\langle \mathbb{C}^I |$, except the base case is $S_I^{(w_0)} = \mathbb{C}^I$ with $I = \{1, \dots, k\}$. \square

Remark: The diagonalization basis of $A(u)$ is usually called the F -basis (F for factorizing), because it factorizes the action of the R -matrix. It is related to the notion of (factorizing) Drinfeld twist.

Example. in $Gr(1, 2)$,

$$C(x_1) | \circ \circ \rangle = (x_1 - x_2) | \bullet \circ \rangle + | \circ \bullet \rangle = [0]$$

$$C(x_2) | \circ \circ \rangle = | \circ \bullet \rangle = [\infty]$$

$$\langle \circ \circ | B(x_1) = \langle \bullet \circ | = [0]$$

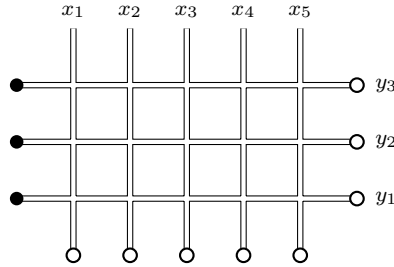
$$\langle \circ \circ | B(x_2) = (x_2 - x_1) \langle \circ \bullet | + \langle \bullet \circ | = [\infty]$$

4.4. **Applications.** The result of the previous section has beautiful applications. Let us name three.

First, consider the following bra-ket

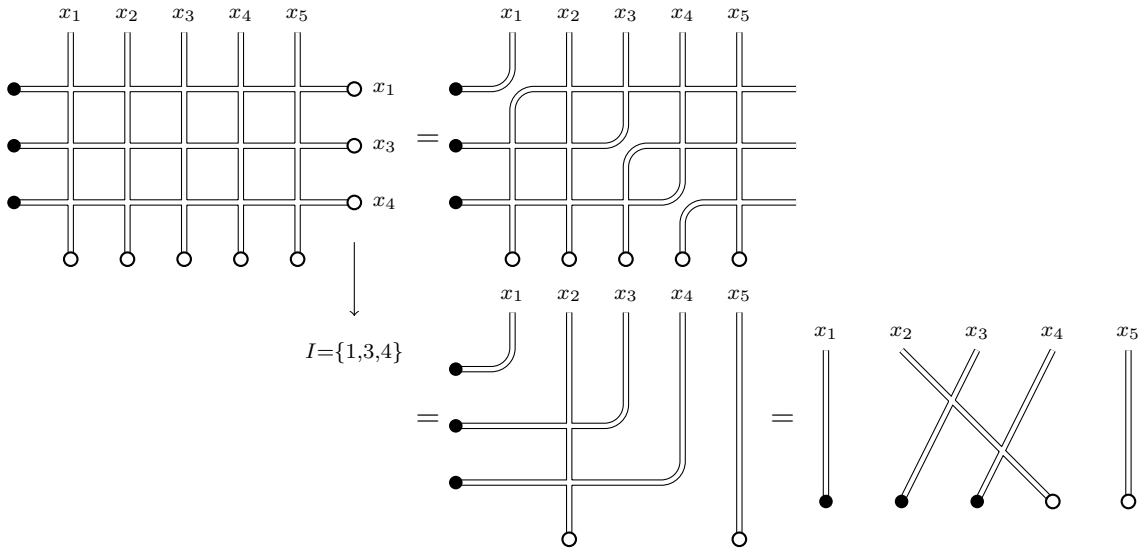
$$\langle \emptyset | B(y_1) \dots B(y_k) | s_J \rangle$$

where recall that $| s_J \rangle$ is just the standard basis vector with occupied sites in J . Graphically, this is the following partition function:



where top endpoints are occupied at J . When we specialize $y_i = x_{I_i}$ for some k -subset I , this is the scalar product of the fixed point \mathbb{C}^I and of s_J , i.e., it is the restriction of the Schubert class s_J to this fixed point. Comparing with equivariant localization formulae, we conclude that $\langle \emptyset | B(y_1) \dots B(y_k) | s_J \rangle$ is the factorial Schur polynomial associated to J (and the Bethe roots y_i play the role of Chern roots). This gives an explicit combinatorial formula for this polynomial as the partition function of the 5-vertex model (this formula is equivalent to a sum over Semi-Standard Young Tableaux).

Secondly, in the proof of the result, we've found that every fixed point can be obtained from a known fixed point (namely $\mathbb{C}^{\{n-k+1, \dots, n\}}$ or $\mathbb{C}^{\{1, \dots, k\}}$ depending on whether one uses kets or bras), by application of the R -matrix. Alternatively, one can start directly from the definition of $\langle \mathbb{C}^I |$ as $\langle \emptyset | B(y_1) \dots B(y_k)$ and replace $y_i = x_{I_i}$:



In the last picture, we recognize the diagram of the minimal (Grassmannian) permutation sending $\{1, \dots, k\}$ to I . Fixing the top endpoints to say the subset J of occupied vertices, this gives a combinatorial formula for the Schubert class s_J restricted to the fixed point \mathbb{C}^I , or equivalently the Schur polynomial s_J specialized at $y_i = x_{I_i}$. This is a special case of the so-called Anderson–Jantzen–Soergel–Billey formula (the general formula can be obtained in the exact same fashion, but in a higher rank model, see the section on generalizations).

Finally, we note that according to general equivariant localization formulae, we find the norm of the Bethe states:

$$\langle \mathbb{C}^I | \mathbb{C}^J \rangle = \delta_{IJ} E_I = \delta_{IJ} \prod_{\substack{i=1 \\ i \in I}}^n \prod_{\substack{j=1 \\ j \notin I}}^n (x_i - x_j)$$

4.5. Geometric interpretation of monodromy matrix elements. The reason that $A(u)$ is singled out among the four operators made out of the monodromy matrix (as the one that is diagonal in the fixed point basis) can be understood as follows. Consider the two simple geometric operations

$$\begin{aligned} f, f' : Gr(k, n) &\rightarrow Gr(k+1, n+1) \\ f : V &\mapsto V \times \mathbb{C} = V \oplus \mathbb{C}^{\{n+1\}} \\ f' : V &\mapsto \mathbb{C} \times V = wf(V) \end{aligned}$$

where in the first definition, we view \mathbb{C}^n as embedded in \mathbb{C}^{n+1} in the natural way, and w is the permutation $(2, \dots, n+1, 1)$. We define the torus action on $Gr(k+1, n+1)$ by giving to basis vectors weights (x_1, \dots, x_n, u) (resp. (u, x_1, \dots, x_n)) for f (resp. f'), making both maps equivariant. The operations f_\star (resp. f'^\star) correspond respectively to adding a \bullet to the right (resp. removing a \bullet to the left, or producing zero if there isn't one).¹ Note that no such interpretation exists for f^\star and f'_\star , due to the “asymmetric” definition of Schubert varieties as $B = B_+$ -orbits rather than B_- -orbits.

From the geometric definition of the R -matrix, w is the Weyl group element naturally associated to the monodromy matrix:

$$T(u) = \begin{array}{c} x_1 \quad x_2 \quad \dots \quad x_n \quad u \\ \begin{array}{c} \diagdown \quad \diagdown \quad \dots \quad \diagdown \quad \diagdown \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \diagup \quad \diagup \quad \dots \quad \diagup \quad \diagup \end{array} \end{array}$$

We can then write:²

$$A(u) = f'^\star T(u) f_\star = f^\star f_\star = f'^\star f'_\star$$

(The last equality just expresses the fact that $f = wf'$ and f' only differ by an automorphism of $Gr(k+1, n+1)$)

The action of $A(u)$ on Schubert classes is nontrivial, but on fixed points it is quite simple: f_\star takes a fixed point \mathbb{C}^I to $\mathbb{C}^{I \cup \{n+1\}}$, and f^\star removes the final entry, thus producing \mathbb{C}^I again, except as always with pullbacks one has a corrective factor which is the ratio of normalizations $E_{I \cup \{n+1\}}/E_I$. Using (3), we recover the eigenvalue (7).

From this we conclude that $A(u)$ is a multiplication operator as follows:³ since f^\star is clearly surjective, given $|x\rangle \in H^*Gr(k, n)$, $|x\rangle = f^\star |y\rangle$, $A(u) |x\rangle = f^\star (f_\star f^\star |y\rangle) = f^\star (f_\star(1) |y\rangle) =$

¹Here I always assume that one is acting on kets – if acting on bras, use f^\star and f'_\star instead.

²In these expressions, there is no need to worry about the action of the Weyl group on $H_T^*(\cdot)$: by choosing different actions on $Gr(k+1, n+1)$, we've made all our maps T -equivariant.

³I thank V. Gorbunov for pointing this out to me.

$(A(u) \cdot 1 | x \rangle$, where 1 is the unit of the ring $H^*Gr(k, n)$. One can compute $A(u) \cdot 1$ explicitly: it is the total Chern class of the complement of the tautological bundle.

The interpretation of the other operators $B(u)$, $C(u)$, $D(u)$, is slightly more complicated. The problem is that addition of a \circ to the right (resp. removal of a \circ to the left), has no obvious geometric interpretation. We therefore use the following trick:

Lemma. *Consider the two natural maps g and h from $Gr(k, 1; n) = \{V \subset W, \dim V = k, \dim(W/V) = 1\}$ to $Gr(k, n)$ and $Gr(k+1, n)$, namely $g : (V, W) \mapsto V$ and $h : (V, W) \mapsto W$. Then $g_* h^* = \sigma_n^+$ and $h_* g^* = \sigma_1^-$, or more explicitly*

$$\begin{aligned} g_* h^* |\cdots \bullet \rangle &= |\cdots \circ \rangle \\ g_* h^* |\cdots \circ \rangle &= 0 \\ h_* g^* |\bullet \cdots \rangle &= 0 \\ h_* g^* |\circ \cdots \rangle &= |\bullet \cdots \rangle \end{aligned}$$

where the \cdots stand for an arbitrary binary string.

Sketch of proof. Consider say $h_* g^*$. This operation is easy to describe homologically. Given a subvariety X of $Gr(k, n)$, consider

$$X^+ = \{W \supset V, \dim W = k+1, V \in X\} \subset Gr(k+1, n)$$

Then

$$h_* g^*[X] = \begin{cases} [X^+] & \text{if } \text{codim } X^+ = \text{codim } X - k \\ 0 & \text{if } \text{codim } X^+ > \text{codim } X - k \end{cases}$$

(up to multiplicity issues which are irrelevant here)

Now consider the case of a basis vector $|I\rangle$, that is of a Schubert variety \overline{S}_I . By definition it is given by the inequalities $\dim(V \cap V_i) \geq d_i$ where (V_i) is the reference flag. Since $W \supset V$, we also have $\dim(W \cap V_i) \geq d_i$. Now use the equality

$$\text{codim } \overline{S}_I = \sum_{i=0}^{n-1} d_i - \frac{k(k-1)}{2}$$

to conclude that \overline{S}_I^+ is contained in the union of Schubert cells S_J which satisfy

$$\begin{aligned} \text{codim } S_J &= \sum_{i=0}^{n-1} \dim(W \cap V_i) - \frac{k(k+1)}{2} \\ &\geq \sum_{i=0}^{n-1} \dim(V \cap V_i) - \frac{k(k+1)}{2} \\ &= \text{codim } \overline{S}_I - k \end{aligned}$$

with equality iff $\dim(V \cap V_i) = \dim(W \cap V_i)$ for $i = 0, \dots, n-1$. By inspection this is only possible if $d_{n-1} = k$, i.e., $1 \notin I$ (the binary string starts with a \circ), in which case J is obtained from I by replacing the first \circ with a \bullet . It is then easy to see that indeed \overline{S}_I^+ contains this S_J . \square

Note that g and h are not only T -equivariant, but actually G -equivariant, which means they commute with the Weyl group action as well. Now we can compose these various

Given an eigenstate $|\Psi\rangle$, factor the corresponding eigenvalue as

$$(9) \quad \mathcal{Q}(u) |\Psi\rangle = \prod_{i=1}^k (y_i - u) |\Psi\rangle$$

(taking into account the leading term $Q(u) \sim (-1)^k u^k$ as $u \rightarrow \infty$).

Setting $u = y_i$ in the TQ relation, we immediately recover the Bethe equations for the y_i . Furthermore, the eigenvalue of $\mathcal{T}(u)$ matches with the one found earlier for the Bethe state.

But we can do better. Rewriting the TQ relation in terms of the y_i leads to the following constraint

$$\frac{\prod_{i=1}^k (u - y_i)}{\kappa + (-1)^k \prod_{i=1}^n (u - x_i)} = (\text{polynomial of degree } n - k \text{ in } u)$$

These are nothing but the defining relations of the quantum equivariant cohomology of $Gr(k, n)$. In particular, setting $\kappa = 0$, we recover the relations of $H_T^*(Gr(k, n))$ (in the earlier notations, $u = -1/t$).

[comment on how Pieri rule is encoded in matrix elements of these two operators...]

Various results that we derived for untilded operators can be analogously derived for tilded ones. For instance. If one introduces the two natural inclusions \tilde{f} and \tilde{f}' of $Gr(k, n)$ into $Gr(k, n + 1)$ by adding one more copy of \mathbb{C} to the ambient space left/right (cf a similar definition of f and f'), with an additional equivariant parameter u , then one has $\tilde{D}(u) = \tilde{f}^* \tilde{f}_* = \tilde{f}'^* \tilde{f}'_*$, and one immediately concludes that $\tilde{D}(u)$ has fixed points \mathbb{C}^I as eigenvectors with eigenvalue $E_{I\mathbb{C}[n+1]}/E_{I\mathbb{C}[n]} = \prod_{i=1}^k (x_{I_i} - u)$, which is consistent with (9) after identification of the y_i with the Chern roots. This allows us to interpret $\tilde{D}(u)$ as the multiplication operator by the total Chern class of the tautological bundle on $Gr(k, n)$.

One can also make a “dual” Bethe Ansatz involving acting with \tilde{B} (resp. \tilde{C}) on the fully occupied ket (resp. bra).

5. OTHER POINTS I NEVER GOT TO

5.1. **Pairing and 180 degree rotation.** ...

5.2. **Determinantal methods.** ...

5.3. **Large size limit: free fermionic CFT.** Back to Schur functions... Fock space, charge, infinite transfer matrices, shift operator. branching rule.

6. GENERALIZATIONS

There are various directions of generalizations, which can be arbitrarily combined. We name four.

6.1. **Higher rank and partial flag varieties.** We can obtain higher rank models by considering partial flag varieties. For example, consider the *two-step flag variety*:

$$Gr(k_1, k_2; n) := \{V_1 \subset V_2 \text{ linear subspaces of } \mathbb{C}^n, \dim_{\mathbb{C}} V_1 = k_1, \dim_{\mathbb{C}}(V_2/V_1) = k_2\}$$

Fixed points are indexed by partitions of $\{1, \dots, n\}$ into subsets I, J, K , with $|I| = k_1$, $|J| = k_2$ (and therefore $|K| = n - k_1 - k_2$), such that $V_1 = \mathbb{C}^I$ and $V_2 = \mathbb{C}^{I \cup J}$. Their B -orbit closures are Schubert varieties. Their classes form the standard basis of a Hilbert space which now has three states per site, i.e., of the form $\mathcal{H} = (\text{some completion of}) \bigotimes_{i=1}^n \mathbb{Q}(x_i)^3$.

This way one defines 9×9 R -matrices, and more generally, for the $(r - 1)$ -step flag variety, $r^2 \times r^2$ R -matrices. Note that these can be related to various representations of the so-called *nil-Hecke algebra*. They can also be thought of as a degenerate limit of $sl(r)$ -invariant integrable models (see cotangent bundle section).

In higher rank, Bethe Ansatz is more complicated (so-called Nested Bethe Ansatz), but in the limit of zero twist – which corresponds to ordinary (non-quantum!) equivariant cohomology – it simplifies, and we obtain very similar formulae to the Grassmannian case. These are all particular cases of so-called (reduced) pipedream formulae for Schubert classes/polynomials [16, 7, 6].

6.2. K -theory. We can replace equivariant cohomology with equivariant K -theory. Schubert varieties have K -classes (classes of their structure sheaves in homological K -theory), and the exact same procedure works, leading to an R -matrix of the form

$$\check{R}^{(21)} = \begin{pmatrix} 1 & & & & \\ & 1 & 1 - z_2/z_1 & & \\ & 0 & z_2/z_1 & & \\ & & & & \\ & & & & 1 \end{pmatrix}$$

for the Grassmannian. This is the so-called *trigonometric* five-vertex model. (The word *trigonometric* comes from the fact that the R -matrix now depends on ratios, rather than differences, of equivariant/spectral parameters; performing the substitution $z_i = e^{x_i}$, the R -matrix depends again on differences of x_i , at the expense of its entries being trigonometric functions of the x_i .)

The rational limit (expressing the limit from K -theory to cohomology, cf the Grothendieck–Riemann–Roch theorem) corresponds to expanding at first nontrivial order in x_i after substituting $z_i = e^{x_i}$.

K -theory also works in connection with higher rank models, and is again related to pipedreams, this time for Grothendieck polynomials [5].

6.3. Cotangent bundle. Perhaps the most important generalization is to go over to the cotangent bundle, say $T^*Gr(k, n)$. This corresponds to lifting from the 5-vertex model (a somewhat degenerate model) to the *6-vertex model*, possibly the most important integrable model of two-dimensional statistical mechanics.

On the geometric side, however, there are complications. The question is to find a natural analogue of the basis of Schubert varieties. A naive guess is to use closures of conormal bundles of Schubert cells. Unfortunately the corresponding R -matrix does not satisfy the “locality” property (i.e., the basis does not display a tensor product structure of the Hilbert space in such a way that \check{R}_i only acts on i^{th} and $(i + 1)^{\text{th}}$ factors of the tensor product). On the physical side however, it leads to interesting connections to models with nonlocal interaction (loop models) [4, 13].

A better solution it to consider the *stable basis* thanks to the stable envelope construction of Maulik–Okounkov [18], ... Relation to Kazhdan–Lusztig theory...

In ordinary cohomology, we get the rational (or $sl(2)$ -invariant) six-vertex model ($\Delta = \pm 1$ depending on sign conventions). In K -theory, we obtain the trigonometric (general Δ) six-vertex model [21].

One can then go beyond cotangent bundles of partial flag varieties and consider arbitrary *Nakakima quiver varieties* [19]... In this context, the Weyl group action point of view is a little too naive (i.e., R -matrices still correspond to change of bases in cohomology, but the

different bases are not related by Weyl group action), though we still get solutions of the Yang–Baxter equation this way.

6.4. Other gauge groups and boundary conditions. All partial flag varieties are related to $GL(n)$ in the sense that they are homogeneous spaces G/P where $G = GL(n)$ and P is some parabolic subgroup. Similarly Nakajima quiver varieties are usually defined with gauge group of type $GL(n)$. Using other (classical) Lie groups as gauge groups should lead to integrable models with different boundary conditions. This should be understood as follows: the R -matrix is defined in terms of the Weyl group of the gauge group. In particular in type A (i.e., $SL(n)$ or $GL(n)$), every edge of the Dynkin diagram is of the form $\bullet_s \text{---} \bullet_t$, leading to the Coxeter relation $(st)^3 = 1$ and ultimately to the usual Yang–Baxter equation. In contrast, in type say C (symplectic group), there is at the boundary of the Dynkin diagram a double edge $\bullet_s \text{---} \bullet_t$, leading to $(st)^4 = 1$ and to a variation of the Yang–Baxter equation called *reflection equation* (or boundary Yang–Baxter equation). Physically, it means that the model has a nontrivial boundary.

It is not fully understood at present how to get, given a solution of YBE, the most general associated solutions of the reflection equation from the geometry.

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