Weingarten matrices and Jucys–Murphy elements

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(see also Matsumoto+Novak: 0905.1992 and Matsumoto: 1001.2345)

Outline of the talk

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1. Weingarten matrices
   - Introduction
   - General formula for $O(N)$
   - General formula for $U(N)$
2. Jucys–Murphy elements
   - Definition
   - Application to $U(N)$ Weingarten matrix
   - Back to the $O(N)$ case

Example (A typical problem)

We wish to compute

$$\int_{O(N)} dU \; P(U_{1,1}, \ldots, U_{N,N})$$

where we integrate over orthogonal matrices $U = (U_{i,j})_{i,j=1,\ldots,N}$, dU is the Haar measure and P is a polynomial.

Useful in

- Lattice gauge theory [Weingarten, 70's].
- Random Matrix Theory (CUE). $N \to \infty$.
- Free probability [Collins, 00's].
- Statistical loop models.
- Invariant theory (next slide).

Representation-theoretic aspect

$$\int_{O(N)} dU\, U_{i_1,j_1} \dots U_{i_{2k},j_{2k}} = \left(\int dU\, \rho(U)\right)_{i,j} \qquad \begin{array}{l} i = (i_1, \dots, i_{2k}) \\ j = (j_1, \dots, j_{2k}) \\ \rho = \text{rep on } (\mathbb{R}^N)^{\otimes 2k} \end{array}$$

$$= \sum_{R\ irrep} \left(\int dU\, \rho(U) P^R\right)_{i,j} \qquad \sum_{R\ irrep} P^R = 1$$

$$= P^{\varnothing}_{i,j}$$

So we are looking for an explicit description of the projector onto the trivial subrepresentation of $(\mathbb{R}^N)^{\otimes 2k}$.

Example ($O(N)$, $2k = 2, 4$)

$\underline{2k = 2:}$

$$\int_{O(N)} dU\, U_{i_1,j_1} U_{i_2,j_2} = \frac{1}{N}\delta_{i_1,i_2}\delta_{j_1,j_2} = \frac{1}{N}\left|1\right\rangle\left\langle 1\right|$$

where

$\left|1\right\rangle =$

$\left\langle 1 \middle| 1\right\rangle = N$

$\underline{2k = 4:}$

$\left|1\right\rangle =$

$\left|2\right\rangle =$

$\left|3\right\rangle =$

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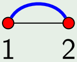
$|1\rangle =$ 

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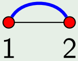
$|1\rangle =$ 

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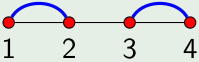
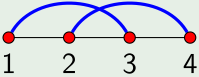
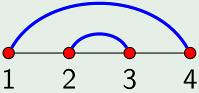
$|1\rangle =$ 

$|2\rangle =$ 

$|3\rangle =$ 

Weingarten matrix as a pseudo-inverse

Question: given a generating set $\{|i\rangle\}$ of a subspace of a Euclidean space, how to write the orthogonal projector P onto this subspace?

Answer: compute the Gramm matrix:

$$G_{ij} = \langle i | j \rangle$$

and take its (pseudo-)inverse:

$$W = W^T \qquad GWG = G \qquad WGW = W$$

Then

$$P = \sum_{i,j} W_{i,j} \, |i\rangle \, \langle j|$$

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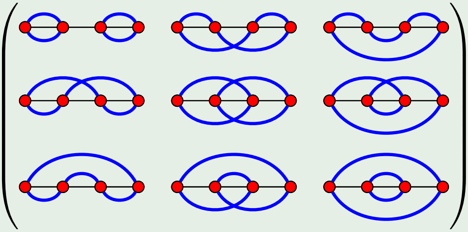
Example ($O(N)$, $2k = 4$, cont'd)

$G =$



$$= \begin{pmatrix} N^2 & N & N \\ N & N^2 & N \\ N & N & N^2 \end{pmatrix}$$

$$W = \frac{1}{N(N-1)(N+2)} \begin{pmatrix} N+1 & -1 & -1 \\ -1 & N+1 & -1 \\ -1 & -1 & N+1 \end{pmatrix}$$

A matching is a set partition into pairs. Call $\mathcal{B}_k$ the set of matchings of $\{1, \ldots, 2k\}$.

$$\int_{O(N)} dU\, U_{i_1,j_1} \ldots U_{i_{2k},j_{2k}} = \sum_{\pi,\pi' \in \mathcal{B}_k} W_{\pi,\pi'} \prod_{(a,b) \text{ paired in } \pi} \delta_{i_a,i_b} \prod_{(a,b) \text{ paired in } \pi'} \delta_{j_a,j_b}$$

where W is the pseudo-inverse of G:

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Remark: $\#\mathcal{B}_k = (2k-1)!!$, $\dim(\mathbb{R}^N)^{\otimes 2k} = N^{2k}$.
But! viewing matchings as fixed-point-free involutions,

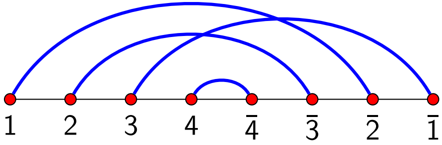
Theorem (Baik, Rains, '01)

Let $\mathcal{B}_k^{(N)}$ be the subset of $\mathcal{B}_k$ of fixed-point-free involutions with no decreasing subsequence of length $N+1$ (i.e. whose RS diagram has at most N rows). Then $\{|\pi\rangle, \pi \in \mathcal{B}_k^{(N)}\}$ is a basis of the invariant subspace of $(\mathbb{R}^N)^{\otimes 2k}$.

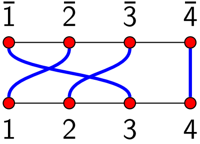
$$\int_{U(N)} dU\, U_{i_1,j_1} \ldots U_{i_k,j_k} \bar{U}_{\bar{\imath}_1,\bar{\jmath}_1} \ldots \bar{U}_{\bar{\imath}_k,\bar{\jmath}_k} = P^{\varnothing}_{i,j}$$

where the projector acts on $V^{\otimes k} \otimes (V^{\star})^{\otimes k}$, $V \cong \mathbb{C}^N$.

Invariants:



or



$$\int_{U(N)} dU\, U_{i_1,j_1} \ldots U_{i_k,j_k} \bar{U}_{\bar{i}_1,\bar{j}_1} \ldots \bar{U}_{\bar{\imath}_k,\bar{\jmath}_k}$$

$$= \sum_{\sigma,\sigma' \in \mathcal{S}_k} W_{\sigma,\sigma'} \prod_{a=1,\ldots,k} \delta_{i_a,\bar{\imath}_{\sigma(a)}} \prod_{a=1,\ldots,k} \delta_{j_a,\bar{\jmath}_{\sigma'(a)}}$$

where W is the pseudo-inverse of G:

$$G = \langle \sigma | \sigma' \rangle = N^{\#\text{cycles of } \sigma^{-1}\sigma'}$$

Remark: $\#\mathcal{S}_k = k!$, $\dim(\mathbb{C}^N)^{\otimes 2k} = N^{2k}$.
But!

Theorem (Baik, Rains, '01)

Let $\mathcal{S}_k^{(N)}$ be the subset of $\mathcal{S}_k$ of permutations with no decreasing subsequence of length $N+1$ (i.e. whose RS diagram has at most N rows). Then $\{|\sigma\rangle, \sigma \in \mathcal{S}_k^{(N)}\}$ is a basis of the invariant subspace of $V^{\otimes k} \otimes (V^\star)^{\otimes k}$.

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Define the Jucys–Murphy elements to be

$$m_1 := 0 \qquad m_j := \sum_{i=1}^{j-1} (ij) \qquad j = 2, \ldots, k$$

They form a maximal commutative subalgebra of $\mathbb{C}[\mathcal{S}_k]$.
If we write $\mathbb{C}[\mathcal{S}_k] = \bigoplus_\lambda M(d_\lambda, \mathbb{C})$, then the m_j are just a bunch of diagonal matrices.
In this subalgebra, the rank 1 projectors (elementary diagonal matrices) are Young's orthogonal idempotents. They are naturally indexed by Standard Young Tableaux with k boxes. We denote them e_T. Note that

$$P_\lambda := \sum_{T \text{ of shape } \lambda} e_T$$

is the central idempotent associated to the Young diagram λ.

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The eigenvalues of m_j are the contents of the boxes j of the tableaux:

$$m_j e_T = e_T m_j = c_T(j) e_j$$

Example

$$m_5\, e_{\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&6\\\cline{1-2} 4\\\cline{1-1}\end{array}} \to \begin{array}{cccc} & & & 3\\ 1 & \boxed{1} & \boxed{3} & \boxed{5}\\ & \boxed{2} & \boxed{6} & \\ & \boxed{4} & & \end{array} = 2\, e_{\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&6\\\cline{1-2} 4\\\cline{1-1}\end{array}}$$

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$$= \quad 2\, e_{\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&6\\\cline{1-2} 4\\\cline{1-1}\end{array}}$$

Jucys' formula

The eigenvalues of the m_j are permutations of each other for all tableaux of a given shape! Therefore symmetric polynomials of the m_j are central. In particular there is the following explicit form for elementary symmetric polynomials:

Theorem (Jucys, '71)

$$\prod_{j=1}^{k}(t+m_j) = \sum_{\sigma\in\mathcal{S}_k} \sigma \; t^{\#\text{cycles of } \sigma}$$

Note that according to the above, we also have

$$\prod_{j=1}^{k}(t+m_j) = \sum_{\lambda\vdash k} P_\lambda \prod_{(i,j)\in\lambda}(t+j-i)$$

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We now notice that the $U(N)$ Gramm matrix G is the matrix of left or right multiplication by $\prod_{j=1}^{k}(N+m_j)$ on $\mathbb{C}[\mathcal{S}_k]$.

Theorem (Collins, '03)

The $U(N)$ Weingarten matrix W is the matrix in both left or right regular representation of the operator

$$W = \sum_{\substack{\lambda \vdash n \\ c_\lambda \neq 0}} c_\lambda^{-1} P_\lambda$$

where

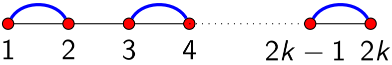
$$c_\lambda := \prod_{(i,j)\in\lambda} (N + j - i)$$

If $G = \prod_{j=1}^{k}(N + m_j)$, then for N sufficiently large

$$W = N^{-k} \prod_{j=1}^{k} (1 + m_j/N)^{-1}$$
$$= \sum_{i=0}^{\infty} \frac{1}{N^{k+i}} (-1)^i h_i(m_1, m_2, \ldots, m_k)$$

where the h_i are the complete symmetric functions.
This provides the $1/N$ expansion of W as $N \to \infty$.

$\mathcal{S}_{2k}$ acts transitively on $\mathcal{B}_k$. Explicitly, $\mathcal{B}_k \cong \mathcal{S}_{2k}/\mathcal{H}_k$ where $\mathcal{H}_k$ is the hyperoctahedral group. Choosing a particular element of $\mathcal{B}_k$:

$\beta_k =$ 

identifies $\mathcal{H}_k$ with the stabilizer of β_k.

Therefore, there is a natural inclusion: $\mathbb{C}[\mathcal{B}_k] \subset \mathbb{C}[\mathcal{S}_{2k}]$ that sends a matching π viewed as a coset to the sum of its elements $\sum_{\sigma\in\pi}\sigma$.

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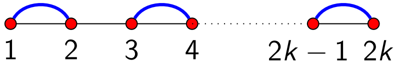
$\beta_k =$ 

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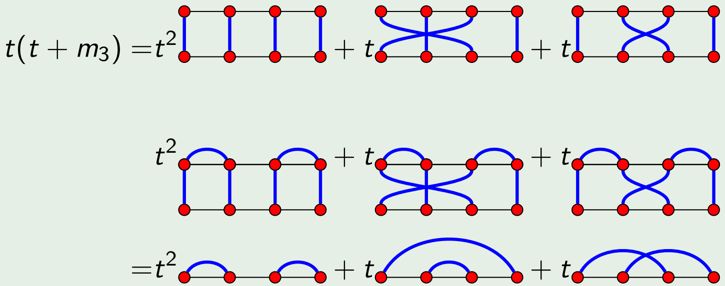
A variation of Jucys' formula

Proposition (PZJ, '09)

There exists a choice of representatives of cosets of $\mathcal{S}_{2k}/\mathcal{H}_k$, that is of $\sigma_\pi \in \pi$ such that

$$\prod_{j=1}^{k}(t + m_{2j-1}) = \sum_{\pi \in \mathcal{B}_k} \sigma_\pi \; t^{\#\text{loops of } \beta_k \cup \pi}$$

Example ($2k = 4$)



Proposition (PZJ, '09)

The $O(N)$ Gramm matrix G is the matrix of $\prod_{j=1}^{k}(N + m_{2j-1})$ acting by multiplication on the right of $\mathbb{C}[\mathcal{B}_k]$.

The restriction to $\mathbb{C}[\mathcal{B}_k]$ implies that one should only consider Young tableaux obtained by the "doubling procedure":

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \longrightarrow \begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 6 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}$$

Proposition (Collins, Matsumoto, '09)

The $O(N)$ Weingarten matrix W is the matrix of

$$\sum_{\substack{\lambda\vdash n\\ c_{\lambda;2}\neq 0}} c_{\lambda;2}^{-1} P_{2\lambda}$$

acting by multiplication on the left or right on $\mathbb{C}[\mathcal{B}_k]$, where

$$c_{\lambda;2} := \prod_{(i,j)\in\lambda} (N+2j-1-i)$$

and also for N sufficiently large,

$$W = \sum_{i=0}^{\infty} \frac{1}{N^{k+i}} (-1)^i h_i(m_1, m_3, \ldots, m_{2k-1})$$

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Example ($2k = 4$)

$$G = N(N+2)P_{\square\square\square\square} + N(N-1)P_{\substack{\square\square \\ \square\square}}$$

and indeed

$$G = \begin{pmatrix} N^2 & N & N \\ N & N^2 & N \\ N & N & N^2 \end{pmatrix}$$

has eigenvalue $N(N+2)$ with multiplicity 1 and $N(N-1)$ with multiplicity 2.

- Symplectic case: $Sp(2N) \cong O(-2N)$.
- More general β-ensemble (use Jack polynomials).
- Quantum group analogues?