Quantum Knizhnik–Zamolodchikov equation, pipe dreams and all that

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Joint work with P. Di Francesco, A. Knutson.

Loop models

- Loop models are prototypical two-dimensional lattice models with nonlocal degrees of freedom.
- Several such models enjoy the property of being exactly solvable.
- Here we focus on one particular model: the Brauer loop model (dense crossing loops). Furthermore we specialize the loop fugacity to 1.
- This model was first investigated in [Martins, Nienhuis, Rietman '98]; surprisingly, degrees of certain algebraic varieties appear in its ground state entries, as noticed in [de Gier, Nienhuis '04].
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  Use of the quantum Knizhnik–Zamolodchikov equation.

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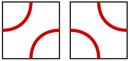
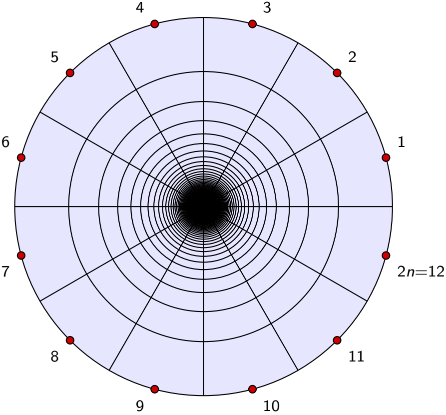
Brauer loop model

 with probability $4/9$

 with probability $1/9$

Probability of connectivity of external vertices?



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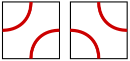 with probability $4/9$

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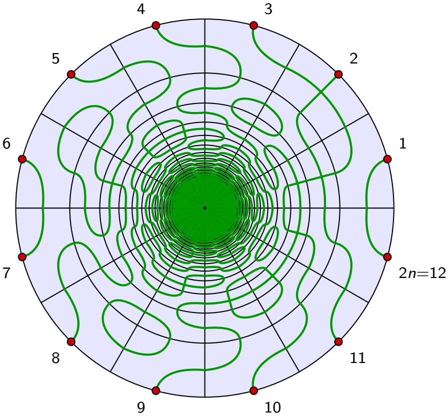

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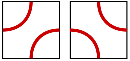 with probability 4/9

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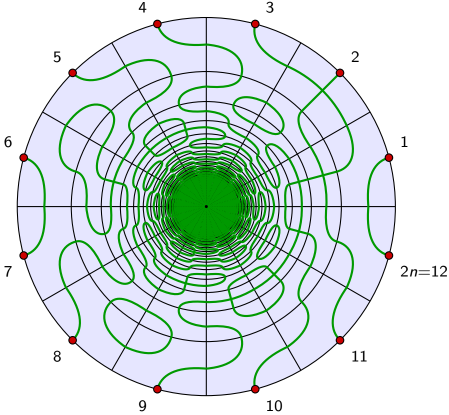

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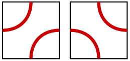 with probability $4/9$

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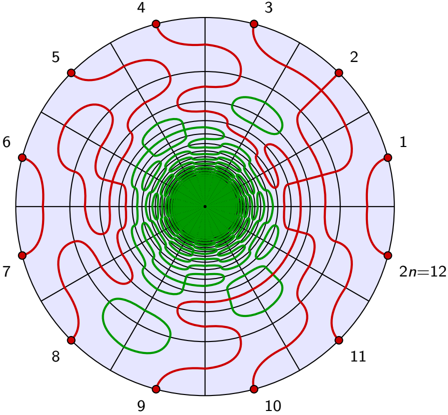
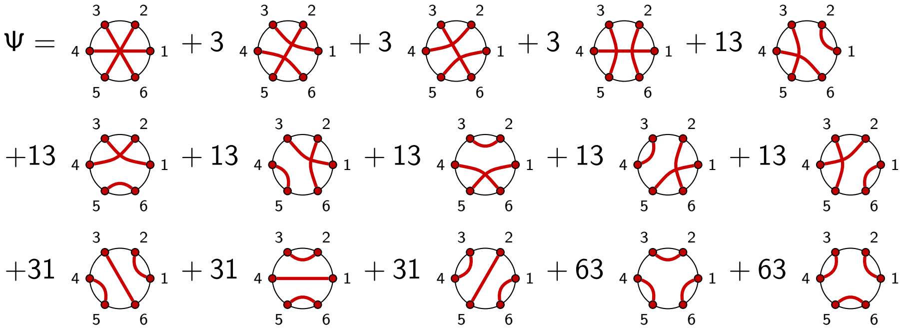


These probabilities can be encoded in a vector Ψ, whose components Ψ_π are indexed by crossing link patterns π, or chord diagrams, or fixed-point free involutions.

Example ($N = 2n = 6$)

Up to normalization,



Inhomogeneous Brauer loop model

Introduce local probabilities dependent on the column i via a parameter x_i respecting integrability of the model (i.e. satisfying Yang–Baxter equation).

 $= \hbar(t - x_i)$  $= \hbar(\hbar - t + x_i)$  $= \dfrac{(t - x_i)(\hbar - t + x_i)}{2}$

The corresponding probabilities $\Psi_\pi(x_1, \ldots, x_{2n})$ can be chosen to be coprime polynomials of degree $2n(n-1)$.

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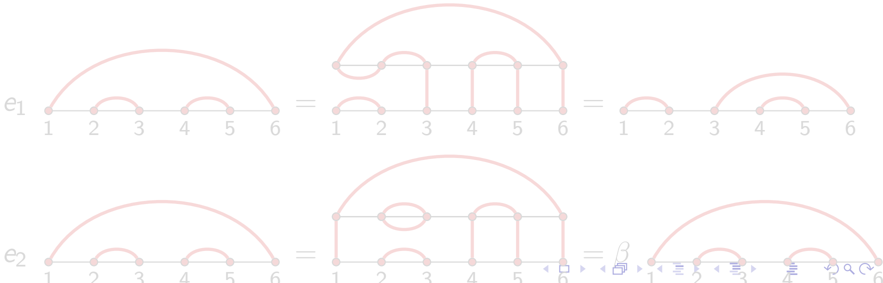
Brauer algebra

- Generators e_i, f_i, $i = 1, \ldots, N-1$ and relations

$$
\begin{array}{lll}
e_i^2 = \beta e_i & e_i e_{i\pm 1} e_i = e_i & e_i e_j = e_j e_i \\
f_i^2 = 1 & (f_i f_{i+1})^3 = 1 & f_i f_j = f_j f_i \\
f_i e_i = e_i f_i = e_i & e_i f_{i+1} f_i = e_i e_{i+1} = f_{i+1} f_i e_{i+1} & e_i f_j = f_j e_i \\
 & e_{i+1} f_i f_{i+1} = e_{i+1} e_i = f_i f_{i+1} e_i & (|i-j| > 1)
\end{array}
$$

- Action on link patterns of size $N = 2n$: rewrite link patterns on a line



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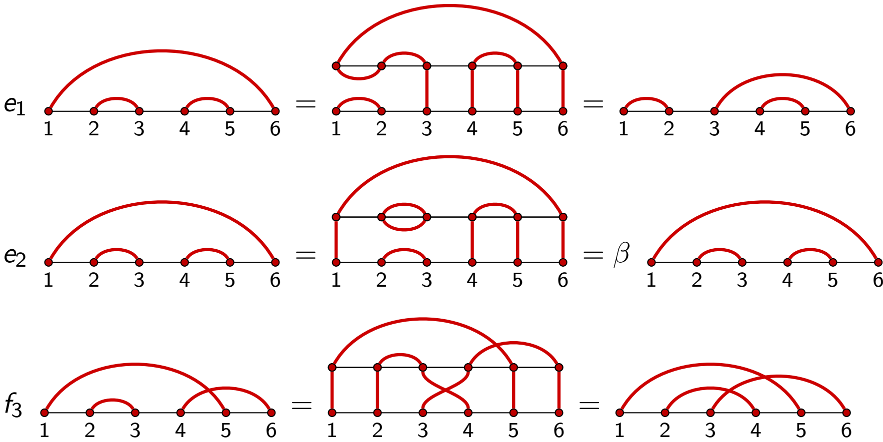

qKZ system

Add one more generator ρ (Affine Brauer) that rotates link patterns.

Define the operator

$$\check{R}_i(u) = \frac{\hbar(\hbar - u) + \hbar\, u e_i + (1 - \beta/2) u (\hbar - u) f_i}{(\hbar + u)(\hbar - (1 - \beta/2) u)}$$

Fix ϵ and consider the following system of equations for the vector $\Psi^{(\epsilon)}$:

$$\check{R}_i(x_i - x_{i+1}) \Psi^{(\epsilon)}(x_1, \ldots, x_N) = \Psi^{(\epsilon)}(x_1, \ldots, x_{i+1}, x_i, \ldots, x_N) \quad i=1,\ldots,N-1$$
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Solution of qKZ system

Theorem (Knutson, Z-J '14)

If $\beta = \frac{2(\hbar-\epsilon)}{2\hbar-\epsilon}$, there exists a solution of the qKZ system such that its components $\Psi_\pi^{(\epsilon)}$ are homogeneous polynomials of degree $2n(n-1)$ with integer coefficients in the variables $\hbar, \epsilon, x_1, \ldots, x_N$. This solution is unique up to scaling.

If $\epsilon = 0$, $\Psi^{(0)}$ coincides with the vector of probabilities Ψ of the Brauer loop model.

In practice, one can show that up to normalization

$$\Psi^{(\epsilon)} = \prod_{\substack{1\le i<j\le 2n \\ j-i<n}} (\hbar + x_i - x_j) \prod_{\substack{1\le i<j\le 2n \\ j-i>n}} (\hbar - \epsilon + x_j - x_i)$$

and then the exchange relation provides a recurrence relation for the $\Psi_\pi^{(\epsilon)}$.

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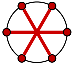
$\Psi^{(\epsilon)}$ 

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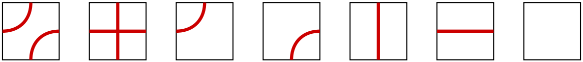
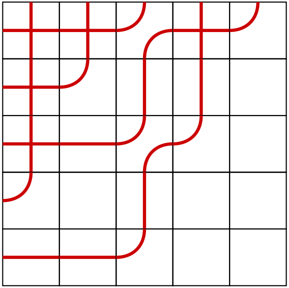
Pipe dreams

- Pipe dreams were introduced in [Fomin Kirillov '96; Bergeron Billey '93] in the context of Schubert polynomials and acquired their present name in [Knutson Miller '05] in their study of Matrix Schubert Varieties.
- They are (in disguise) an exactly solvable lattice model, similar in spirit to loop models. Their integrability was already present in [Fomin, Kirillov '96] and appears prominently in my work [Z-J '09].
- Since they are a degenerate model, we introduce here the corresponding nondegenerate model, which we dub in this context "generic pipe dreams".
- We then find a new, surprising relation between Brauer loop model and generic pipe dreams. Somewhat reminiscent of the Razumov–Stroganov correspondence.

Generic pipe dreams

Consider a square grid with N/W boundaries occupied, S/E boundaries empty, and tiles





Connectivity of the boundary points:

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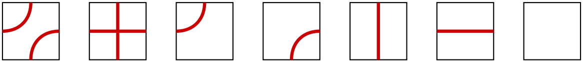

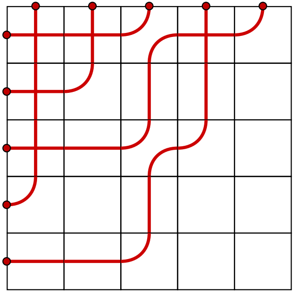

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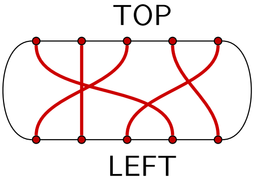
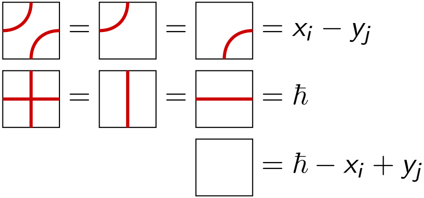


Boltzmann weights

Define weights



depending on row parameter x and column parameter y, and the partition function at fixed connectivity π

$$Z_\pi(x_1,\ldots,x_n;y_1,\ldots,y_n) = \sum_{\substack{\text{generic pipe dreams}\\ \text{connectivity } \pi}} \prod_{i,j=1}^{n} \text{weight}(x_i,y_j)$$

Relation to Brauer qKZ equation

Theorem

For all $\pi \in \mathcal{S}_n$, identified with a fixed-point-free involution of $\{1,\ldots,2n\}$,

$$\Psi_\pi^{(\epsilon)}(x_1,\ldots,x_{2n})$$
$$= \prod_{1\le i<j\le n} (\hbar+x_i-x_j)(\hbar+x_{i+n}-x_{j+n}) Z_\pi(x_1,\ldots,x_n; x_{2n}+\epsilon,\ldots,x_{n+1}+\epsilon)$$

Proof.

Z_π satisfies the (type A) exchange relation for both xs and ys
$\to$ (Brauer!) exchange relation for $i \neq 1, n+1$
This plus the initial condition fixes it uniquely. □

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- However, earlier, it was already observed in [de Gier, Nienhuis '04] and proven in [Di Francesco, Z-J '05] that if one restricts to the "permutation sector", then we can use for the same purpose the so-called upper-upper scheme of [Knutson '03].
- Knutson's motivation was that there exists a Gröbner degeneration from the commuting scheme to one component of the upper-upper scheme.
- Recently, I became interested in the upper-upper scheme again because it is a natural deformation of the union of conormal varieties of Matrix Schubert Varieties.
- This led me to generic pipe dreams. . . (Bethe Ansatz for stable classes)

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$X := \{(M, C) \in \mathrm{Mat}(n, \mathbb{C})^2 : \ MC \text{ lower triangular, } CM \text{ upper triangular}\}$

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(unfortunately this conjecture is wrong: there are more equations)

Remark: the eigenvalues $(MC)_{i,i}$ are the deformation parameters, as they're sent to zero one recovers the union of conormal varieties of MSVs. (and the flat limit of each component gives "stable cycles")

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The Gröbner degeneration

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There exists a Gröbner degeneration of X such that each X_π degenerates into a reduced union

$$X_\pi \rightsquigarrow \bigcup_{P:\ connectivity(P)=\pi} D_P$$

where each irreducible component D_P is indexed with a generic pipe dream with connectivity π.
D_P is a complete intersection with equivariant cohomology class given by the Boltzmann weight of the pipe dream P.

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Application: commuting variety

Define the commuting scheme

$$\mathcal{C} := \left\{(M, C) \in \mathrm{Mat}(n, \mathbb{C})^2 : \; MC = CM\right\}$$

Corollary

The degree of $\mathcal{C}$ is the sum over generic pipe dreams with "identity" connectivity of the weight

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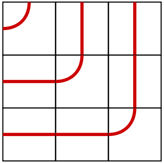
Example: $n = 3$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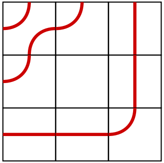
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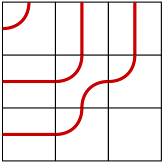
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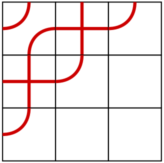
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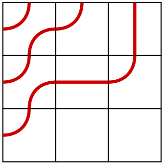
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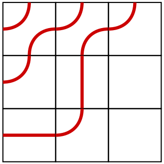
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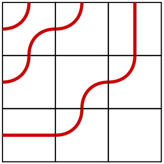
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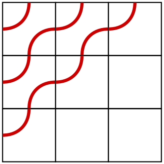
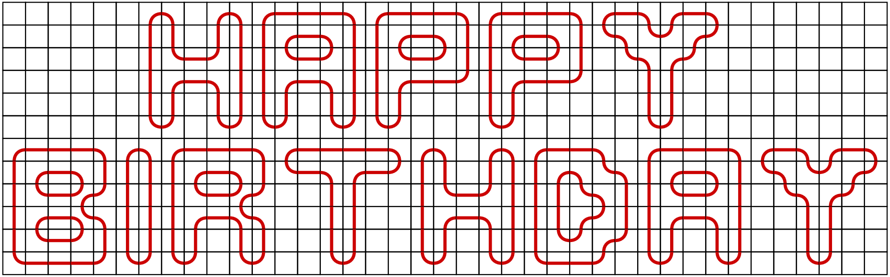
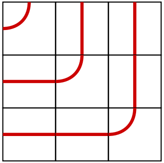

8

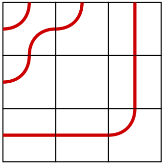
$\rightarrow$ The degree of $\mathcal{C}$ is 31.


HAPPY
BIRTHDAY

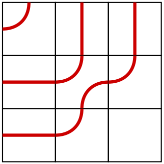
Example: $n = 3$



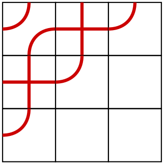
a^6



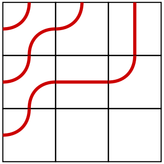
$a^4 b(a+b)$



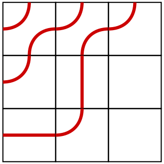
$a^4 b(a+b)$



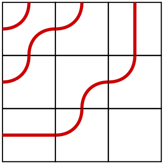
$a^2 b^3 (a+b)$



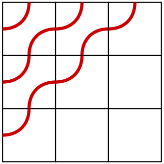
$a^2 b^2 (a+b)^2$



$a^2 b^2 (a+b)^2$



$a^2 b(a+b)^3$



$b^3 (a+b)^3$

$\rightarrow$ The bidegree of $\mathcal{C}$ is $a^6 + 3\,a^5 b + 7\,a^4 b^2 + 9\,a^3 b^3 + 7\,a^2 b^4 + 3\,a\,b^5 + b^6$.