Fully Packed Loops Model:
Integrability and Combinatorics

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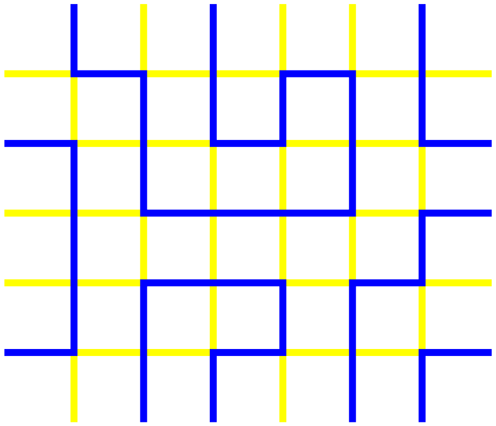
Plan

I The FPL^2 model. Relation to 6v model and ASM.

II Wieland theorem and Razumov–Stroganov conjecture.

III Case of 3 arches and relation to Plane Partitions. Generalizations.

FPL^2 model



$$Z = \sum_{\text{configurations}} n_1^{\#\text{blue loops}} n_2^{\#\text{yellow loops}}$$

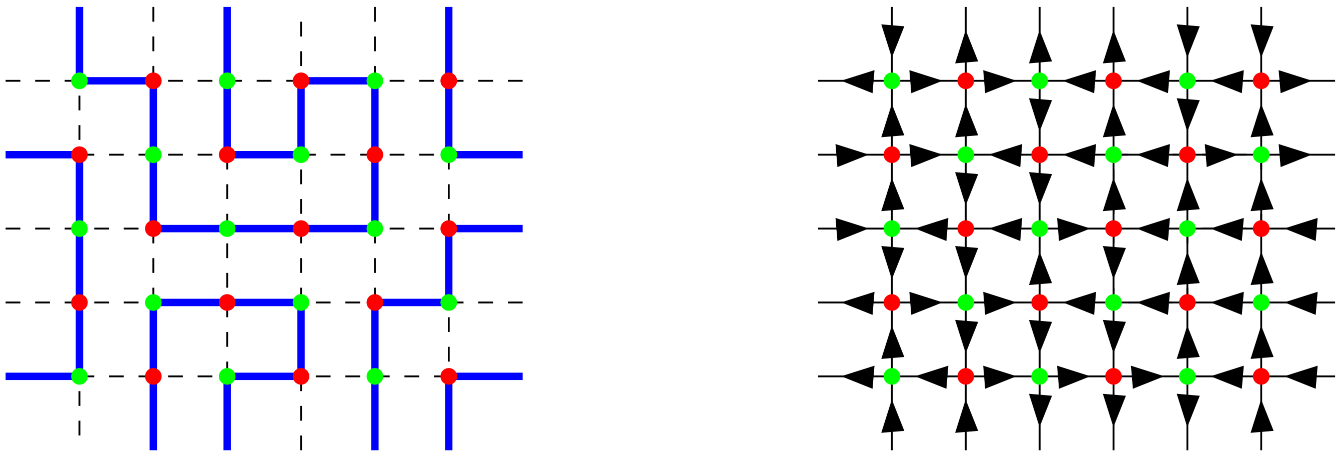
Kondev & Henley ('95).

Jacobsen & Kondev ('98): Coulomb gaz approach

Critical for $n_i = 2\cos\pi e_i$, $i = 1, 2$:

$$c = 3 - 6\frac{e_1^2}{1-e_1} - 6\frac{e_2^2}{1-e_2}$$

Connection to the 6-vertex model



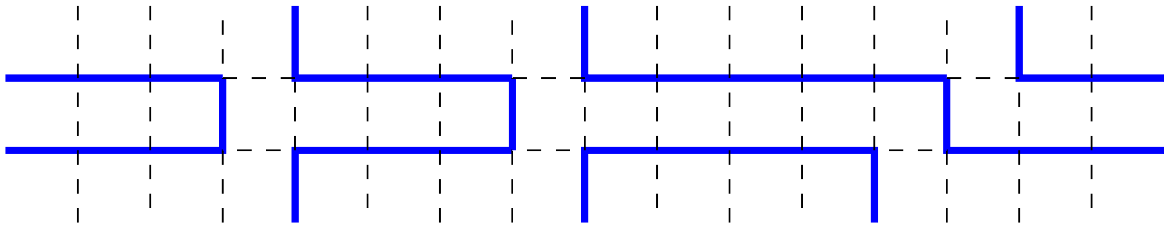
Arrows of occupied (resp. empty) edges go from green to red (resp. red to green).

Integrability for $n_1 = n_2$

Coordinate Bethe Ansatz (Nienhuis and Dei Cont)

Algebraic Bethe Ansatz: (Jacobsen + PZJ)

Consider the **double row** transfer matrix T with PBC on a strip of width $2L$:



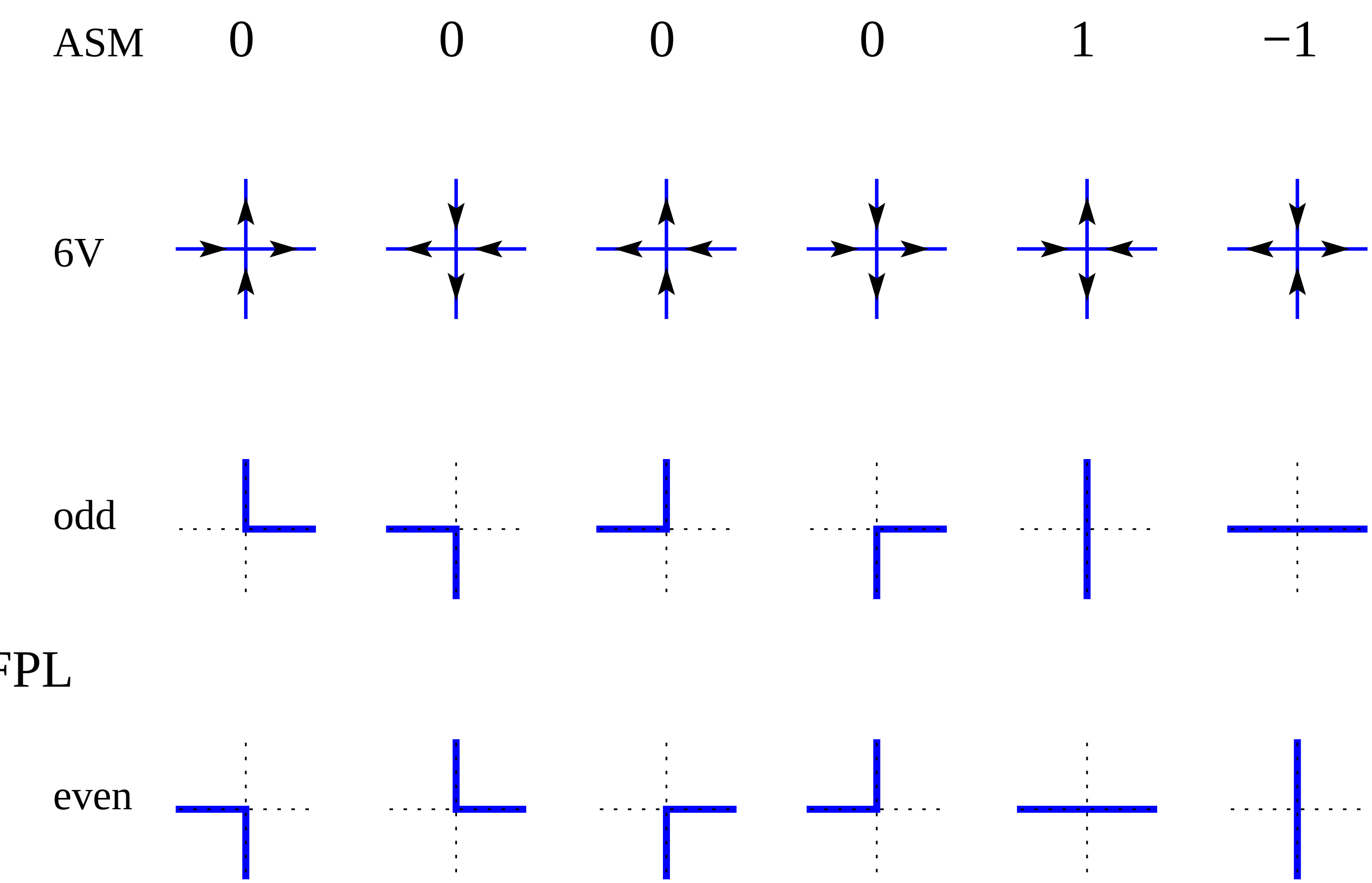
$$T = \mathrm{tr}\, R_1 \dots R_L \Omega$$

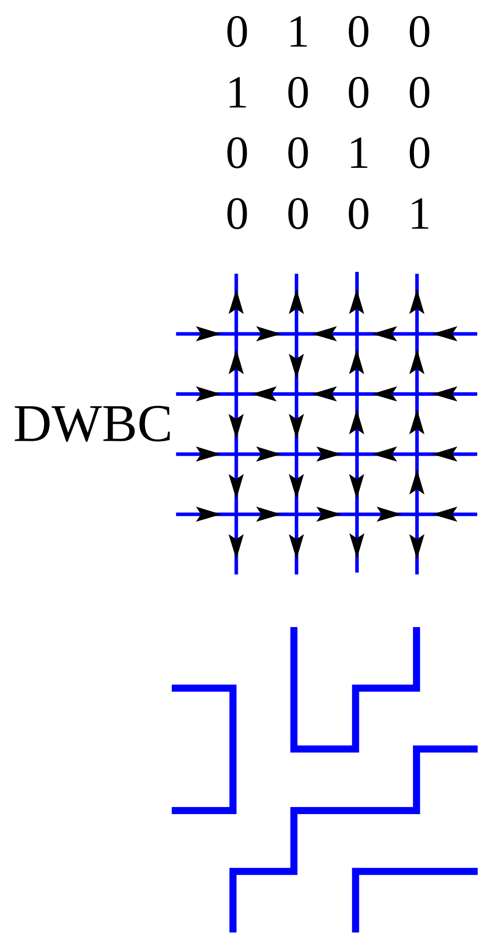
where R_i is the R-matrix associated to $U_q(sl(4))$ and two representations $\square \otimes \begin{smallmatrix}\square\\ \square\end{smallmatrix}$; and Ω is the usual twist to take care of winding loops.

$n_1 = n_2 = -q - q^{-1}$

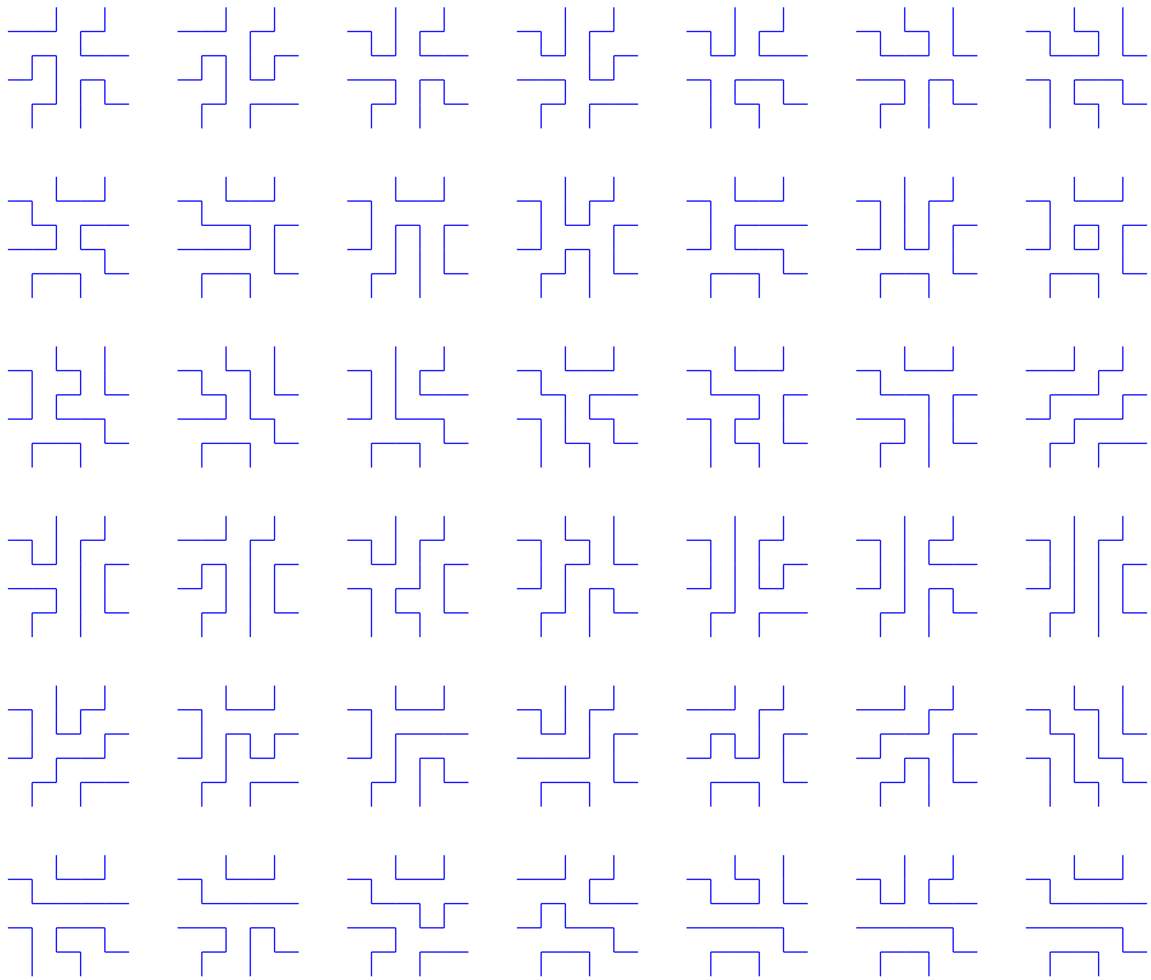
Confirms central charge…

ASM ↔ 6 Vertex ↔ FPL^2



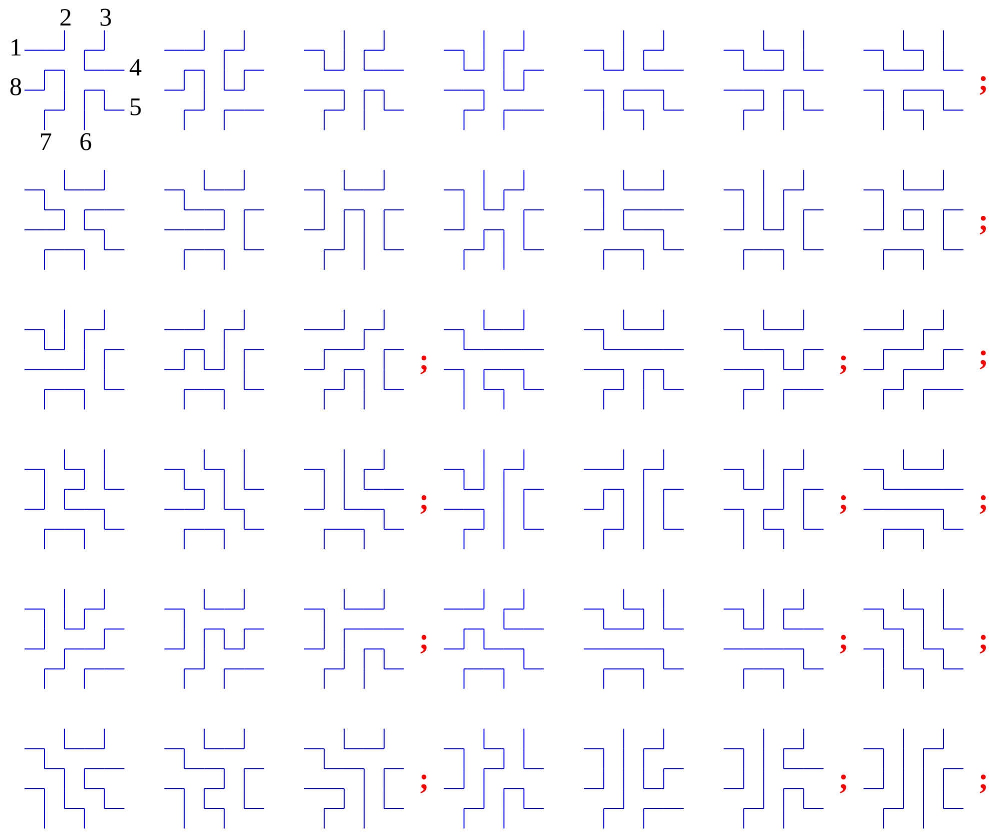


The 42 FPL on a 4×4 grid



For a given size n, FPL configurations fall into Connectivity Classes (or "link patterns") π of their external links.

There are $C_n = \frac{(2n)!}{(n+1)!n!}$ (Catalan number) distinct link patterns π. For example, for $n = 4$, 14 classes

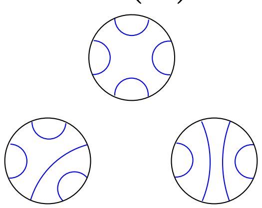


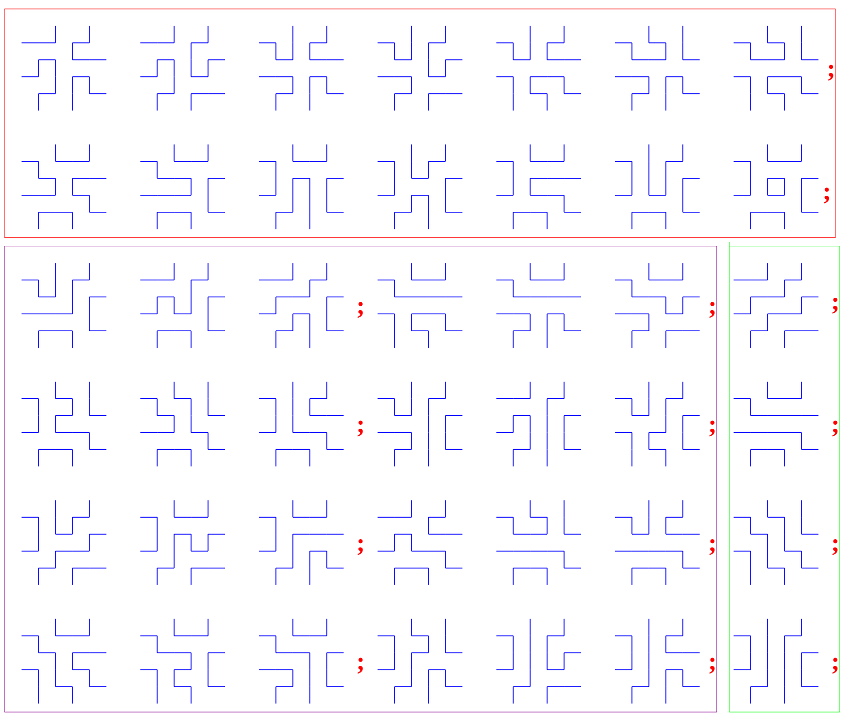
We want the numbers $A_n(\pi)$ of FPL configurations pertaining to π.

Dihedral symmetry of the $A_n(\pi)$

Theorem [Wieland (2000)]
If π and π' are obtained from one another by $\sigma \in D_n$, the dihedral group, then $A_n(\pi) = A_n(\pi')$.

For $n = 4$,
3 independent
$A_4(\pi)$





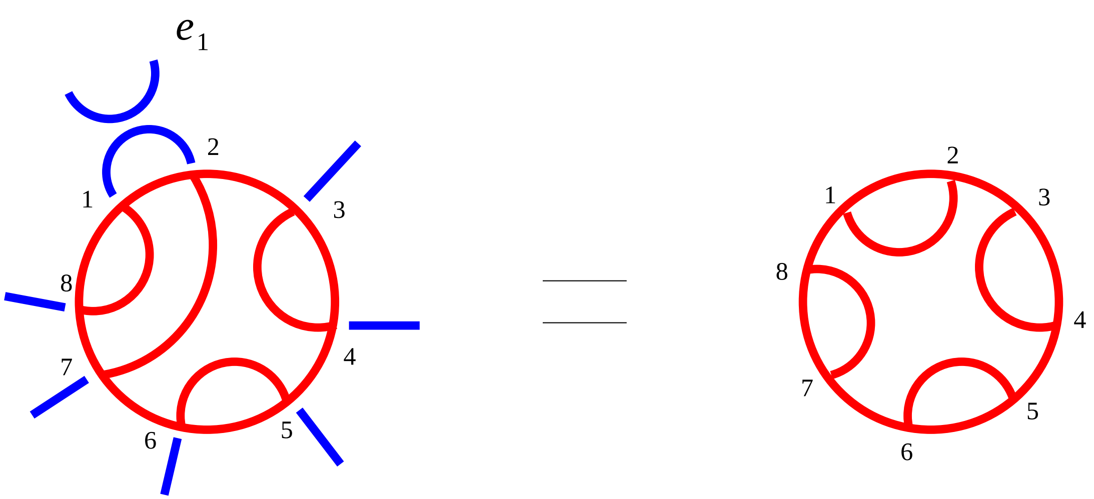
Periodic Temperley–Lieb algebra

Generators e_i, $i = 1, \ldots, 2n$

$$e_i^2 = \beta e_i; \qquad e_i e_j = e_j e_i \quad \text{if} \quad |i - j| > 1; \qquad e_i e_{i\pm1} e_i = e_i$$

with $\beta = 1$.

Consider periodic boundary conditions $e_{2n+1} \equiv e_1$:



Relation to XXZ chain

If $\beta = \omega + \omega^{-1}$, change of basis:

$$\underbrace{k \quad \ell} = \omega^{1/2} |\uparrow\rangle_k |\downarrow\rangle_\ell + \omega^{-1/2} |\downarrow\rangle_k |\uparrow\rangle_\ell$$

$$-\mathcal{H} = -\frac{1}{2} \sum_{j=1}^{2n} \left(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z \right) + \text{twisted B.C.}$$

with $\Delta = -\beta/2$.

$\Rightarrow$ For $\beta = 1$, the loop model is equivalent to a sector of the $\Delta = -1/2$ XXZ spin chain. In particular the ground states coincide.

We are interested in a loop model with the "Hamiltonian" $-\mathcal{H} = -\sum_{i=1}^{2n} e_i$. In the basis $|\pi_a\rangle$, $\mathcal{H}$ is a matrix with non negative integer entries. Want to find its Perron-Frobenius eigenvector (the ground state of $\mathcal{H}$)

$$|\Psi\rangle = \sum_a \psi_a |\pi_a\rangle$$

Easy lemmas:

* $\mathcal{H}$ has the left eigenvector $(1,\dots,1)$ with eigenvalue $2n$
* this is the highest eigenvalue (multiplicity one)
* the coresponding right eigenvector has ≥ 0 rational ($\rightarrow$ integers) components ψ_a

Razumov–Stroganov main conjecture

The Perron-Frobenius eigenvector of $\mathcal{H} = \sum_{i=1}^{2n} e_i$, i.e. the solution of $\mathcal{H}|\Psi\rangle = 2n|\Psi\rangle$, is

$$|\Psi\rangle = \sum_a A_n(\pi_a)|\pi_a\rangle$$

i.e. its components are the $A_n(\pi)$ (with proper normalization) [R&S 2001].

Other types of b.c. on TL $\leftrightarrow$ different symmetry classes of ASM/FPL

* periodic, odd number of sites: connection with half turn symmetric ASM/FPL
* open, even number of sites: connection with vertically symmetric ASM/FPL

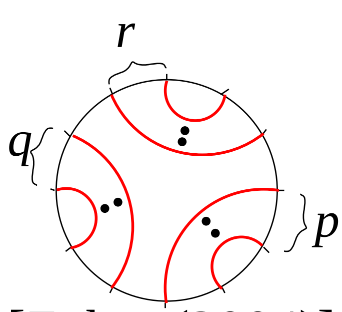
[Razumov-Stroganov 2001; Pearce, de Gier & Rittenberg 2001, . . ., Mitra, Nienhuis, de Gier & Batchelor 2004]

Other conjectures

⋆ Exact expressions for correlation functions in the TL model / spin chain [RS, Nienhuis–Mitra, etc]

⋆ Explicit expression for some coefficients $A_n(\pi)$.

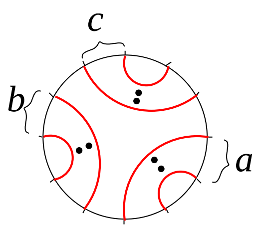
Introduce the "superfactorial" $m \overset{2}{!} := \prod_{r=1}^{m} r!\ , \quad (-1)\overset{2}{!} = 0\overset{2}{!} = 1.$



$$= \frac{(p+q+r-1)\overset{2}{!}\,(p-1)\overset{2}{!}\,(q-1)\overset{2}{!}\,(r-1)\overset{2}{!}}{(p+q-1)\overset{2}{!}\,(q+r-1)\overset{2}{!}\,(r+p-1)\overset{2}{!}} \qquad p,q,r, \geq 0$$

[Zuber (2004)]

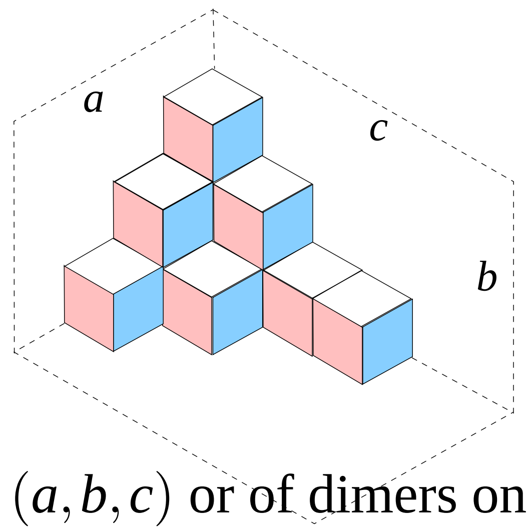
Relation to Plane Partitions



$$= \frac{(a+b+c-1)\overset{2}{!}\,(a-1)\overset{2}{!}\,(b-1)\overset{2}{!}\,(c-1)\overset{2}{!}}{(a+b-1)\overset{2}{!}\,(b+c-1)\overset{2}{!}\,(c+a-1)\overset{2}{!}} \qquad a,b,c,\geq 0$$

Unexpectedly, this is also the number of plane partitions in a box $a \times b \times c$ (MacMahon formula)

$$\cdots = \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2} = \#$$

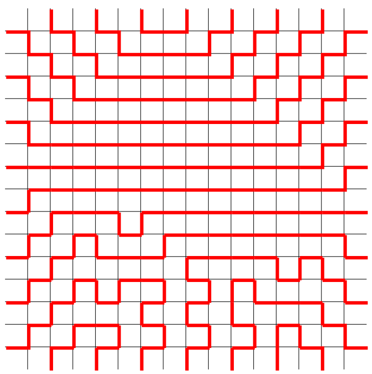


a.k.a. the number of tilings of a hexagon of sides (a,b,c) or of dimers on a honeycomb grid.

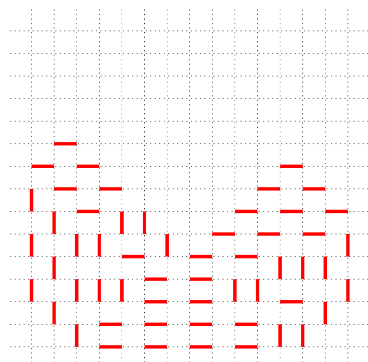
Theorem [PDF-PZJ-JBZ (2003)] There is an explicit bijection between FPL with three sets of a, b, c nested arches and Plane Partitions in a Box of size $a \times b \times c$

Main idea [de Gier (2002)] : boundary conditions on FPL fix a certain number of edges. Choice of the remaining ones amounts to a tiling problem

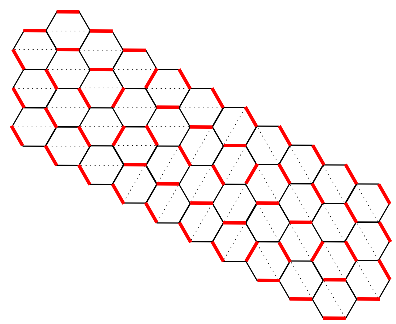
Take for example a FPL with $a = 2,\ b = 3,\ c = 10.$



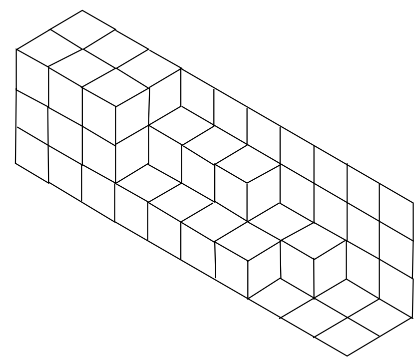
FPL



active part of FPL



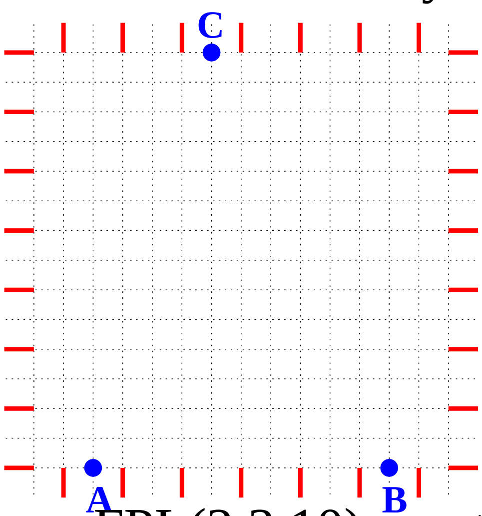
dimers



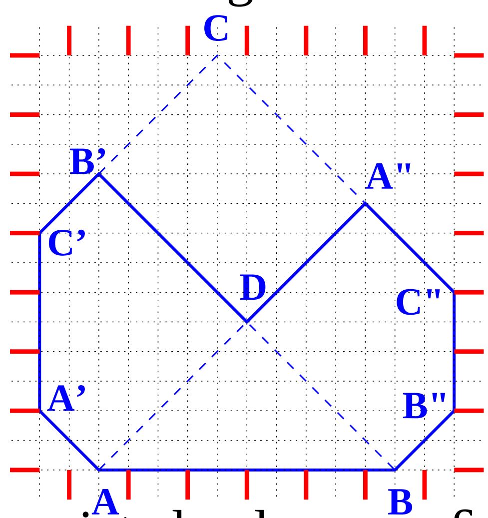
tiling

`http://ipnweb.in2p3.fr/lptms/membres/pzinn/fpl`

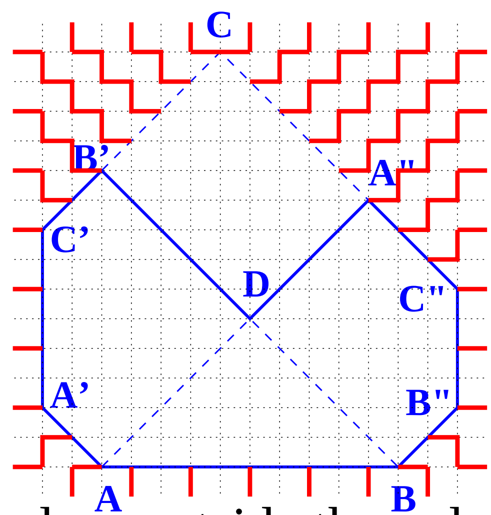
Use a lemma by de Gier to fix edges:



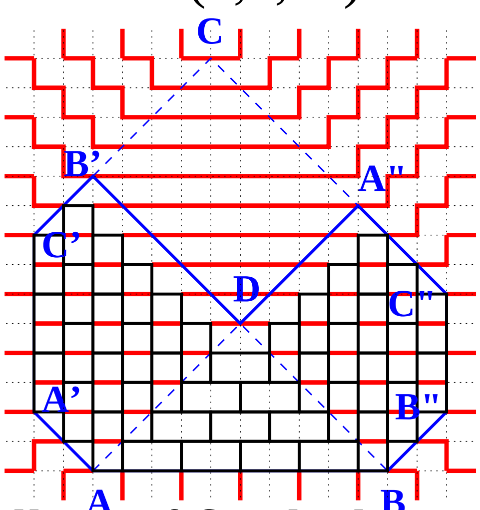
FPL(2,3,10)



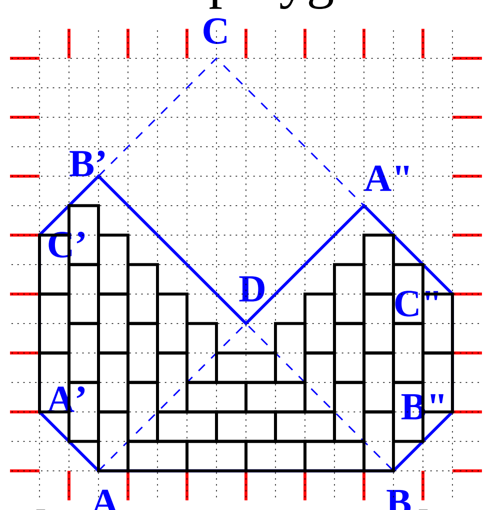
the associated polygon



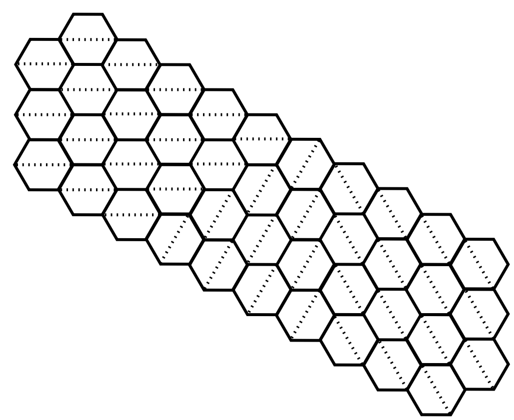
fixing edges outside the polygon



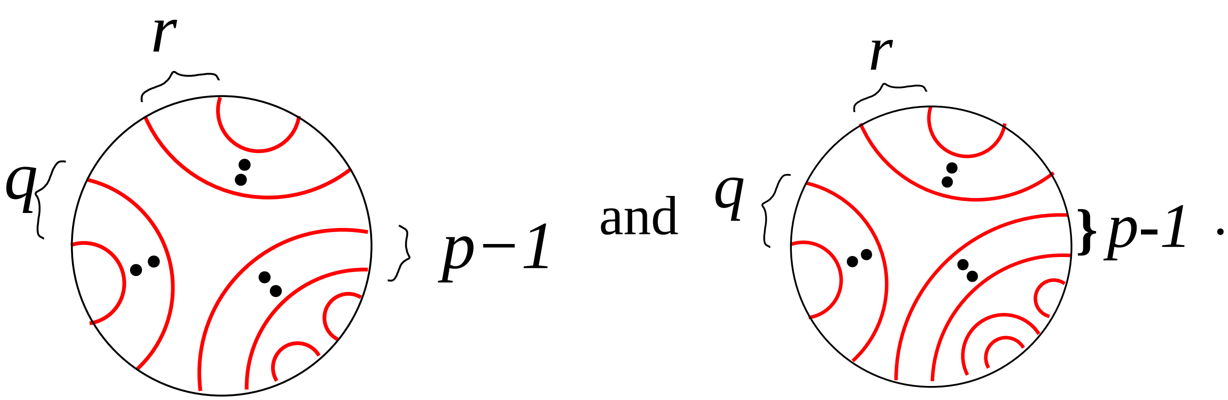
full set of fixed edges



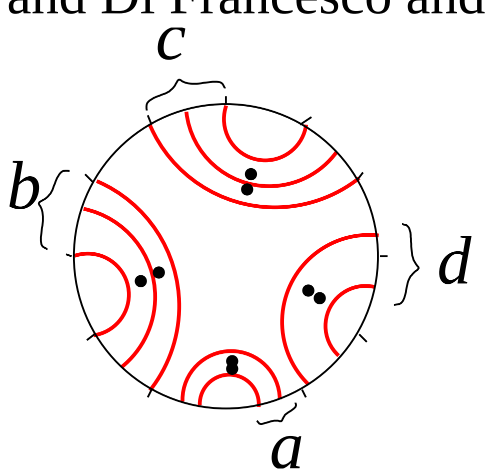
dominos


dominos → hexagons

By a refinement of this line of arguments, Caselli and Krattenthaler have proved formulae for

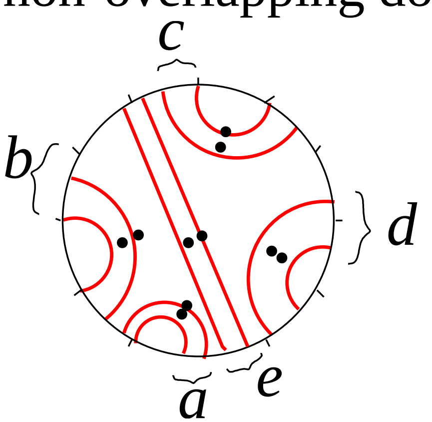


and Di Francesco and Zuber have found (complicated) formulae for

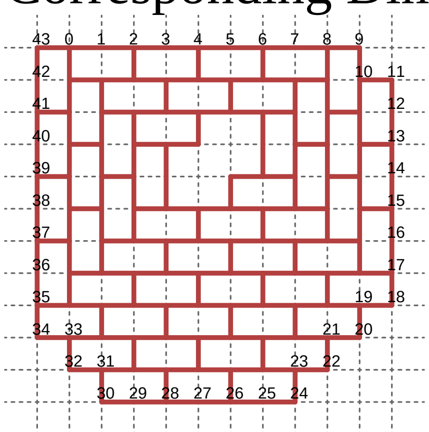


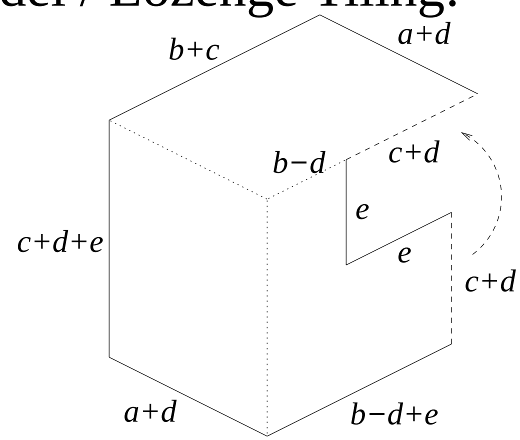
Fully Packed loops and Lozenge Tilings

Most general connectivity pattern for which the “active domain” is a set of non-overlapping dominoes: (up to a Wieland rotation) [PDF–PZJ–JBZ (un)]

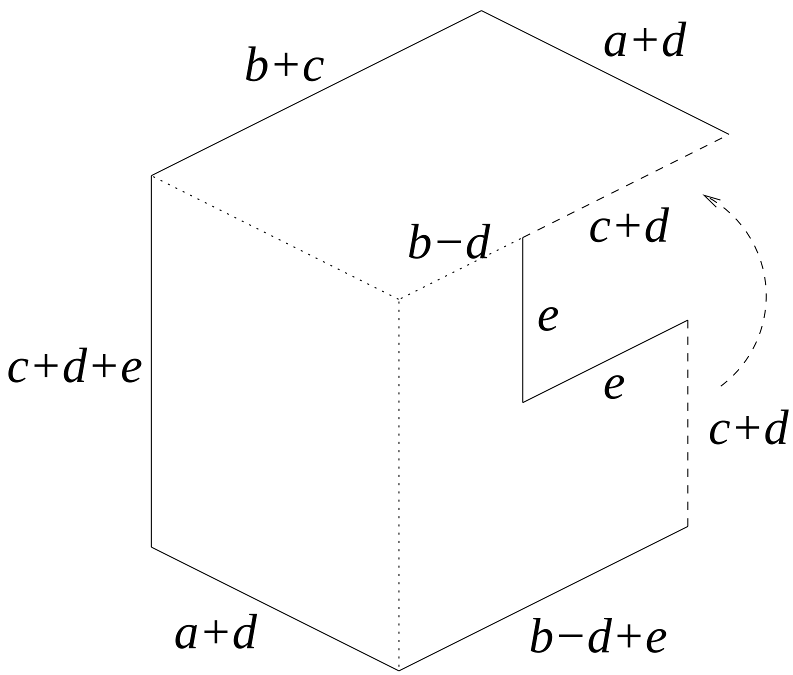


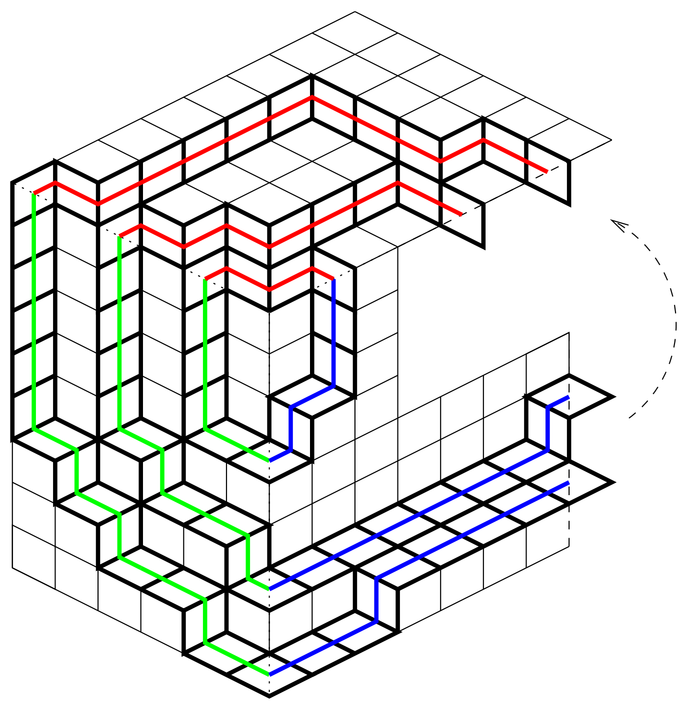
Corresponding Dimer Model / Lozenge Tiling:





Lozenge Tilings and Non-Intersecting Paths



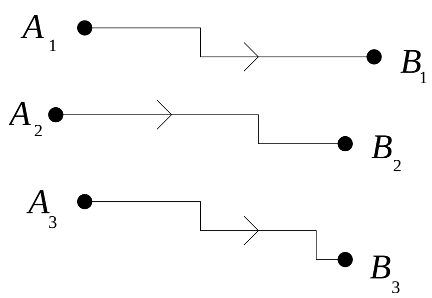


(d winding loops and any number of trivial loops)

Non-Intersecting Paths and Free Fermions

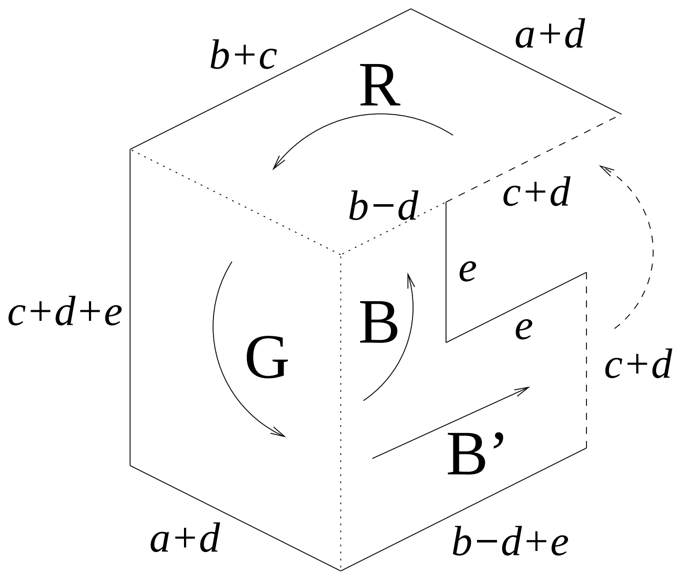
Gessel–Viennot theorem: (directed) non-intersecting paths are free fermions

$$\mathrm{NIP}(A_1,\ldots A_n \to B_1,\ldots,B_n) = \det(A_i \to B_j)$$



Application:

$$\#\text{ lozenge tilings} = [x^d]\det(RG(B+xB'))$$



Concluding remarks

- Can all the FPL numbers be expressed as numbers of tilings / non-intersecting paths?
- Proof of the Razumov–Stroganov conjecture?
- Combinatorics $\leftrightarrow$ integrability…