05/03

Matrix Models and Knot Theory

P. Zinn-Justin

References:
◇ P. Zinn-Justin, J.-B. Zuber, math-ph/9904019, math-ph/0002020, math-ph/0303049
◇ P. Zinn-Justin, math-ph/9910010, math-ph/0106005.
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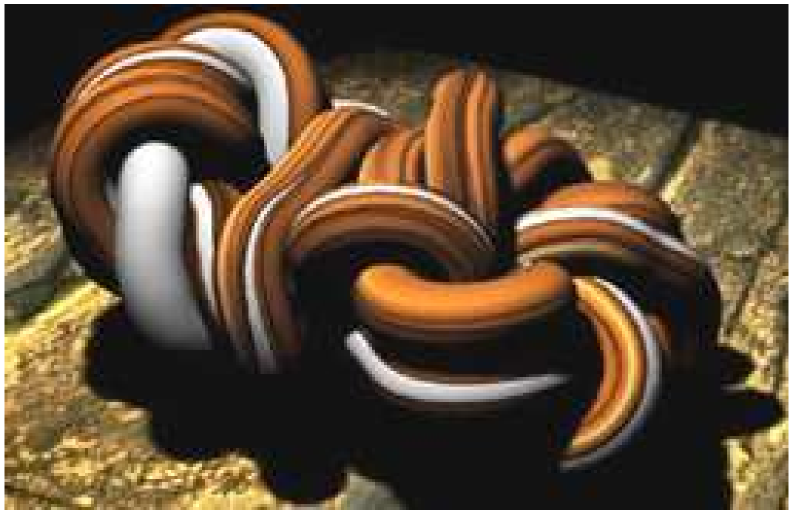
◇ G. Schaeffer, P. Zinn-Justin, math-ph/0304034.

- Classification and Enumeration of Knots, Links, Tangles.
- Feynman diagrams. $O(n)$ matrix model and renormalization.
- Universality and conjectures on asymptotic counting.
- Phase diagram of $O(n)$-symmetric 2D statistical models.
- Numerical check: Monte Carlo.
- Virtual link diagrams and Links on thickened surfaces.
- Renormalization and the generalized flyping conjecture for virtual alternating links.

A bit of History...

The Gordian knot: (Piotr Pieranski, 2001)



• Knots represented by their projection: **diagrams** (Tait, 1876):

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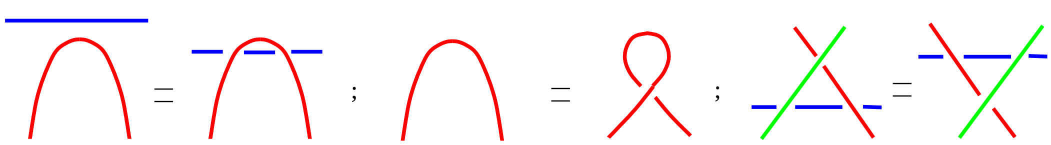
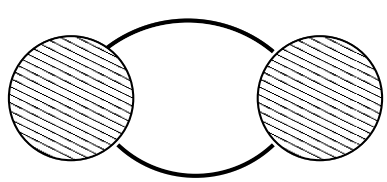
• Two diagrams represent the knot/link/tangle iff they are related by a sequence of Reidemeister moves: (Reidemeister, 1932)

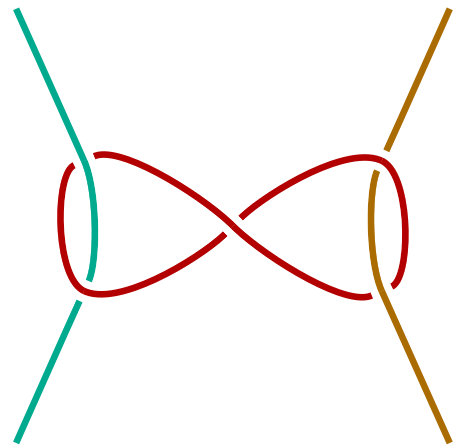


• All knots are connected sums of **prime knots** (Schubert, 1949):

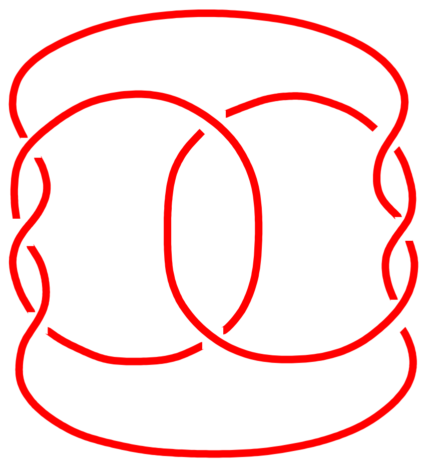


Knots, links and tangles

Links are collections of knots: **tangles** have strings coming out:

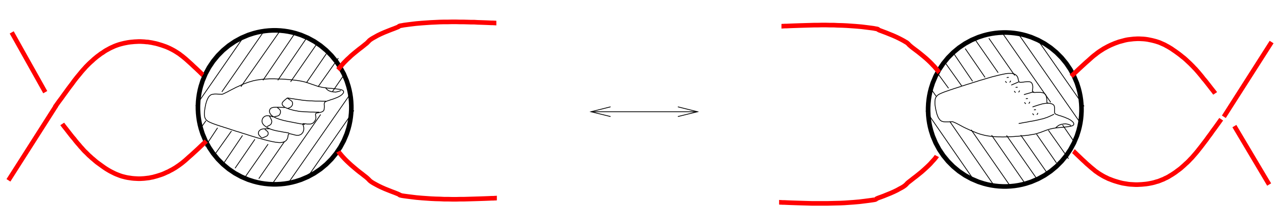


• Alternating vs Non-Alternating:

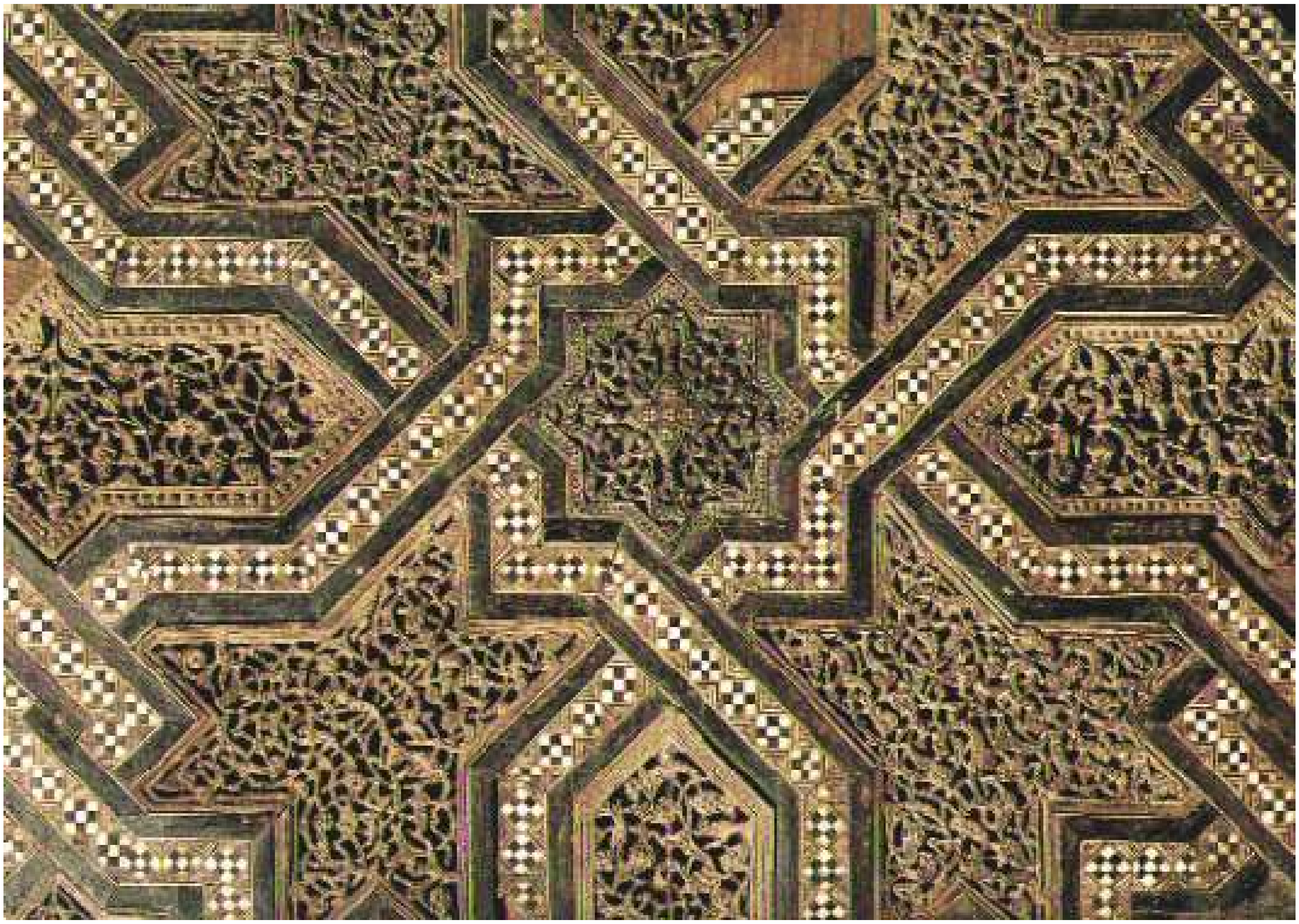


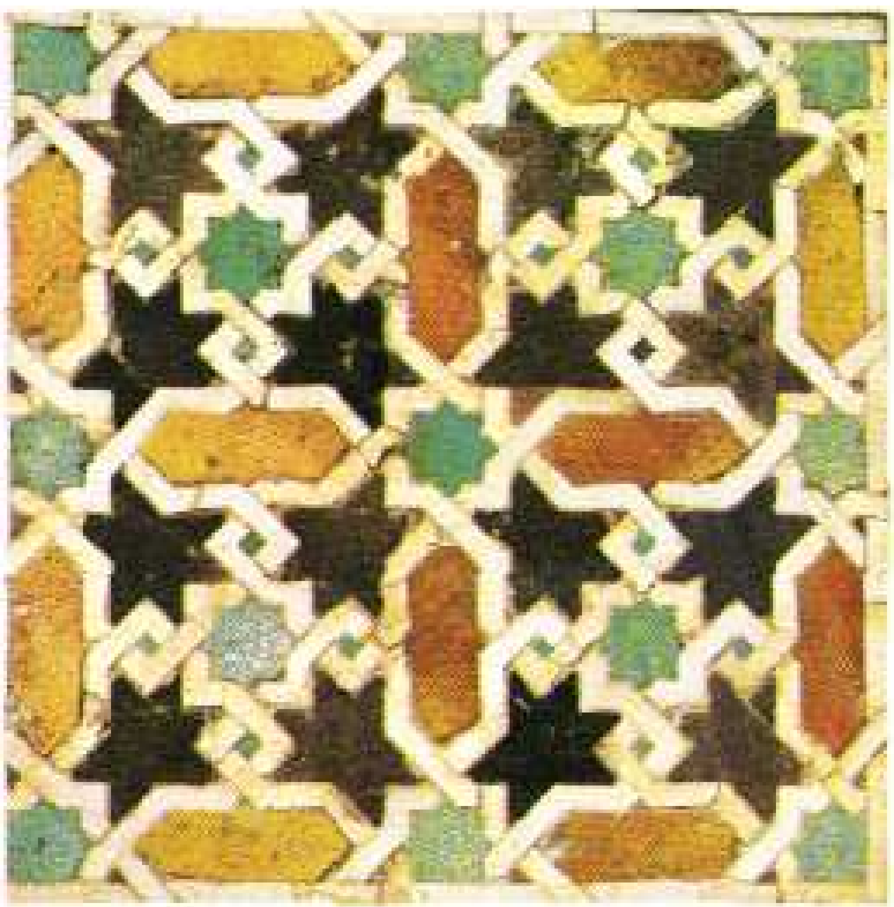
Tait's flyping conjecture: (Tait, 1898)

Two reduced alternating diagrams represent the same object iff they are related by a sequence of flypes:



Proved by Menasco and Thisthlethwaite ('91).



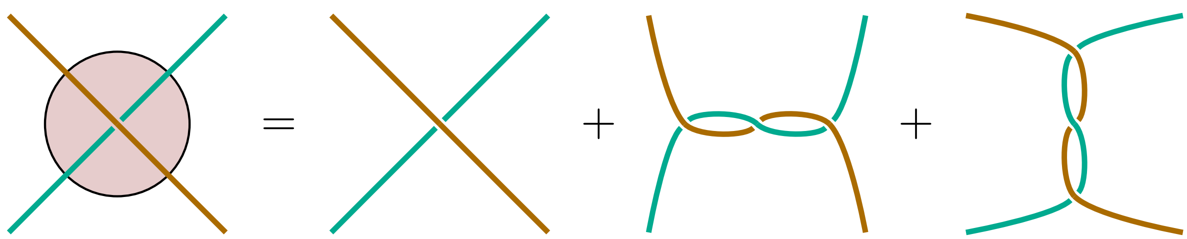


What is the problem?

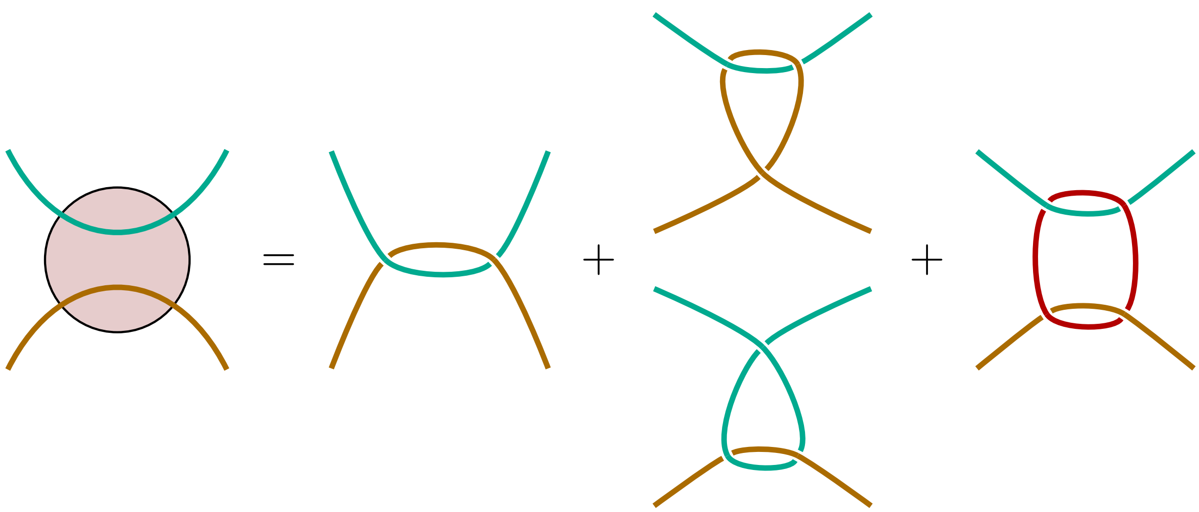
We want to enumerate (prime) alternating tangles with a given number of connected components and crossings:

$$\Gamma(n,g) = \sum_{k,p=1}^{\infty} a_{k;p} g^p n^k$$

Example: tangles with four external legs. Two types:



$$\Gamma_1(n,g) = g + g^3 + g^3 + \cdots$$



$$\Gamma_2(n,g) = g^2 + g^3 + ng^4 + \cdots$$

n = "number of colors": a diagram with k closed loops can be drawn in n^k ways.

Feynman diagrams

Gaussian integral over real variables x_i, $A = A^T > 0$ def. matrix:

$$\int \mathrm{d}^d x \, e^{-\frac{1}{2}\sum x_i A_{ij} x_j} = \frac{(2\pi)^{n/2}}{\det^{\frac{1}{2}} A}$$

$$\int \mathrm{d}^n x \, \mathrm{e}^{-\frac{1}{2}\sum x_i A_{ij} x_j + \sum b_i x_i} = \frac{(2\pi)^{n/2}}{\det^{\frac{1}{2}} A} \, \mathrm{e}^{\frac{1}{2}\sum b_i A^{-1}_{ij} b_j}$$

Differentiate w.r.t. b_i

$$\langle x_{k_1} x_{k_2} \cdots x_{k_\ell} \rangle = \frac{\int d^n x \; x_{k_1} x_{k_2} \cdots x_{k_\ell} e^{-\frac{1}{2}x.A.x}}{\int d^n x \, e^{-\frac{1}{2}x.A.x}}$$

$$= \frac{\partial}{\partial b_{k_1}} \cdots \frac{\partial}{\partial b_{k_\ell}} \mathrm{e}^{\frac{1}{2}b.A^{-1}.b}\Big|_{b=0}$$

$$= \sum_{\substack{\text{all distinct} \\ \text{pairings } P \text{ of the } k}} A^{-1}_{k_{P_1} k_{P_2}} \cdots A^{-1}_{k_{P_{\ell-1}} k_{P_\ell}}$$

Non-Gaussian integral: power series ("perturbative") expansion.

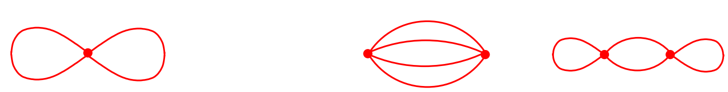
Example: $(d = 1)$

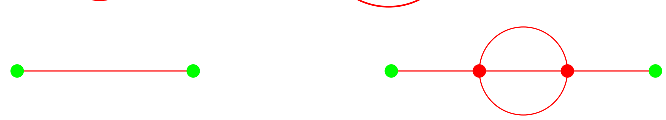
$$Z = \int dx \, \mathrm{e}^{-\frac{1}{2}Ax^2 + \frac{g}{4!}x^4} = \left(\frac{2\pi}{A}\right)^{\frac{1}{2}} \sum_{p=0}^{\infty} \frac{g^p}{p!} \int dx \left(\frac{x^4}{4!}\right)^p \mathrm{e}^{-\frac{1}{2}Ax^2}$$

$$= \left(\frac{2\pi}{A}\right)^{\frac{1}{2}} \sum_{p=0}^{\infty} \sum_{\substack{\text{graphs } G \text{ with } 2p \text{ lines} \\ \text{and } p \text{ 4-valent vertices}}} \frac{g^p}{|\mathrm{Aut}\, G|} A^{-2p}$$

$\log Z =$ **connected** Feynman diagrams

$$= \frac{g}{8A^2} + \frac{g^2}{A^4}\left(\frac{1}{2.4!} + \frac{1}{2^4}\right) + \cdots$$

$=$ 

$\langle x^2 \rangle =$ 

Matrix Integrals: Feynman Rules

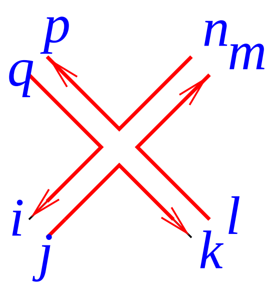
$N \times N$ Hermitean matrices M,

$dM = \prod_i dM_{ii} \prod_{i<j} d\Re e M_{ij}\, d\Im m M_{ij}$

$$Z = \int dM \mathrm{e}^{N[-\frac{1}{2}\,\mathrm{tr}\, M^2 + \frac{g}{4}\,\mathrm{tr}\, M^4]}$$

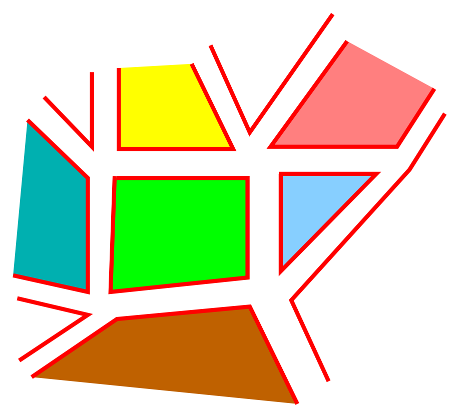
Feynman rules:

propagator  $= \frac{1}{N}\delta_{i\ell}\delta_{jk}$

4-valent vertex  $= gN\delta_{jk}\delta_{\ell m}\delta_{np}\delta_{qi}$

Count powers of N in a connected diagram:

- each vertex $\to N$;
- each double line $\to N^{-1}$;
- each loop $\to N$.



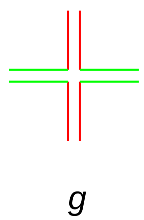
$$\#\text{vert.} - \#\text{lines} + \#\text{loops} = \chi_{\text{Euler}}(\Sigma)$$

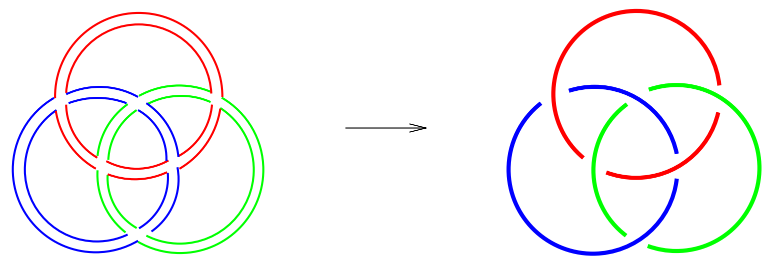
't Hooft (1974):

$$\log Z = \sum_{\text{conn. surf.}\Sigma} N^{2-2\text{genus}(\Sigma)} \frac{g^{\#\text{vert.}(\Sigma)}}{\text{symm. factor}}$$

A Matrix Model for Alternating Link Diagrams

$$Z^{(N)}(n,g) = \int \prod_{a=1}^{n} \mathrm{d}M_a \, \mathrm{e}^{N \,\mathrm{tr}\left(-\frac{1}{2}M_a^2 + \frac{g}{4}(M_a M_b)^2\right)}$$





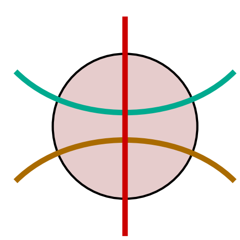
The large N free energy $F(n,g)$ and correlation functions are double generating series in n, g (number of connected components, number of crossings).

$F(n,g)$ counts link diagrams (weighted by their symmetry factors):

$$F(n,g) = \lim_{N\to\infty} \frac{\log Z^{(N)}(n,g)}{N^2} = \sum_{k,p=1}^{\infty} f_{k;p} g^p n^k$$

The correlation functions count tangle diagrams:

$$\lim_{N\to\infty} \left\langle \frac{1}{N} \mathrm{tr}(M_1 M_2 M_3 M_2 M_1 M_3) \right\rangle_c =$$



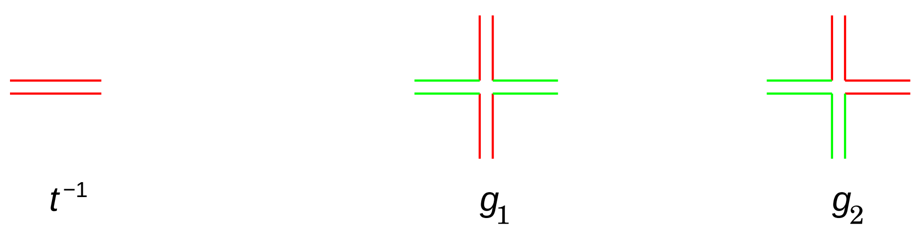
From tangle diagrams to tangles: Renormalization

General idea: removal of the redundancy associated to multiple equivalent diagrams acts like a "finite renormalization" on the model.

• Reduced diagrams ⇒ renormalization of the quadratic term in the action.

• Taking into account the flyping equivalence renormalizes the quartic term. However, there are **two** four-vertex interactions compatible with the $O(n)$-symmetry → more general $O(n)$ model:

$$Z^{(N)}(n,t,g_1,g_2) = \int \prod_{a=1}^{n} \mathrm{d}M_a \; \mathrm{e}^{N\,\mathrm{tr}\left[-\frac{t}{2}M_a^2 + \left(\frac{g_1}{4}M_aM_bM_aM_b + \frac{g_2}{2}M_aM_aM_bM_b\right)\right]}$$



t, g_1 and g_2 are functions of the renormalized coupling constant g, chosen such that the correlation functions are the appropriate generating series in g of the number of alternating links.

Exactly solved cases

• $n = 1$: the counting of alternating tangles, and more

Usual one-matrix model:

$$Z^{(N)}(t, g_0) = \int \mathrm{d}M \, \mathrm{e}^{N \, \mathrm{tr}\left(-\frac{t}{2}M^2 + \frac{g_0}{4}M^4\right)}$$

with $g_0 = g_1 + 2g_2$.

Renormalization equations recombine into a fifth degree equation:

$$32 - 64A + 32A^2 - 4\frac{1 + 2g - g^2}{1 - g}A^3 + 6gA^4 - gA^5 = 0$$

Correlation functions are given in terms of its solution. In particular, if $\left\langle \frac{1}{N} \mathrm{tr}\, M^{2\ell} \right\rangle_c = \sum_{p=0}^{\infty} a_p g^p$ is the generating function of **prime alternating tangles with 2ℓ legs**, then

$$a_p \overset{p\to\infty}{\sim} cst \, g_c{}^{-p} p^{-5/2}$$

with $g_c = \frac{\sqrt{21001}-101}{270}$ ($g_c^{-1} \approx 6.1479$).

($\ell = 2$: Sundberg & Thistlethwaite '98)

$\Rightarrow$ The number f_p of prime alternating links grows like

$$f_p \sim cst \, g_c{}^{-p} p^{-7/2}$$

(Schaeffer & Kunz-Jacques, '01)

• $n = -2 \ldots$

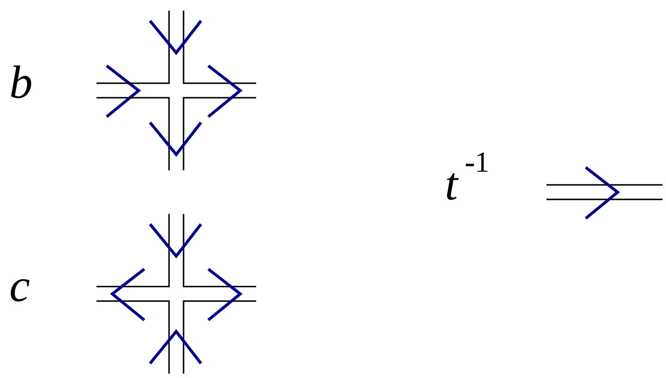
• $n = 2$: the counting of oriented alternating tangles (P.Z-J. & J.-B. Zuber)

$$Z^{(N)}(t,g_1,g_2) = \int \mathrm{d}M_1 \mathrm{d}M_2 \, \mathrm{e}^{N \,\mathrm{tr}\left[-\frac{t}{2}(M_1^2+M_2^2) + \frac{g_1+2g_2}{4}(M_1^4+M_2^4) + \frac{g_1}{2}(M_1M_2)^2 + g_2 M_1^2 M_2^2 \right]}$$

Introduce a complex matrix $X = \frac{1}{\sqrt{2}}(M_1 + iM_2)$:

$$Z^{(N)}(t,b,c) = \int \mathrm{d}X \mathrm{d}X^\dagger \, \mathrm{e}^{N \,\mathrm{tr}\left(-tXX^\dagger + bX^2X^{\dagger 2} + \frac{1}{2}c(XX^\dagger)^2\right)}$$

with $b = g_1 + g_2$ and $c = 2g_2$. Feynman rules:



Six-vertex model on random lattices. This model has been exactly solved (P.Z-J.; I. Kostov).

$\Rightarrow$ Generating function of (prime, alternating) tangles given by transcendental equation. Asymptotics:

$$a_p \overset{p\to\infty}{\sim} \mathrm{cst}\, g_c^{-p} p^{-2} (\log p)^{-2}$$

with $g_c^{-1} \approx 6.2832$.

Conjectures on the asymptotic behavior

Links $\sim$ discretized surfaces with random geometry

$\rightarrow$ 2D quantum gravity...

Conjecture: For $|n| < 2$, the matrix model is in the universality class of a 2D field theory with spontaneously broken $O(n)$ symmetry, coupled to gravity.

The large size limit is described by a CFT with $c = n - 1 \Rightarrow$ (KPZ)

$$a_p(n) \sim \mathrm{cst(n)}\, g_c(n)^{-p}\, p^{\gamma(n)-2}$$

$$f_p(n) \sim \mathrm{cst(n)}\, g_c(n)^{-p}\, p^{\gamma(n)-3}$$

$$\gamma = \frac{c - 1 - \sqrt{(1-c)(25-c)}}{12}$$

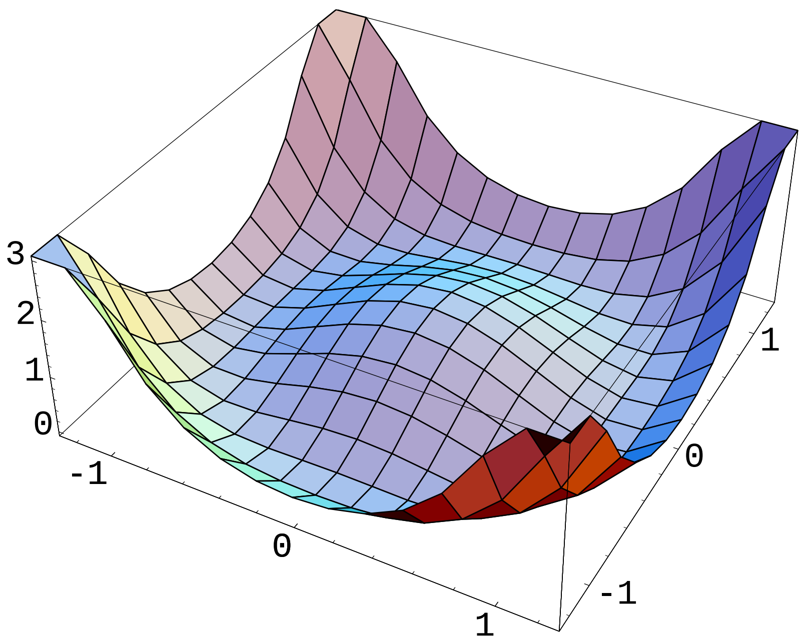
In particular, knots correspond to the limit $n \rightarrow 0$:

$$f_p(0) \sim \mathrm{cst}\, g_c^{-p} p^{-\frac{19+\sqrt{13}}{6}}$$

$O(n)$ Spontaneous Symmetry Breaking

Continuum limit of a $O(n)$-symmetric statistical model = $O(n)$-symmetric field theory.

Spontaneous symmetry breaking? $O(n) \to O(n-1)$



$\to$ σ-model on S^{n-1}. Renormalization group equation:

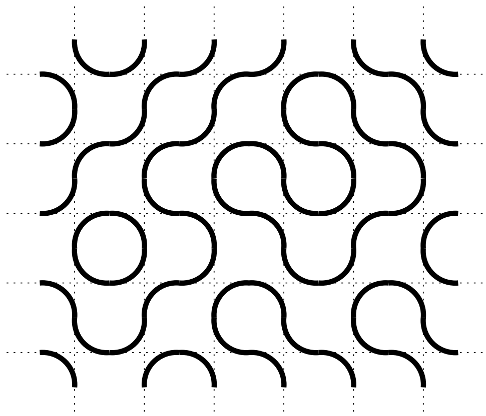
$$L\frac{\mathrm{d}}{\mathrm{d}L}g = (n-2)g^2 + O(g^3)$$

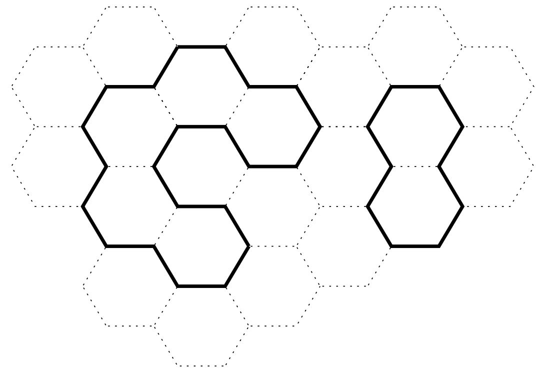
- Perturbation marginally relevant for $n > 2$.
- Perturbation marginally irrelevant for $n < 2$.

$\to$ Statistical model for this phase?

$O(n)$ Statistical Lattice Models

Model of non-crossing loops: (weight n per loop)

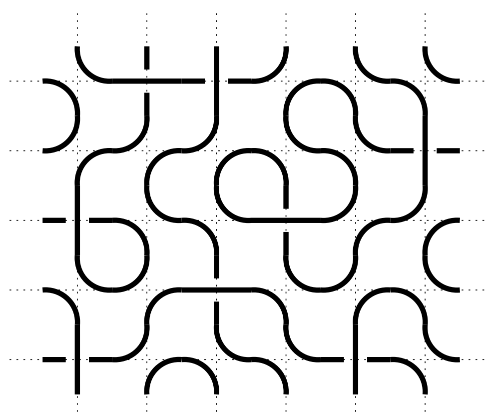




Nienhuis ('82): critical (dense phase) with central charge

$$c = 1 - 6(\sqrt{e} - 1/\sqrt{e})^2 \qquad n = -2\cos(\pi e)$$

Read & Saleur: study of supersymmetric models → different models! Jacobsen, Read & Saleur: numerical evidence that non-crossing loops model unstable to perturbation by crossings:



Conjecture central charge for $n \in [-2, 2]$:

$$c = n - 1$$

KPZ formula and asymptotic enumeration

link diagrams $\equiv$ statistical model on a random lattice.

Continuum limit = CFT coupled to gravity.

Knizhnik Polyakov Zamolodchikov. David. Distler Kawai. ('89)

$$Z^{(h)} \sim \mathcal{A}^{(1-h)(\gamma-2)-1}$$

where

$$\gamma = \frac{c - 1 - \sqrt{(1-c)(25-c)}}{12}.$$

Here, $p \sim A$, f_p = number of links $\sim Z$.

$$f_p^{(h)}(n) \sim \mathrm{e}^{\sigma(n)\, p + ((1-h)(\gamma(n)-2)-1)\, \log p + \kappa(n)}$$

where $\sigma(n)$, $\kappa(n)$ are non-universal parameters.

Conjecture: $c = n - 1$.

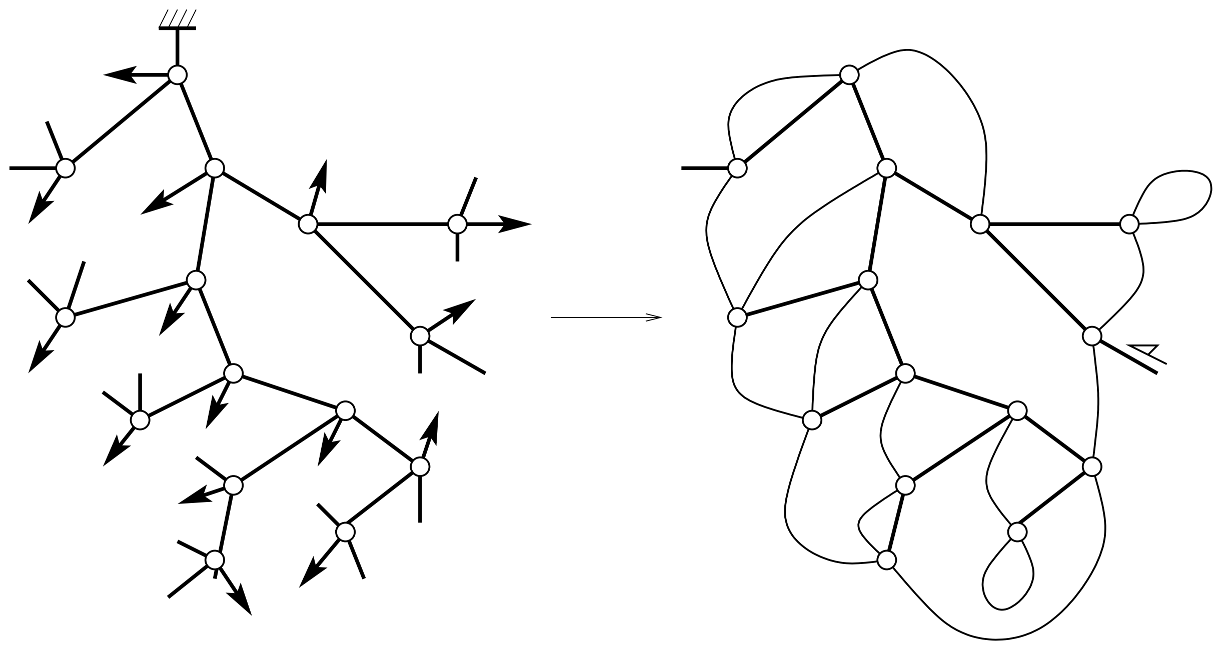
In particular for knots ($n = 0$), $\gamma = -\frac{1+\sqrt{13}}{6}$.

(Also, formulae for dimensions of operators)

Monte Carlo: random sampling of planar maps

(G. Schaeffer and P. Z.-J., '03)

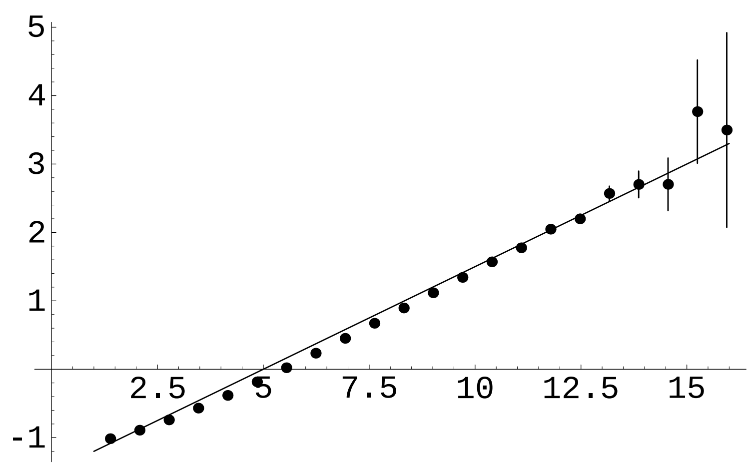
Schaeffer's bijection between trees and planar maps:



Results in an algorithm to produce random planar maps in linear time. Used to sample maps up to $p = 10^7$ vertices.

Test quantity: $\gamma' \equiv \frac{\mathrm{d}\gamma}{\mathrm{d}n}_{|n=1} = 3/10$ according to the conjecture.

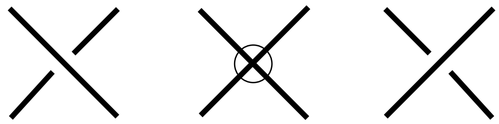
Very good agreement:



Virtual Link Diagrams

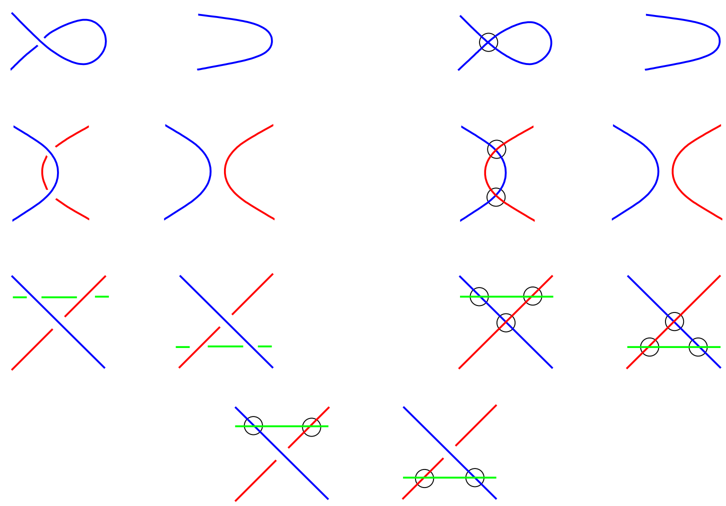
"Classical" link diagrams = embedded planar graphs (planar maps).

Kauffman's definition of virtual link diagrams:

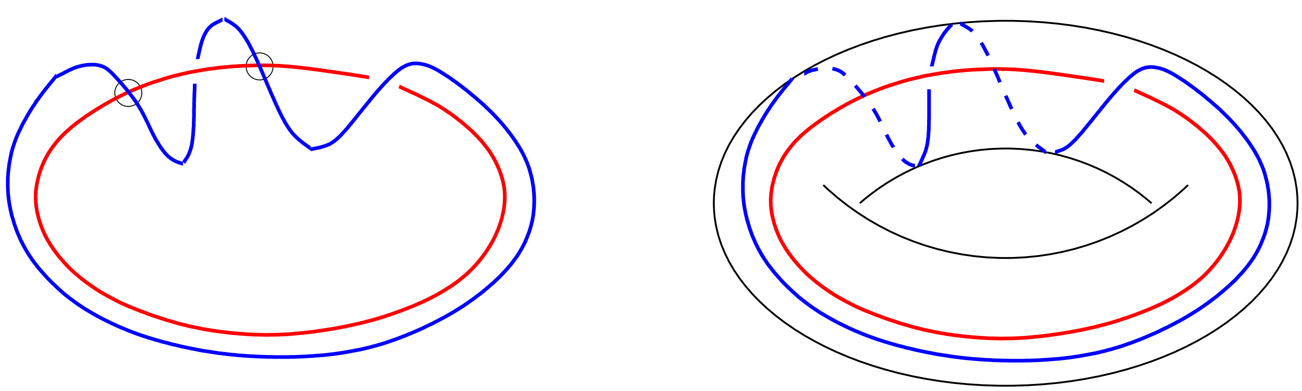


Idea: we're trying to draw non-planar graphs in a planar way...

Reidemeister moves:

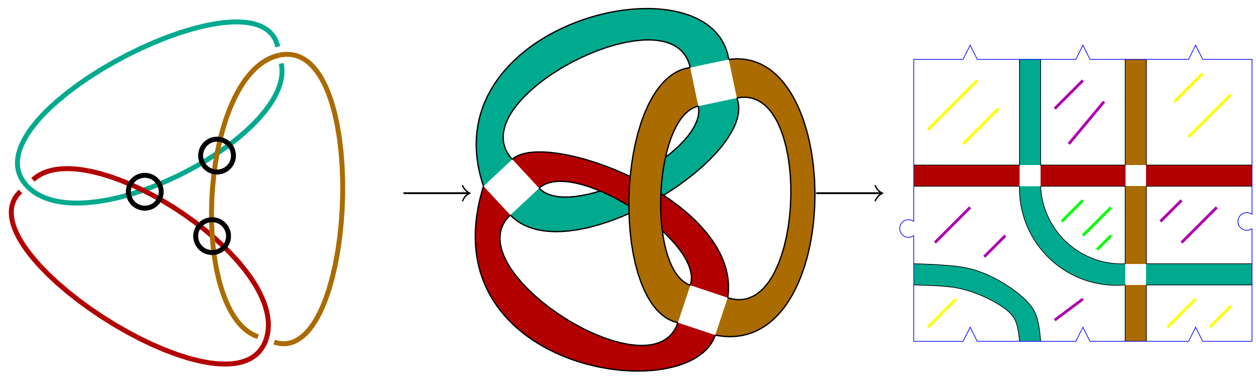


More precisely, we want to imagine them as links in **thickened surfaces** $\Sigma \times I$



Thickened Surfaces and Non-Planar Maps

To each virtual link diagram one can associate an abstract surface:



Problem: what is the relation with knot theory on the corresponding thickened surface?

• How can one compare diagrams when they each live on their own surface?

• What is the meaning of classical / virtual Reidemeister moves in this context?

Carter, Kamada, Saito ('00). Kuperberg ('02).

Virtual links are links embedded in thickened surfaces $\Sigma \times I$ up to orientation-preserving homeomorphisms of the surface Σ and up to addition/removal of handles.

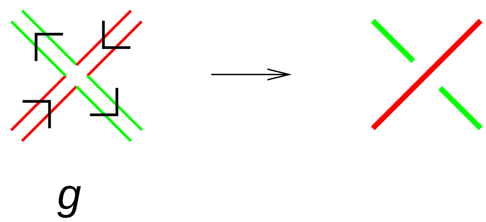
NB: Higher genus surface possess homeomorphisms that are non-isotopic to the identity! (torus: Dehn twists)

A matrix model for virtual alternating links

Matrix models produce abstract discretized surfaces. How to recover under/over-crossings?

We concentrate on alternating diagrams ⇒ complex matrix model:

$$Z^{(N)}(n,g) = \int \prod_{a=1}^{n} \mathrm{d}M_a \mathrm{d}M_a^\dagger \, \mathrm{e}^{N \,\mathrm{tr}\left(-M_a M_a^\dagger + \frac{g}{2}(M_a M_b^\dagger)^2\right)}$$



NB: in genus > 0, not every quadrangulation is bipartite!!!

$$\log Z^{(N)}(n,g) = \sum_{h\geq 0, k\geq 1, p\geq 1} f_{k;p}^{(h)} N^{2-2h} g^p n^k$$

Triple generating function of alternating link diagrams living on asbtract surfaces.

"Renormalization" ? General conjecture: the same combinatorial process used for classical alternating links and tangles works in higher genus.

$$Z^{(N)}(n,t,g_1,g_2) = \int \prod_{a=1}^{n} \mathrm{d}M_a \mathrm{d}M_a^\dagger$$

$$\mathrm{e}^{N \,\mathrm{tr}\left[-t M_a M_a^\dagger + \left(\frac{g_1}{2} M_a M_b^\dagger M_a M_b^\dagger + g_2 M_a M_a^\dagger M_b M_b^\dagger\right)\right]}$$

Step 1: reduced diagrams of prime links

Conjecture 1: Reduced alternating diagrams have minimal crossing number. (proved?)

Conjecture 2: Reduced alternating diagrams have minimal genus.

a) b)

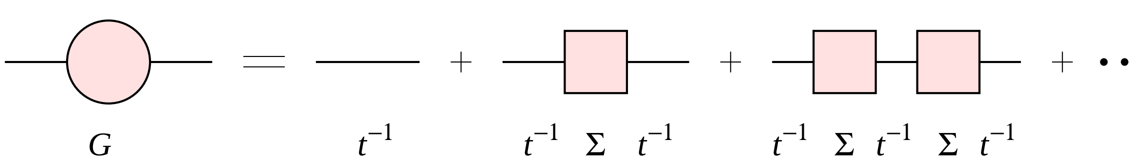
Forbidding *at the level of diagrams* decompositions of type b) removes irrelevant crossings and restricts to prime links / tangles. Clearly this amounts to imposing on the two-point function $G \equiv \lim_{N\to\infty} \left\langle \frac{1}{N} \operatorname{tr} M_a^2 \right\rangle$:

$$G = 1$$

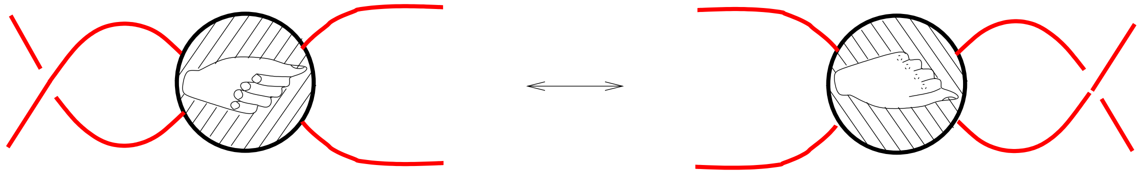
This is equivalent to

$$t = 1 + \sigma \tag{1}$$

where σ counts **one-particle-irreducible** diagrams, via the equation $G = \frac{1}{t-\sigma}$:



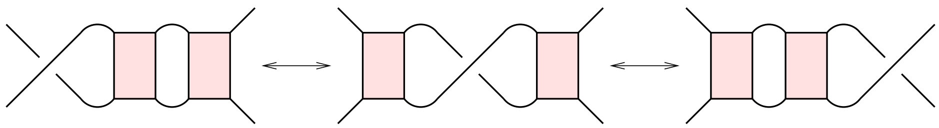
Step 2: flypes



Conjecture 3: Tait's flyping conjecture also holds for virtual alternating links and tangles.

NB: only **planar** flypes are allowed!

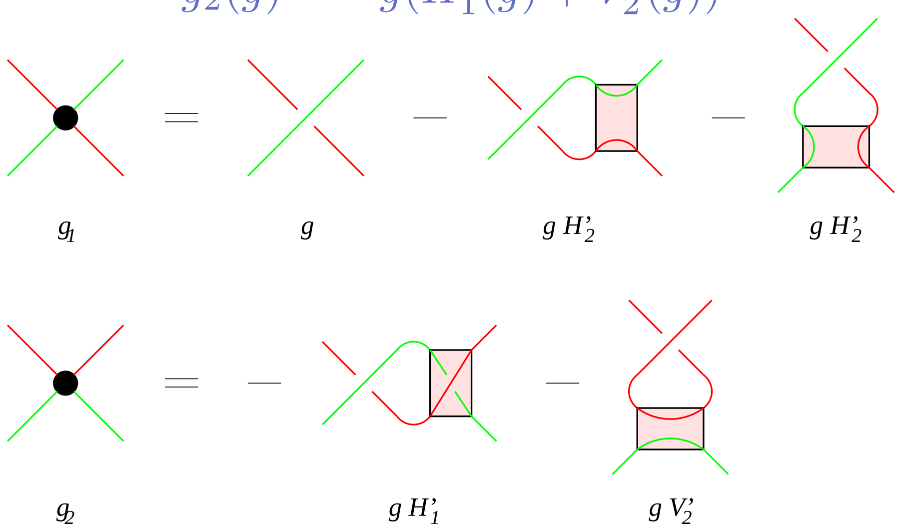
One must first decompose flypes into **elementary** flypes:



Introduce generating functions $H_1'(g)$, $H_2'(g)$, $V_2'(g)$ for H-2PI planar tangles. Then the equations fixing $g_1(g)$, $g_2(g)$ are:

$$g_1(g) = g(1 - 2H_2'(g)) \tag{2}$$

$$g_2(g) = -g(H_1'(g) + V_2'(g)) \tag{3}$$



Analytic results at $n = 1$

$$Z_N(g,t) = \int \mathrm{d}M\mathrm{d}M^\dagger \mathrm{e}^{N\,\mathrm{tr}\left[-tMM^\dagger + \frac{g}{2}(MM^\dagger)^2\right]}$$

$$\log Z_N(g,t) = \sum_{h=0}^{\infty} N^{2-2h} F^{(h)}(g,t)$$

First few terms of the expansion were computed. ($h = 1$: Morris. $h = 2, 3$: Akemann & Adamietz). In terms of A solution of

$$A = 1 + 3gA^2$$

We find:

$$F^{(0)}(g,1) = \log A - \frac{1}{12}(A-1)(9-A)$$

$$F^{(1)}(g,1) = -\frac{1}{24}\log\frac{(2-A)(2+A)^3}{27}$$

After renormalization, we find that the number of prime virtual alternating links of genus h grows like

$$f_p^{(h)} \overset{p\to\infty}{\sim} c\, g_c^{-p} p^{5/2(h-1)-1} \qquad h = 0, 1, 2, 3$$

with $g_c = \frac{\sqrt{21001}-101}{270}$ ($g_c^{-1} \approx 6.1479$).