$O(1)$ loop model and Combinatorics

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LPTHE, Université Paris 6

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Outline of the talk

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1. Definition of the model
   - The $O(n)$ loop model
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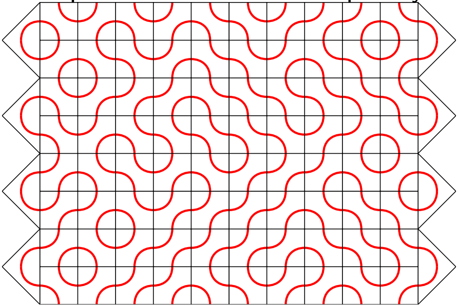
   - Properties of the ground state
   - Introduction of inhomogeneities
2. Quantum Knizhnik–Zamolodchikov equation
   - Temperley–Lieb algebra
   - Back to the $O(1)$ model ground state
   - Vertex operators and qKZ
   - Integral formulae for qKZ solutions
3. Relation to combinatorics
   - Totally Symmetric Self-Complementary Plane Partitions
   - Alternating Sign Matrices
   - Exact result for some observables

The $O(n)$ loop model is a two-dimensional statistical lattice model with *non-local* Boltzmann weights and observables. We use here the square lattice and "Completely Packed Loops":



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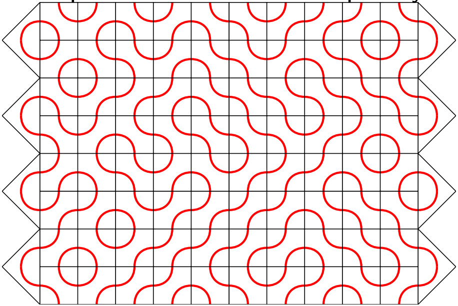

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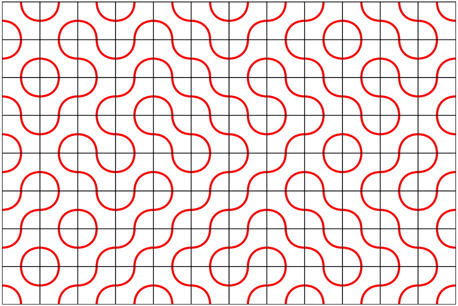

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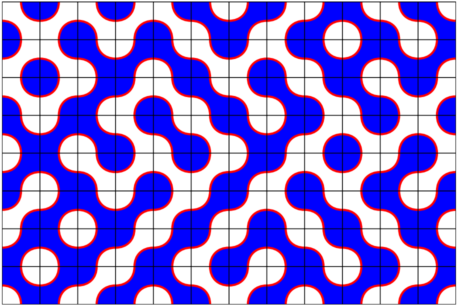

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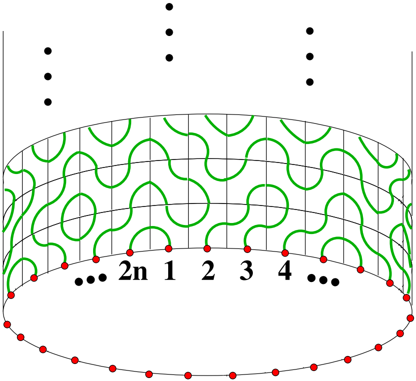
 with probability p,  with probability $1-p$. $(0 < p < 1)$



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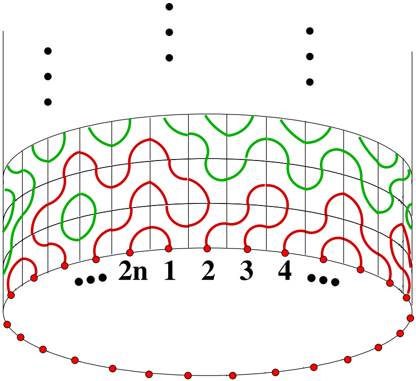

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The connectivity of the external vertices can be encoded into a link pattern = a planar pairing of $2n$ points on a circle. There are $c(n) = (2n)!/n!/(n+1)!$ possible link patterns.

Example

In size $L = 2n = 8$,

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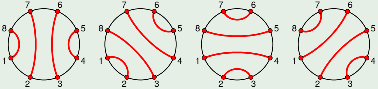

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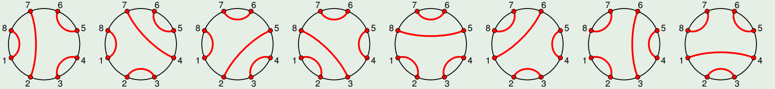

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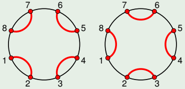

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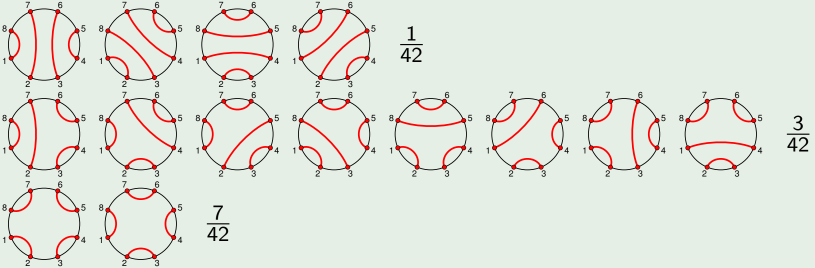

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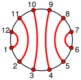
Observations [Batchelor, de Gier, Nienhuis '01]

Normalize Ψ so that the smallest entries, with patterns of the type



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1. All entries are Ψ are (positive) integers. [Di Francesco, PZJ '07]
2. The largest entries of Ψ correspond to patterns of the type

   and are equal to A_{n-1}.

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3. The sum of entries of Ψ is $\langle 1|\Psi\rangle = A_n$. [Di Franccesco, PZJ '04]

where

$$A_n = \frac{1!4!7!\cdots(3n-2)!}{n!(n+1)!(n+2)!\cdots(2n-1)!} = 1, 2, 7, 42, 429\ldots$$

Remark: the combinatorial meaning of *every* entry is the subject of the Razumov–Stroganov conjecture.

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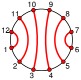
Normalize Ψ so that the smallest entries, with patterns of the type 

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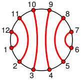

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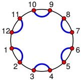

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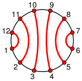

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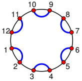

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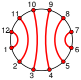

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   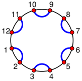

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Inhomogeneous $O(1)$ loop model

Consider the probabilistic model (on the cylinder) with probabilities p_i which depend on the column $i = 1, \ldots, 2n$ of the plaquette.

Parametrize the probabilities as $p_i = \frac{z_i - q\,t}{t - q\,z_i}$, $q = e^{2i\pi/3}$. z_i are the spectral parameters.

The ground state is now a function of these z_i:

$$\Psi(z_1, \ldots, z_{2n}) = \sum_{\pi} \Psi_\pi(z_1, \ldots, z_{2n})\ \pi$$

where the $\Psi_\pi(z_1, \ldots, z_{2n})$ can be chosen to be polynomials.

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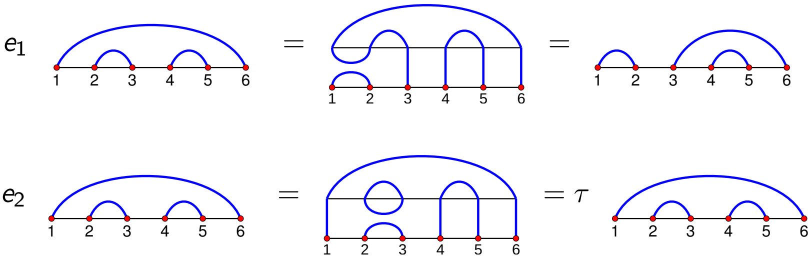
The Temperley–Lieb algebra $\mathrm{TL}_L(\tau)$ (a quotient of the Hecke algebra) is defined by generators e_i, $i = 1, \ldots, L-1$, and relations

$$e_i^2 = \tau e_i \qquad e_i e_{i\pm1} e_i = e_i \qquad e_i e_j = e_j e_i \qquad |i-j| > 1$$

Define the action of Temperley–Lieb generators e_i on link patterns of size $L = 2n$: (for convenience link patterns are drawn in the half-plane)

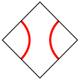
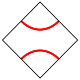


where the weight of a closed loop is τ.

R-matrix

Set $\tau = -q - 1/q$, and define the R-matrix:

$$\check{R}_i(u) = \frac{(q\,u - q^{-1})I + (u-1)e_i}{q - q^{-1}u}$$

where $I =$  and $e_i =$ .

It satisfies the Yang–Baxter equation:

$$\check{R}_i(u)\check{R}_{i+1}(uv)\check{R}_i(v) = \check{R}_{i+1}(v)\check{R}_i(uv)\check{R}_{i+1}(u)$$

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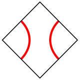
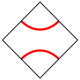
where $I =$  and $e_i =$ .

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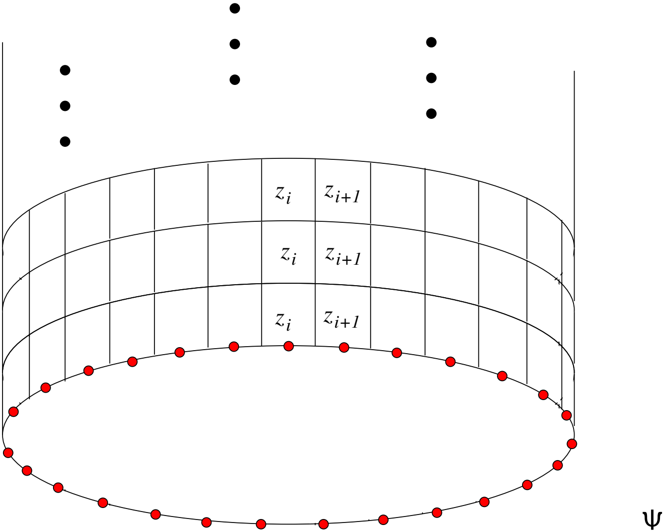

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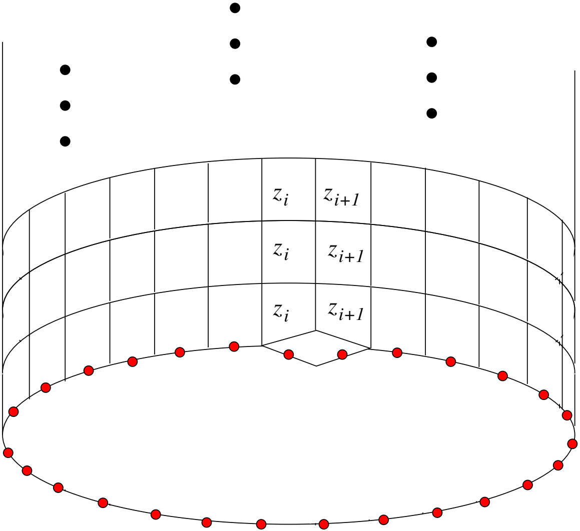

$\check{R}_i(z_{i+1}/z_i)\Psi$

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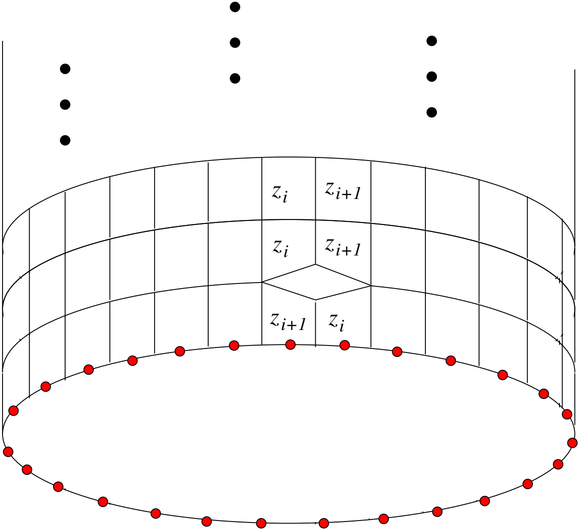

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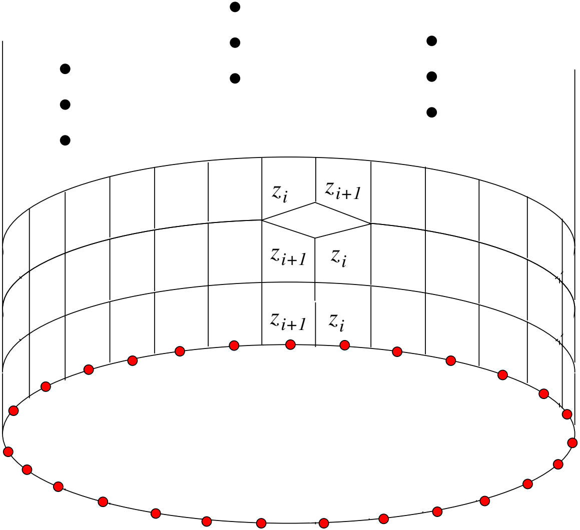

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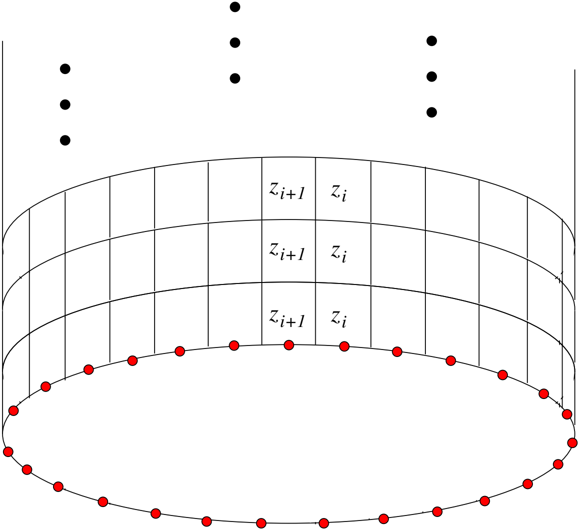

$\Psi(\ldots, z_{i+1}, z_i, \ldots)$

- Exchange relation:

$$\check{R}_i(z_{i+1}/z_i)\Psi(z_1,\ldots,z_{2n}) = \Psi(z_1,\ldots,z_{i+1},z_i,\ldots,z_{2n})$$

cheating slightly: there is a priori a scalar factor which is not easy to get rid of.

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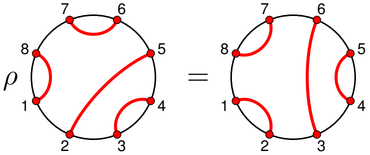

The quantum Knizhnik–Zamolodchikov equation. is a system of equations that appears:

- in the study of form factors of integrable models [Smirnov, '86]
- in the representation theory of quantum affine algebras [Frenkel, Reshetikhin '92]
- in the study of correlation functions of integrable models [Jimbo, Miwa et al, '93]
- in relation to representation theory of affine Hecke algebra and DAHA [Cherednik, Pasquier. . .]

If $\Phi(z) = (\Phi^{(0)}(z), \Phi^{(1)}(z))$ is the vertex operator (type I/II) associated to level 1 highest weight modules of $U_q(\widehat{sl(2)})$, then $\hat\Psi(z_1, \ldots, z_{2n}) = \langle 0| \Phi(z_1) \cdots \Phi(z_{2n}) |0\rangle$ satisfies the following system of equations:

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$$\rho^{-1}\hat\Psi(z_1, \ldots, z_{2n}) = \hat\Psi(z_2, \ldots, z_{2n}, s\, z_1)$$

with $s = q^6$ (level 1).

For $q = \exp(2\pi i/3)$, this coincides with the relations satisfied by the $O(1)$ loop model ground state, and in fact $\Psi = \hat\Psi$ (up to change of basis).

Vertex operators can be expressed in terms of a (q-deformed) bosonic field:
$[b_m, b_n] = \frac{[2m][m]}{m}\delta_{m,-n}$, $[b_0, \beta] = 2$

$$\Phi^{(0)}(z) = e^{\beta} z^{b_0} : e^{\sum_{n\in\mathbb{Z}_{\neq 0}} q^{-|n|/2} \frac{b_n}{[2n]} z^{-n}} :$$

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Homogeneous limit for generic q

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Example ($2n = 4$)

Ψ $= (q\,z_1 - q^{-1} z_2)(q\,z_3 - q^{-1} z_4) \longrightarrow 1$

Ψ $= (q\,z_2 - q^{-1} z_3)(q^{-2} z_4 - q^{2} z_1) \longrightarrow \tau$

where $\tau = -q - q^{-1}$.

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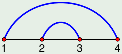
Ψ 

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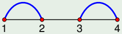
Ψ 

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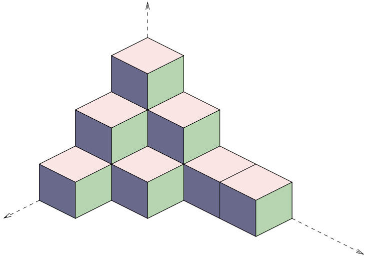

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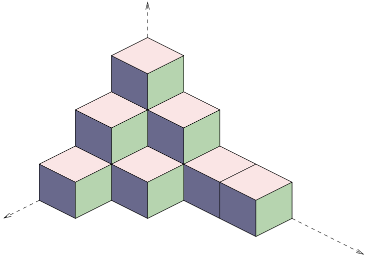

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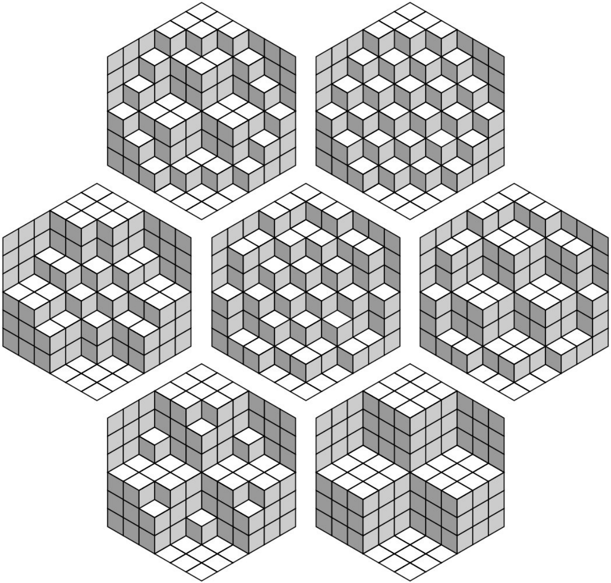


Physically, plane partitions are free fermions and their enumeration produces a Pfaffian or a determinant.
Here the corresponding computation can be carried out [Andrews, '94], and we find that the number of TSSCPPs of size n is given by the sequence

$$A_n = \frac{1!4!7!\cdots(3n-2)!}{n!(n+1)!(n+2)!\cdots(2n-1)!} = 1, 2, 7, 42, 429\ldots$$

After some computations [Di Francesco, PZJ '07] (using a formula from [Zeilberger '07] reproved in [Fonseca, PZJ '08]):

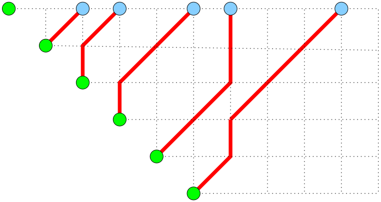
$$\sum_{\pi} \Psi_\pi \big|_{homogeneous}$$

$$= \sum_{0\le r_1<r_2<\cdots<r_n} \tau^{n(n-1)-\sum_j r_j} \det\left[\binom{2i-r_j}{i}\right]_{1\le i\le n, 0\le j\le n-1}$$

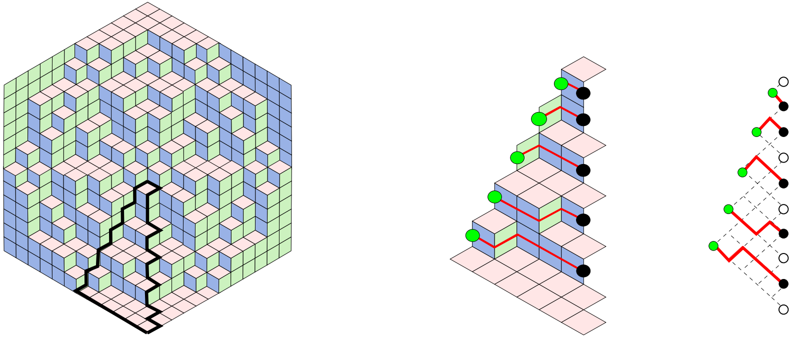
LGV formula for non-intersecting paths

As a special case of the Lindström–Gessel–Viennot formula, $\det\left[\binom{2i-r_j}{i}\right]_{1\le i\le n, 0\le j\le n-1}$ = # non-intersecting lattice paths from $(i,-i)$ to $(r_j,0)$ with moves $(1,0)$ and $(1,1)$, so that

$$\sum_\pi \Psi_\pi\big|_{homogeneous} = \sum_{\text{endpoints}} \sum_{\text{NILPs}} \tau^{\#\ up\ steps}$$



Totally Symmetric Self-Complementary Plane Partitions



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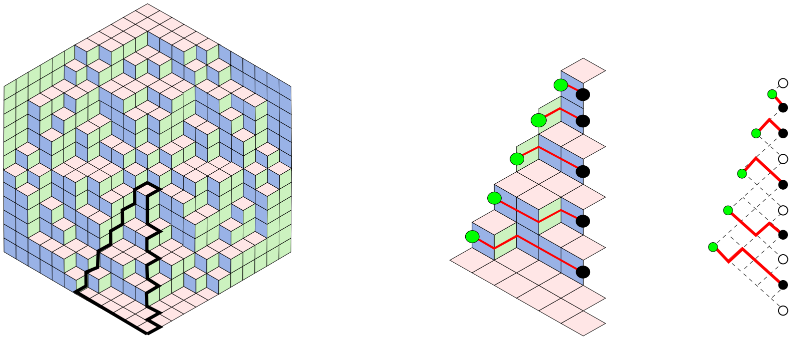

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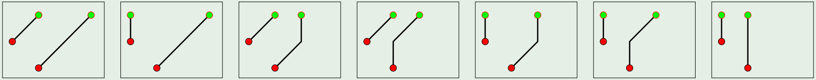
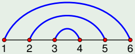
Example ($2n = 6$)

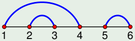
There are $A_3 = 7$ TSSCPPs:



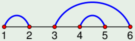
$1 \quad \tau \quad \tau \quad \tau \quad \tau^2 \quad \tau^2 \quad \tau^3$

Ψ 

$\big|_{homogeneous} = 1$

Ψ 

$\big|_{homogeneous} = \tau^2$

Ψ 

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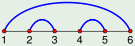
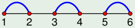
Ψ 

$\big|_{homogeneous} = 2\tau$

Ψ 

$\big|_{homogeneous} = \tau^3 + \tau$

so that $\quad \sum_\pi \Psi_\pi \big|_{homogeneous} = 1 + 3\tau + 2\tau^2 + \tau^3.$

Zeilberger ['94], in a famous 80-page computation, showed that the number of ASMs of size n is equal to the the number of TSSCPPs of size $2n \times 2n \times 2n$, and thus given by the same sequence

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Zeilberger's 80 page theorem reduced to a few simple points:

- Introduce the same integral as before at $q = e^{\pm 2i\pi/3}$:

$$Z_n(\underline{z}) = \prod_{1\le i<j\le 2n} (q\, z_i - q^{-1} z_j) \oint \cdots \oint \prod_{\ell=1}^{n} \frac{\mathrm{d}w_\ell (q\, w_\ell - z_{2\ell-1})}{2\pi i}$$

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In 1986, Mills, Robbins and Rumsey conjectured that the doubly refined ASM counting

$$\sum_{\text{ASMs}} x^{\text{position of 1 of 1st row}} y^{\text{position of 1 of last row}}$$

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One can write integral formulae for these weighted sums and show that they agree [Fonseca, PZJ '08]; the simple refinement was already pointed out in [Razumov, Stroganov, PZJ, '07].

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Partial sums as observables [Fonseca, PZJ '09]

What are interesting observables? (e.g. from the point of view of percolation)

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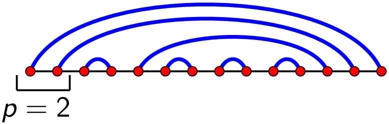

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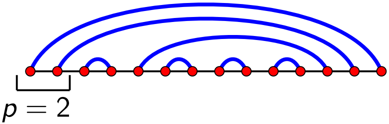

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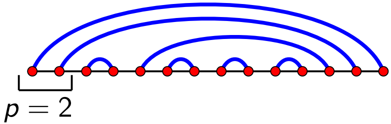

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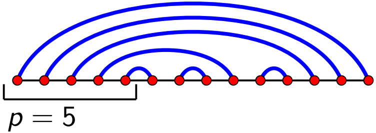

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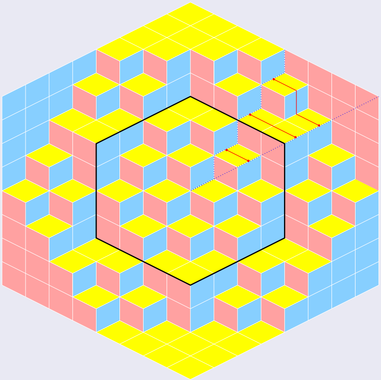

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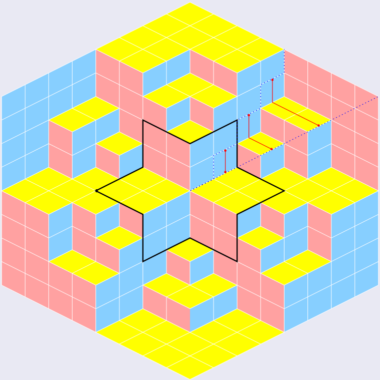

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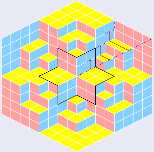

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where $n = p + r$. Asymptotically,

probability(no pairings among first p points)

$$\propto 4^{-p(3p+2)/4} (3\sqrt{3})^{p(p+1)/2} p^{7/72}$$

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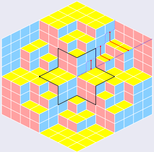

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