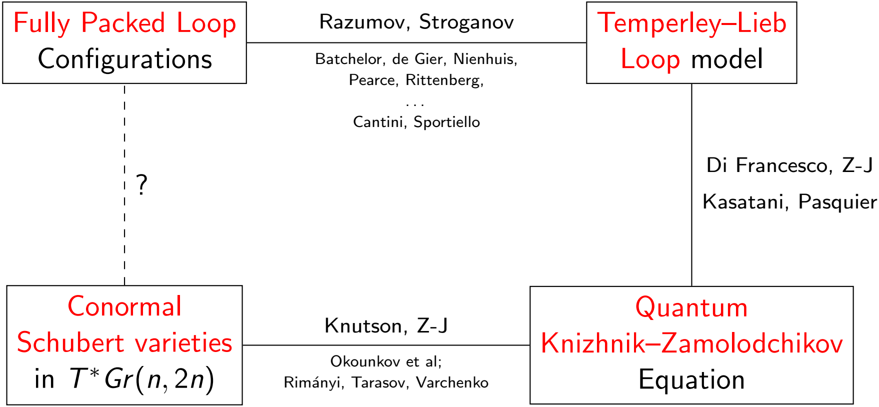
Razumov–Stroganov correspondences and the geometry of Schubert varieties

P. Zinn-Justin

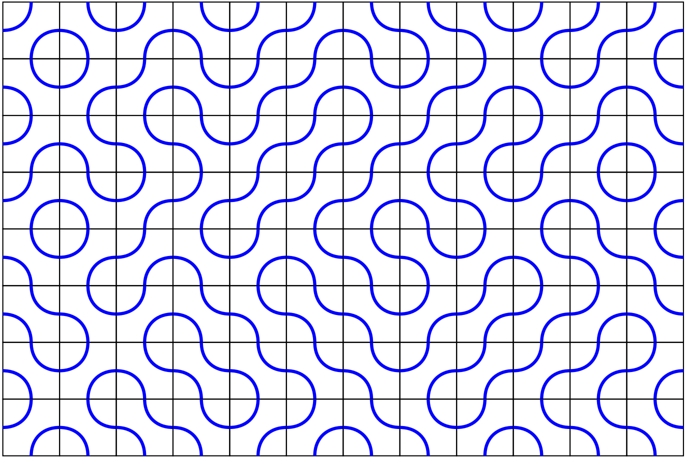
October 20, 2017

Séminaire Lotharingien de Combinatoire

Introduction



The Temperley–Lieb Loop model (equivalent to a model of *critical bond percolation*):



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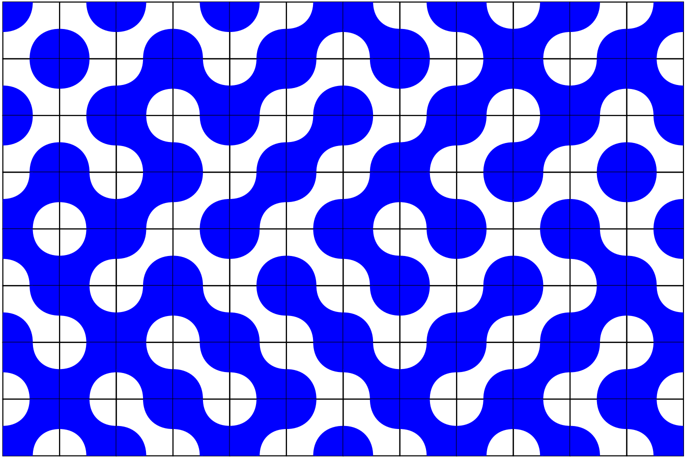

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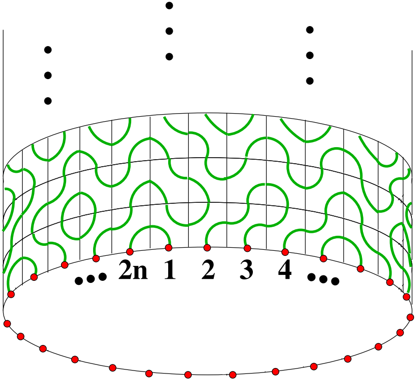
 with probability p,  with probability $1 - p$. $(0 < p < 1)$



Case of the half-infinite cylinder geometry (“periodic boundary conditions”)

Probability law of the connectivity of the external vertices?

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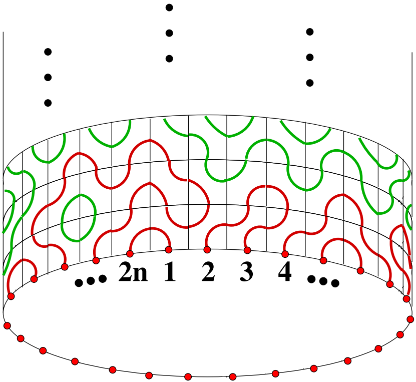


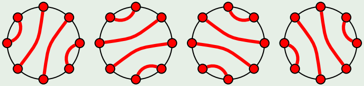
Case of the half-infinite cylinder geometry ("periodic boundary conditions")

Probability law of the connectivity of the external vertices?

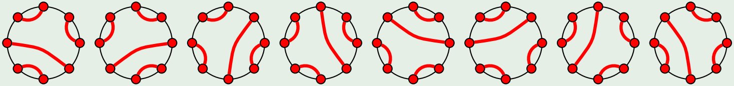
The connectivity of the external vertices can be encoded into a link pattern = a planar pairing of $2n$ points on a circle.

Example

In size $L = 2n = 8$,



$\frac{1}{42}$



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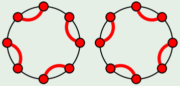
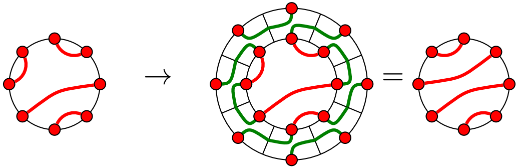


$\frac{7}{42}$

Relation to Markov process on link patterns

Using a transfer matrix formalism, one can reformulate the computation of these probabilities in terms of a Markov process on link patterns (dependent on p):



Then the vector $|\Psi\rangle = \sum_\pi Prob(\pi)|\pi\rangle$ is the steady state eigenvector:

$$T(p)|\Psi\rangle = |\Psi\rangle$$

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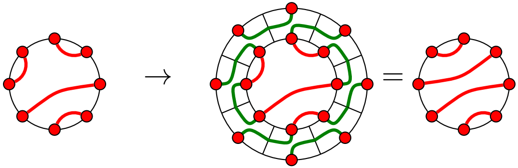

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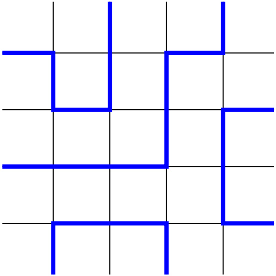
A Fully Packed Loop configuration (FPL) on a $n \times n$ square grid:



Thus, FPL configurations are in bijection with ASMs and with 6-vertex configurations with DWBC. Their number is

$$A_n = \frac{1!4!7!\cdots(3n-2)!}{n!(n+1)!(n+2)!\cdots(2n-1)!} = 1, 2, 7, 42, 429\ldots$$

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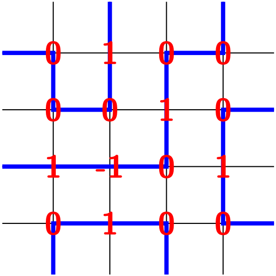

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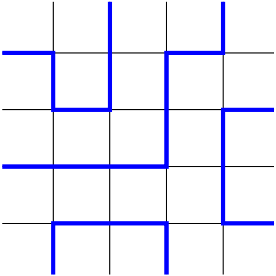

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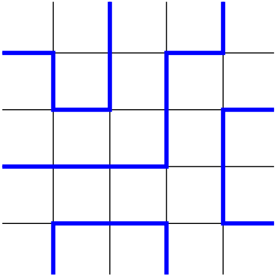

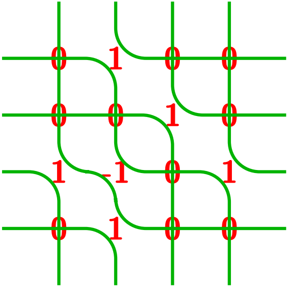

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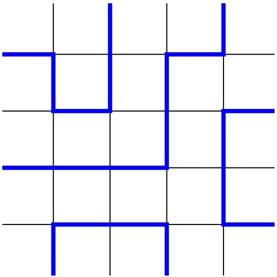
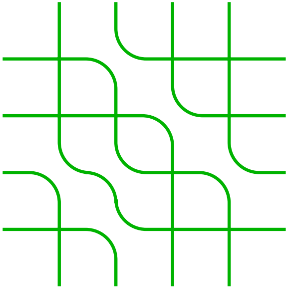


$$\begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{matrix}$$



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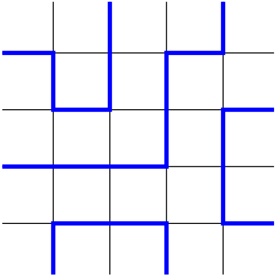

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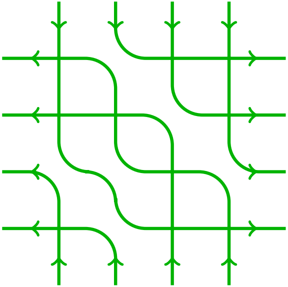

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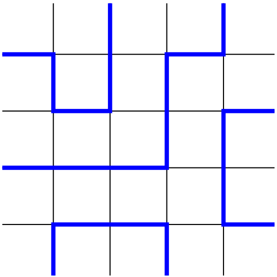

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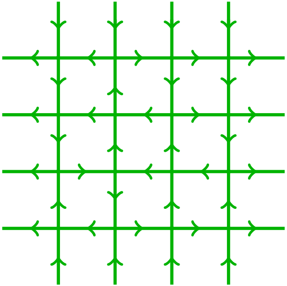

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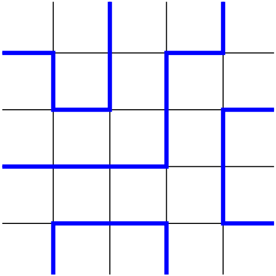

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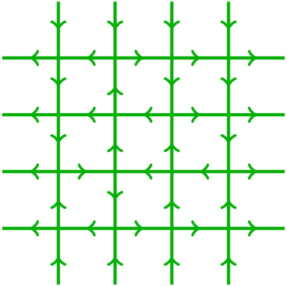

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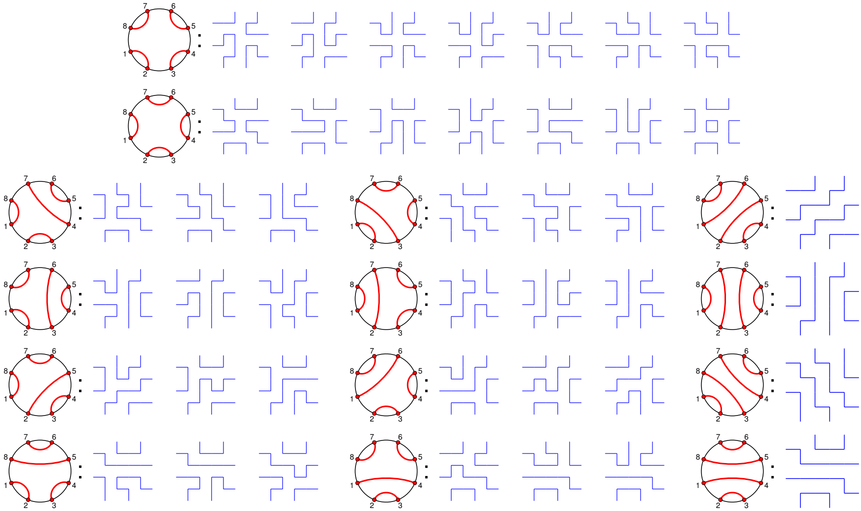
It is natural to group FPLs by connectivity of their endpoints: cf



Denote by $A(\pi)$ the number of FPLs with connectivity described by the link pattern π. Razumov and Stroganov observed (2001), and then Cantini and Sportiello proved (2010), that $A(\pi)$ is exactly the (unnormalized) probability of pattern π in the model of loops with the geometry of the cylinder.

In other words $|\Psi\rangle = \sum_\pi A(\pi)|\pi\rangle$ is the (unnormalized) steady state of the Markov process of loops:

$$T(p)|\Psi\rangle = |\Psi\rangle$$

Remark: there are (still conjectural!) variations: other types of b.c. on TL $\leftrightarrow$ different symmetry classes of ASM/FPL [Batchelor, de Gier & Nienhuis '01; Razumov-Stroganov '01; Pearce, de Gier & Rittenberg '01, ...]

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Consider the probabilistic model (on the cylinder) with probabilities p_i depending on the column $i = 1, \ldots, 2n$, which we parameterize as $p_i = \frac{t - q\, z_i}{z_i - q\, t}$, $1 - p_i = q^2 \frac{t - z_i}{z_i - q\, t}$, $q = e^{2\pi i/3}$.

This inhomogeneous model is still integrable, the z_i are the spectral parameters.

The corresponding steady state is $|\Psi(z_1, \ldots, z_{2n})\rangle$.

$$T(z_1, \ldots, z_{2n}|t)|\Psi(z_1, \ldots, z_{2n})\rangle = |\Psi(z_1, \ldots, z_{2n})\rangle$$

- Polynomiality.
  $|\Psi(z_1,\ldots,z_{2n})\rangle$ can be normalized in such a way that its components are homogenous polynomials of total degree $n(n-1)$ and of partial degree at most $n-1$ in each z_i.
- Factorization and symmetry.
  The components possess various linear factors and properties of symmetry by exchange of variables.
  In particular, their sum is a symmetric polynomial of all z_i.
- Recursion relations.
  Components of $|\Psi(z_1,\ldots,z_{2n})\rangle$ satisfy linear recursion relations; their sum is entirely determined by these:

$$\sum_{\pi} \Psi_\pi(z_1,\ldots,z_{2n}) = IK_n(q;z_1,\ldots,z_{2n}) = Schur(\ydiagram{4,4,3,3,2,2,1,1};z_1,\ldots,z_{2n})$$

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Motivation

The quantum Knizhnik–Zamolodchikov equation is a system of equations that appears:

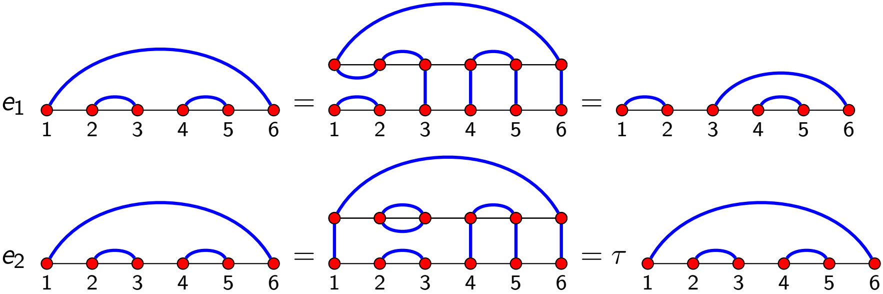
- in the study of form factors of integrable models [Smirnov, '86]
- in the representation theory of quantum affine algebras [Frenkel, Reshetikhin '92]
- in the study of correlation functions of integrable models [Jimbo, Miwa et al, '93]
- in relation to representation theory of affine Hecke algebra and DAHA [Cherednik, Pasquier, '90s]
- As we shall see now, it can also be applied to the Temperley–Lieb loop model [Di Francesco, PZJ, '05]

The Temperley–Lieb algebra

The Temperley–Lieb algebra $\mathrm{TL}_L(\tau)$ (a quotient of the Hecke algebra) is defined by generators e_i, $i = 1, \ldots, L-1$, and relations

$$e_i^2 = \tau e_i \qquad e_i e_{i\pm1} e_i = e_i \qquad e_i e_j = e_j e_i \qquad |i-j| > 1$$

Define the action of Temperley–Lieb generators e_i on link patterns:



where the weight of a closed loop is τ.

Introduce the rotation operator ρ such that $\rho e_i \rho^{-1} = e_{i+1}$. This allows to define an extra element

$$e_L = \rho e_{L-1} \rho^{-1} = \rho^{-1} e_1 \rho$$

Together the $e_1, \ldots, e_L$ form a representation of the *affine* Temperley–Lieb algebra.

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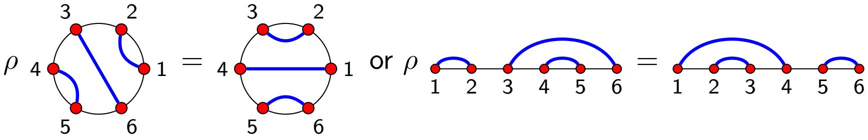
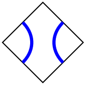
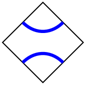


The R-matrix

Write $\tau = -q - 1/q$, and define the R-matrix to be

$$\check{R}_i(u) = \frac{(q\,u - q^{-1})I + (u-1)e_i}{q - q^{-1}u}$$

Graphically, $\check{R}_i = \frac{q\,u - q^{-1}}{q - q^{-1}u}$  $+ \frac{u-1}{q-q^{-1}u}$  acting on i^{th} and $(i+1)^{\text{th}}$ sites, u being the ratio of spectral parameters at sites i and $i+1$.

It satisfies the Yang–Baxter equation

$$\check{R}_i(u)\check{R}_{i+1}(uv)\check{R}_i(v) = \check{R}_{i+1}(v)\check{R}_i(uv)\check{R}_{i+1}(u)$$

and the unitarity equation

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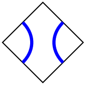
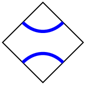
Graphically, $\check{R}_i = \frac{q\,u - q^{-1}}{q - q^{-1}u}$  $+ \frac{u-1}{q-q^{-1}u}$  acting on i^{th} and $(i+1)^{\text{th}}$ sites, u being the ratio of spectral parameters at sites i and $i+1$.

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The qKZ system

Consider the following system of equations for $|\Psi\rangle$, function of $z_1, \ldots, z_L, q, q^{-1}$ with values in the space of linear combinations of link patterns: $(i = 1, \ldots, L-1)$

$$\check{R}_i(z_i/z_{i+1})|\Psi(z_1,\ldots,z_L)\rangle = |\Psi(z_1,\ldots,z_{i+1},z_i,\ldots,z_L)\rangle \qquad (1)$$

$$\rho\,|\Psi(z_1,\ldots,z_L)\rangle = c\,|\Psi(s\,z_L,z_1,\ldots,z_{L-1})\rangle \qquad (2)$$

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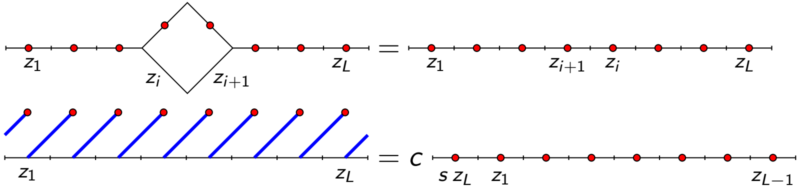
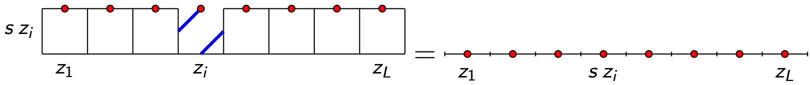


The qKZ equation

By combining Eqs. (1) and (2), one can make one spectral parameter z_i wind around the cylinder:



resulting in an equation of the form

$$S_i(z_1,\ldots,z_L)|\Psi(z_1,\ldots,z_L)\rangle = |\Psi(z_1,\ldots,s\,z_i,\ldots,z_L)\rangle, \qquad i=1,\ldots,L$$

Level 1 Polynomial solution of qKZ

Fact: in size $L = 2n$, for $s = q^6$ (level 1), there exists a polynomial solution of degree $n(n-1)$, unique up to normalization.

Example ($2n = 4$)

Ψ $= (q\, z_2 - q^{-1} z_1)(q\, z_4 - q^{-1} z_3)$

Ψ $= (q\, z_3 - q^{-1} z_2)(q^{-2} z_1 - q^{2} z_4)$

Remark: connection to nonsymmetric Macdonald polynomials [Kasatani, Takeyama] ($s \to q, q \to t^2$)

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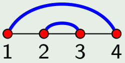
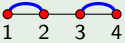
Ψ 

$= (q\, z_2 - q^{-1} z_1)(q\, z_4 - q^{-1} z_3)$

Ψ 

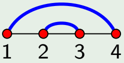
$= (q\, z_3 - q^{-1} z_2)(q^{-2} z_1 - q^2 z_4)$

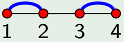
Remark: connection to nonsymmetric Macdonald polynomials [Kasatani, Takeyama] ($s \to q, q \to t^2$)

Level 1 Polynomial solution of qKZ

Fact: in size $L = 2n$, for $s = q^6$ (level 1), there exists a polynomial solution of degree $n(n-1)$, unique up to normalization.

Example ($2n = 4$)

Ψ  $= (q\, z_2 - q^{-1} z_1)(q\, z_4 - q^{-1} z_3)$

Ψ  $= (q\, z_3 - q^{-1} z_2)(q^{-2} z_1 - q^2 z_4)$

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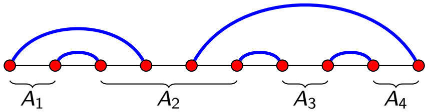
Factorization and symmetry

Given a link pattern π, one can separate vertices into maximal groups of neighbors that are not paired with each other:



Then any solution of the qKZ system satisfies

$$\Psi_\pi = \prod_k \prod_{\substack{i,j \in A_k \\ i<j}} (q\, z_j - q^{-1} z_i)\, \Phi_\pi$$

where Φ_π is symmetric in each set of variables $\{z_i, i \in A_k\}$.
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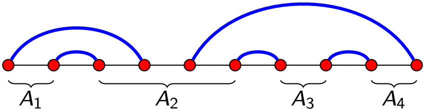

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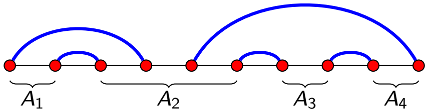

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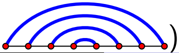
(This immediately implies uniqueness of Ψ, building the entries inductively starting from )

Connection to the loop model steady state

Set $q = e^{\pm 2\pi i/3}$, i.e., $\tau = 1$. $L = 2n$.

Then $|\Psi\rangle$ coincides with the (unnormalized) steady state of the Markov process introduced earlier.

Proof: because $s = 1$, the qKZ equation becomes an eigenvector equation for $S_i(z_1, \ldots, z_{2n}) = T(z_1, \ldots, z_{2n} | t = z_i)$. By Lagrange interpolation, $|\Psi\rangle$ is an eigenvector of $T(z_1, \ldots, z_{2n} | t)$ for all t.

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Summary of generalizations

$|\Psi(z_1,\ldots,z_{2n};q)\rangle$ = solution of qKZ

$\xrightarrow{q=e^{2\pi i/3}}$ $|\Psi(z_1,\ldots,z_{2n})\rangle$ = steady state of the inhomogeneous loop model

$\xrightarrow{z_i=1}$ $|\Psi(q)\rangle = |\tilde\Psi(\tau)\rangle$

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$z_i{=}1$ (left arrow down); $\binom{q=e^{2\pi i/3}}{\tau=1}$ (right arrow down)

$|\Psi\rangle$ = steady state of the homogeneous loop model

What is the combinatorial meaning of the level 1 polynomial solution of qKZ for generic q? In particular, what can one say about the homogeneous limit $z_i = 1$?

Example ($2n = 4$)

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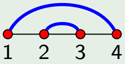

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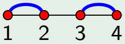

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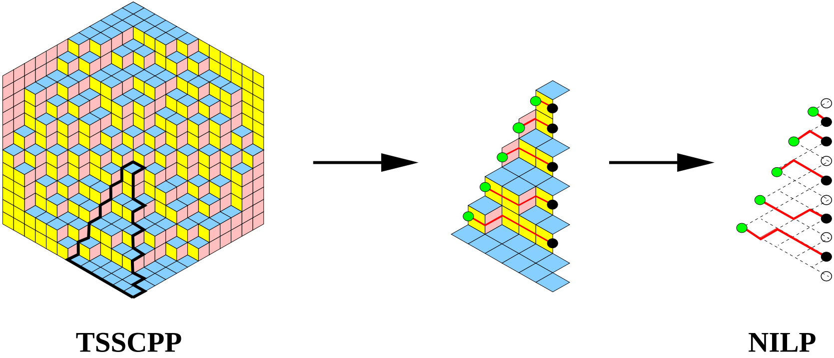
In general, one observes that the components are always polynomials of τ with non-negative integer coefficients.

Totally Symmetric Self-Complementary Plane Partitions

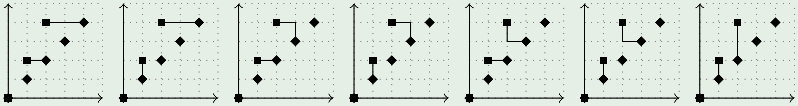


$$\sum_{\pi} \Psi_\pi|_{homogeneous} = \sum_{\text{TSSCPPs}} \tau^{\#\ pink\ \ lozenges}$$

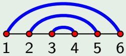
Remark: #TSSCPPs = #ASMs = A_n. (but no known bijection!)

Example ($2n = 6$)

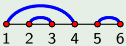
There are $A_3 = 7$ TSSCPPs:



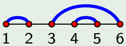
$1 \quad \tau \quad \tau \quad \tau^2 \quad \tau \quad \tau^2 \quad \tau^3$

Ψ 

$\big|_{homogeneous} = 1$

Ψ 

$\big|_{homogeneous} = \tau^2$

Ψ 

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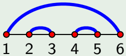
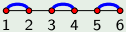
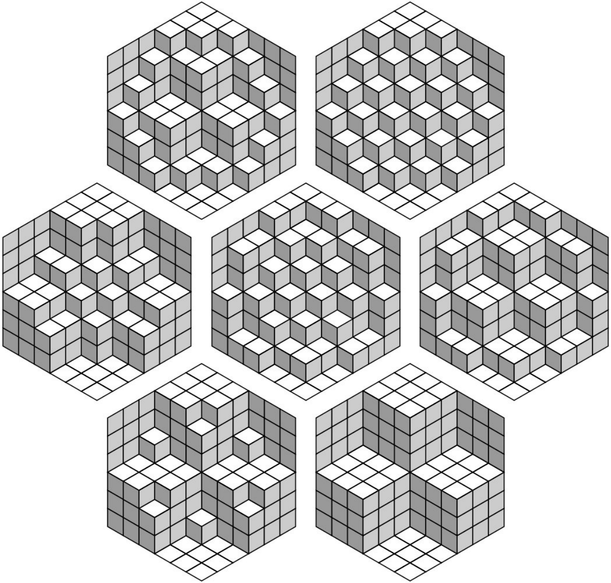
Ψ 

$\big|_{homogeneous} = 2\tau$

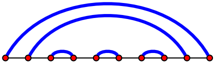
Ψ 

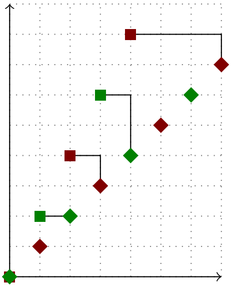
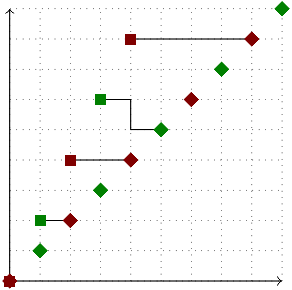
$\big|_{homogeneous} = \tau^3 + \tau$



Some particular components

[Fonseca + Z-J, '09, fixing small mistakes] Consider link patterns

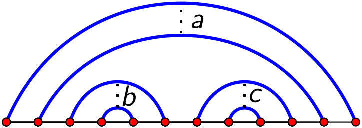
of the form $\pi_a =$ , with a arches atop a series of $n - a$ small arches. Then $\Psi_{\pi_a}|_{homogeneous} = \sum_{\text{paths}} \tau^{\text{vertical steps}}$.

$(a = 1)$  $(a = 2)$ 

(parity constraint; the first $a + 1$ steps are frozen horizontally).

The (a, b, c) case

Now consider link patterns of the form



Then $\Psi_{(a,b,c)} = \tau^{bc}|PP(a, b, c)|$ where $PP(a, b, c)$ is the set of lozenge tilings of a $a \times b \times c$ hexagon, or plane partitions of $c \times b$ with maximal part a. [conjectured by Zuber for FPLs, $\tau = 1$; proven by DF, Z-J, Zuber, '03]

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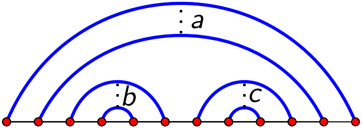

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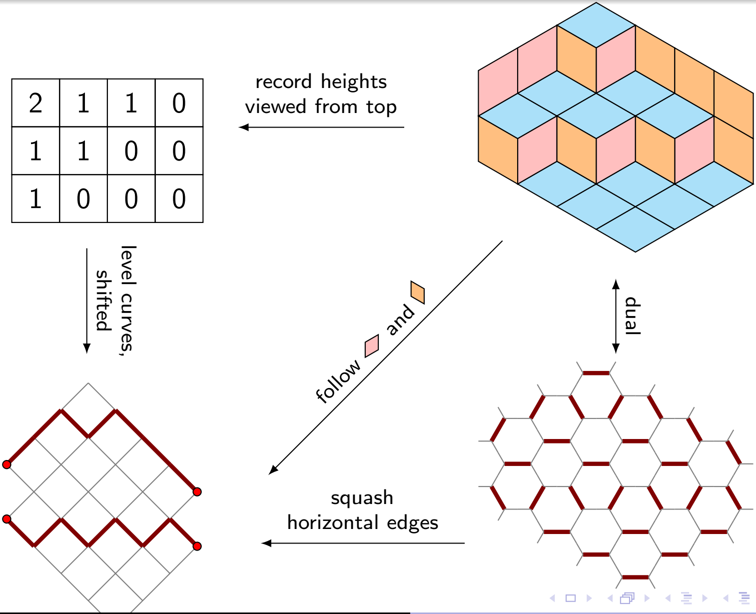
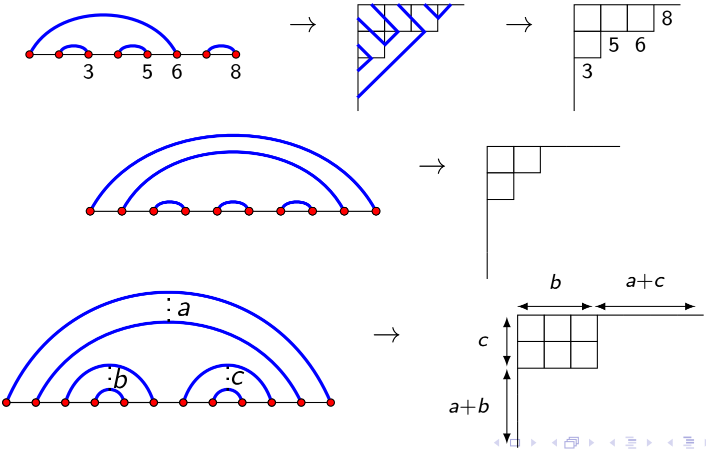
2	1	1	0
1	1	0	0
1	0	0	0



Young diagrams

There is an injective mapping from link patterns to Young diagrams in a $n \times n$ square $\cong$ subsets of $\{1, \dots, 2n\}$ of cardinality n:



Polynomiality

For a given Young diagram λ, $\Psi_{\lambda,n}(\tau)$ is a polynomial of both n and τ, of degree $|\lambda|$ in both.
[at $\tau = 1$, conjectured by Zuber and proven by Caselli, Krattenthaler, Lass, Nadeau for FPLs; for any τ, proven by Fonseca + Z-J]

In fact, the leading term in τ is known explicitly in terms of the subset s:

$$\Psi_{\lambda,n} \overset{\tau\to\infty}{\sim} \tau^{|\lambda|} \det\left[\binom{i-1}{\bar{s}_i - j}\right]_{i,j=1,\dots,n} = \tau^{|\lambda|} \det\left[\binom{n-i}{n-s_i+j}\right]_{i,j=1,\dots,n}$$

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(a,b,c) and Borel–Weil

$|PP(a,b,c)|$ is also the dimension of the representation $b \times a$ of $GL(b+c)$.

Why this choice among possible permutations of $\{a,b,c\}$?
Because if one reintroduces spectral parameters,

$$\Psi_{(a,b,c)} \overset{q\to 0}{\sim} (monomial)\; s_{b\times a}(z_{a+b+1},\dots,z_{a+2b+c})$$

where s_λ denotes the Schur polynomial with Young diagram λ (note the consistency with the symmetry property of $\Psi_{(a,b,c)}$).

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(a,b,c) and Borel–Weil cont'd

Geometrically, this representation occurs as follows: Consider the Grassmannian

$$Gr(b, b+c) = \{V \subset \mathbb{C}^{b+c} : \dim V = b\}$$

This is a projective variety, which has a sheaf $O(a)$ whose space of global sections has dimension $|PP(a, b, c)|$ (and in fact, carries the representation $b \times a$ of $GL(b+c)$).

Plücker relations and NILPs

Let us translate this into combinatorics.
Explicitly, $Gr(b, b+c)$ can be written in terms of coordinates and equations. The coordinates are the Plücker coordinates p_s indexed by subsets s of $\{1, \ldots, b+c\}$ of cardinality b. The equations are certain quadratic relations called Plücker relations.

Example:

$$Gr(2,4) = \{[p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}] : \; p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0\}$$

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Plücker relations and NILPs cont'd

The global sections of $O(a)$ are simply homogeneous polynomials of degree a in the p_s.

Each lozenge tiling in $PP(a, b, c)$, or equivalently each a-tuple of NILPs can be described by the locations of down steps of NILPs; they form a subsets s_α, $\alpha = 1, \ldots, a$, of $\{1, \ldots, b + c\}$ of cardinality b.

We can therefore associate to each element of $PP(a, b, c)$ a monomial of degree a: $\prod_{\alpha=1}^{a} p_{s_\alpha}$.

Theorem: These monomials form a basis of the degree a part of the projective coordinate ring of $Gr(b, b + c)$ (or equivalently, form a basis of global sections of $O_{Gr(b,b+c)}(a)$).

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In other words, the Plücker relations allow to express any monomial of degree a as a linear combination of those of the form above (NILPs)!

Example: $|PP(2,2,2)| = 20 = \binom{6+1}{2} - 1$:

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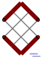
 $p_{12}p_{13} \leftrightarrow$  $p_{12}^2 \leftrightarrow$  $p_{14}p_{23} \leftrightarrow$ ? ?=  − .

Beyond Grassmannians?

It is natural to ask if such an interpretation of the leading $\tau \to \infty$ behavior of $\Psi_{\lambda,n}$ works for any λ.

We're looking for varieties X^λ indexed by partitions λ, and that possess an invariance under $\prod_k GL(|A_k|)$ (due to the symmetry property of $\Psi_{\lambda,n}$).

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It is natural to ask if such an interpretation of the leading $\tau \to \infty$ behavior of $\Psi_{\lambda,n}$ works for any λ.

We're looking for varieties X^λ indexed by partitions λ, and that possess an invariance under $\prod_k GL(|A_k|)$ (due to the symmetry property of $\Psi_{\lambda,n}$).

(and such that if $\lambda = c \times b$, $X^\lambda \cong Gr(b, b+c)$).

Schubert varieties

We define Schubert varieties X^λ inside $Gr(n, 2n)$ as follows: Recall that
$Gr(n, 2n) = \{[p_s,\ s \subset \{1, \ldots, 2n\},\ |s| = n] :\ \text{Plücker relations}\}.$
Also recall that such subsets are in bijection with Young diagrams inside the $n \times n$ square; pointwise $\geq$ corresponds to inclusion $\subset$ of Young diagrams.

Then

$$X^\lambda = \{[p_s] \in Gr(n, 2n) : p_s = 0 \text{ unless } s \subset \lambda\}$$

It is known that $\dim X^\lambda = |\lambda|$.

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Grassmannians inside Grassmannians!

Example: if $\lambda = c \times b$, subsets $s \subset \lambda$ are exactly of the form

$$\{\tilde{s}_1 + a + b, \ldots, \tilde{s}_b + a + b, a + 2b + c + 1, \ldots, 2n\}$$

where $\tilde{s}$ is a subset of $\{1, \ldots, b + c\}$ of cardinality b.

One can check the Plücker relations also agree, so that $p_s \mapsto p_{\tilde{s}}$ gives the isomorphism $X^{c \times b} \cong Gr(b + b + c)$.

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Coherent sheaves on Schubert varieties

Are there coherent sheaves on X^λ such that the leading $\tau \to \infty$ behavior of $\Psi_{\lambda,n}$ is its number of global sections?

The obvious guess (use $O(a)$ sheaves, i.e., polynomials of degree a in the projective coordinates) works for our two series of examples,

but fails for e.g. !

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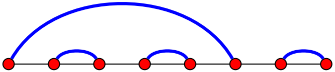
but fails for e.g. !

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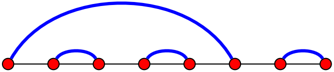
but fails for e.g. !

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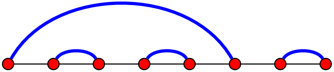
but fails for e.g. !

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Divisors of Schubert varieties

Divisors of X^λ are in one-to-one correspondence with (co)homology classes of subvarieties of codimension 1: they are exactly (integer linear combinations of) the Schubert varieties with one less box.

Example:

To each such divisor one can associate a sheaf (dual of its ideal sheaf). Its global sections are rational functions on X^λ with prescribed order of pole/zero on each divisor X^μ.

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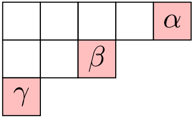

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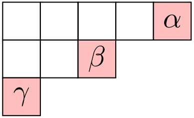

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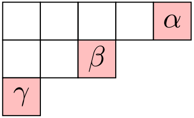

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Global sections

Given integers $a_r \geq 0$ for each corner r of λ, consider reverse plane partitions, i.e., tableaux which are weakly increasing along rows and columns, with entries ≥ 0 which are less or equal than the entries a_r at each corner r.

Theorem: a basis of global sections of the sheaf associated to the integers $a_r \geq 0$, r corner of λ, is given by associating to each tableau as above the product of Plücker coordinates associated to level curves of the tableau [and dividing by the appropriate power of p_λ itself].

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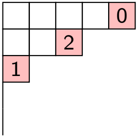
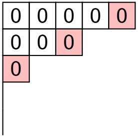
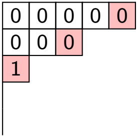


$\sim$ subset (35689)

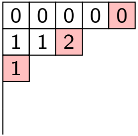


$= 1$



$= p_{45689} / p_{35689}$

...



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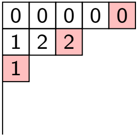

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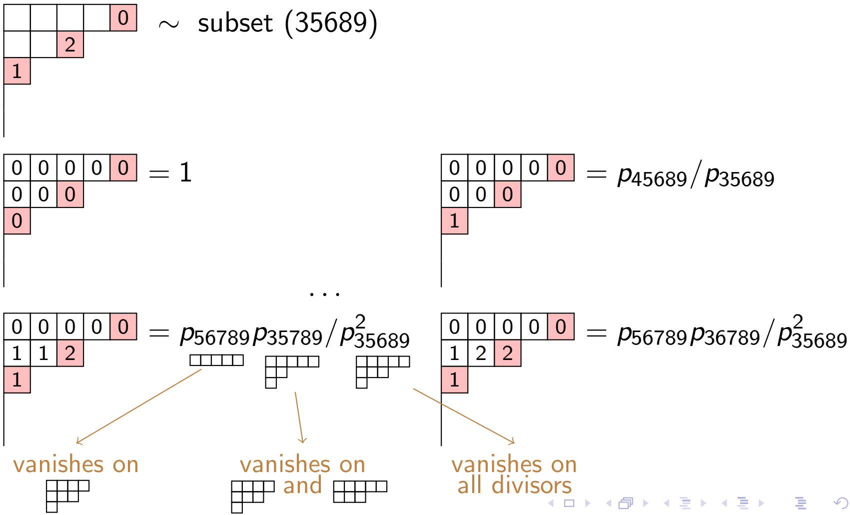

General linear group action and character

The natural action of $GL(2n)$ on $Gr(n, 2n)$ restricts to an action of $\prod_k GL(|A_k|)$ on X^λ. This means global sections of any sheaf on X^λ carry a representation of $\prod_k GL(|A_k|)$ (up to an overall twist), and we can compute the character of the space of its global sections.

Combinatorially, pad reverse partitions with zeros above and $a = \max_r a_r$ below; then the character is given (up to an overall monomial) by

$$\sum_{RPP} \prod_{\boxed{\alpha \atop \beta}} z^{\beta-\alpha}_{c(\boxed{\alpha})+n}$$

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Example of character

For example, for $\begin{array}{|c|c|}\hline & 1 \\ \hline 1 & \\ \cline{1-1}\end{array}$ $(n = 2)$,

$$\begin{array}{|c|c|}\hline 0 & 0 \\ \hline 0 & \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 0 & \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 1 & \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 1 & \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & \\ \cline{1-1}\end{array}$$

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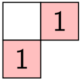
For example, for  ($n = 2$),

⋮	⋮
0	0
0	**0**
0	1
1	1
⋮	⋮

$z_1 z_3$

⋮	⋮
0	0
0	**1**
0	1
1	1
⋮	⋮

$z_1 z_4$

⋮	⋮
0	0
0	**0**
1	1
1	1
⋮	⋮

$z_2 z_3$

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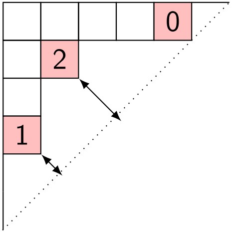
Application

Now given a link pattern and its associated Young diagram, define a sheaf σ_λ by choosing integers to be the distance to the diagonal:



By application of the Lindström–Gessel–Viennot formula, we find that the number of global sections / reverse plane partitions is

$$\det\left[\binom{i-1}{\bar{s}_i - j}\right]_{i,j=1,\ldots,n} = \det\left[\binom{n-i}{n-s_i+j}\right]_{i,j=1,\ldots,n}$$

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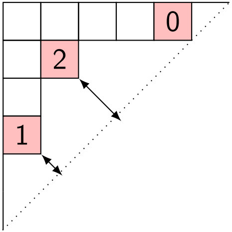

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Remarks

- Link patterns (i.e., Young diagrams inside the triangle) are exactly the cases where all numbers are nonnegative.
- For any Young diagram in the $n \times n$ square, there is no higher sheaf cohomology $\Rightarrow$ we are computing the pushforward to a point of σ_λ in K-theory.
- Polynomiality in n becomes obvious (pushforward to a point is always a polynomial of the overall shift of the integers).
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The extra circle action

Note that the variables z_i, $i = 1, \ldots, 2n$ are the formal parameters associated to computing the character w.r.t. the Cartan torus $(\mathbb{C}^\times)^{2n}$ of $GL(2n)$ (or one of its subgroups $\prod_k GL(|A_k|)$).

In order to reintroduce the parameter q, it is natural to enhance our geometric setting to incorporate an extra circle $\mathbb{C}^\times$ action.

Idea: replace X^λ with the total space of a vector bundle over X^λ:

$$\begin{aligned} CX^\lambda &\to X^\lambda \\ (x, \vec{v}) &\mapsto x \end{aligned}$$

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Digression: Hall–Littlewood polynomials

This extra circle action is standard in geometric representation theory.

In fact, if we extend the Borel–Weil construction, e.g., on $Gr(b, b+c)$, by replacing $Gr(b, b+c)$ with its cotangent bundle $T^*Gr(b, b+c)$, and then taking the same $O(a)$ sheaf (obtained by pullback from $Gr(b, b+c)$), then the corresponding character would be nothing but the (dual) Hall–Littlewood polynomial (with Young diagram $c \times b$).

Cotangent bundle of the Grassmannian

Here we do something different: we consider the cotangent bundle of the ambient space, that is $Gr(n, 2n)$.

This cotangent bundle has a very simple explicit description:

$$T^* Gr(n,2n) = \{(V,u) \in Gr(n,2n) \times \mathrm{End}(\mathbb{C}^{2n}) : \mathrm{Im}\, u \subset V \subset \mathrm{Ker}\, u\}$$

(justification: the fiber of TGr at V lives in $\mathrm{Hom}(V, \mathbb{C}^{2n}/V)$; so the dual fiber is $\mathrm{Hom}(\mathbb{C}^{2n}/V, V)$ which is naturally a subspace of $\mathrm{End}(\mathbb{C}^{2n})$ where $u|_V = 0 \Leftrightarrow V \subset \mathrm{Ker}\, u$ and $\mathrm{Im}\, u \subset V$.)

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We can then define CX^λ to be the conormal variety of the Schubert variety X^λ. (in short: conormal Schubert variety)

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$$CX^\lambda = \overline{\{(V, u) \in T^*Gr(n, 2n) : V \in X^\lambda_{smooth} \text{ and } u \perp T_V X^\lambda\}}$$

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K-theoretic pushforward

Theorem (Knutson, Z-J, '16)

Consider the (pullback of the) same sheaf σ_λ as before on CX^λ. Then its (localized, equivariant) K-theoretic pushforward to a point, which is equal to the $\mathbb{C}^\times \times (\mathbb{C}^\times)^{2n}$ character of the space of its global sections, is proportional to Ψ_λ, with the identification of the extra scaling action parameter: $t = q^{-2}$.

Relation to Okounkov theory

Okounkov et al define the R-matrix geometrically as a matrix of change of basis in the (localized, equivariant) K-theory ring $K_T^{loc}(T^* Gr(n, 2n))$; here implementing the natural Weyl group action ($\mathcal{S}_{2n}$) on the ambient space $T^* Gr(n, 2n)$.

However this matrix depends (nontrivially!) on a choice of basis of this ring. The choice made here (certain sheaves on conormal Schubert varieties) produces the R-matrix of the Temperley–Lieb loop model (which was our starting point).

Okounkov prefers the use of the so-called stable basis, which leads to the R-matrix of the six-vertex model. However in the Grassmannian case, the change of basis is very easy (maximal parabolic Kazhdan–Lusztig polynomials), so that the corresponding integrable models are easily shown to be equivalent.

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Back to (a, b, c)

Explicitly, in the case $\lambda = (a, b, c)$, the equations take the form

$$CX^{\lambda} = \{[p_s], (u_{ij}) : \ quad(p_s),\ bil(p_s, u_{ij}) = 0\}$$

where the matrix $u = (u_{ij})$ is restricted to be of the form

$$u = \begin{array}{c} \\ a+b \\ b+c \\ c+a \end{array} \begin{array}{c} \begin{array}{ccc} a+b & b+c & c+a \end{array} \\ \begin{pmatrix} 0 & B & \star \\ & 0 & C \\ & & 0 \end{pmatrix} \end{array}$$

(the upper-right block has not been named since its entries never occur in any equation.)

(a, b, c) cont'd

The sheaf $O(a)$ simply consists of polynomials in the p_s and the entries of B and C which are of degree a in the p_s (note that the degree in B and C is free, but incurs a weight of t in the computation of the character = generating/Hilbert series).

As before, one can eliminate the Plücker relations by considering only $\prod_{\alpha=1}^{a} p_{s_\alpha}$ where the s_α are in bijection with lozenge tilings. The dependence on B and C remains arbitrary, modulo the bilinar relations $bil(p_s, B_{ij}) = 0$ and $bil'(p_s, C_{ij}) = 0$.

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Example: $a = b = c = 1$

$b = c = 1$ means $Gr(b, b + c) \cong \mathbb{P}^1$, i.e., two projective coordinates p_1 and p_2 and no Plücker relations.

The bilinear equations read

$$\begin{pmatrix} p_1 & p_2 \end{pmatrix} \begin{pmatrix} B_{12} & B_{22} & C_{11} & C_{12} \\ B_{11} & B_{21} & -C_{21} & -C_{22} \end{pmatrix} = 0$$

By combining these equations, we can find

$$p_s(BC)_{ik} = 0, \qquad s, i, k = 1, 2$$

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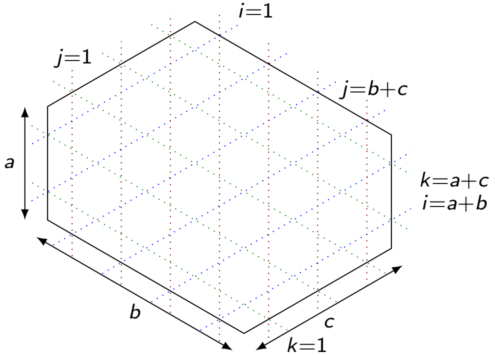
Coordinates on lozenge tilings

Introduce the following redundant coordinate system on lozenge tilings:



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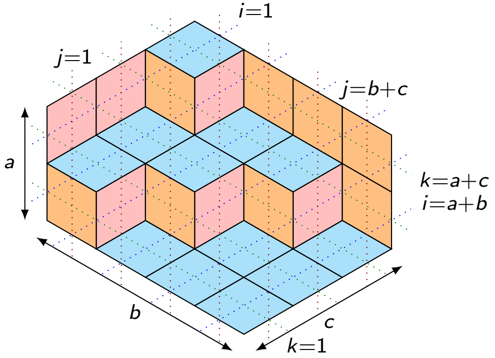

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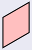
(a, b, c) and Gröbner degeneration

Theorem (Knutson, Z-J, '16)

There exists a Gröbner degeneration of the space of global sections of σ_λ (where $\lambda = (a, b, c)$), such that the equations take the following form:
For each monomial $\prod_{\alpha=1}^{a} p_{s_\alpha}$ and its associated lozenge tiling, the equations are

- $\prod_{\alpha=1}^{a} p_{s_\alpha} B_{ij} = 0$ *for each lozenge*  *at location* (i, j).
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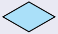
- $\prod_{\alpha=1}^{a} p_{s_\alpha} (BC)_{ik} = 0$ *for each lozenge*  *at location* (i, k).

(this is a slight simplification. . . equations above actually define toric varieties)

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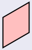
- $\prod_{\alpha=1}^{a} p_{s_\alpha} B_{ij} = 0$ *for each lozenge*  *at location* (i, j).
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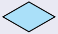
- $\prod_{\alpha=1}^{a} p_{s_\alpha} (BC)_{ik} = 0$ *for each lozenge*  *at location* (i, k).

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Example cont'd: $a = b = c = 1$

The degenerated bilinear equations read

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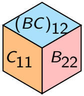

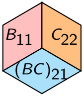

(a, b, c) and Gröbner degeneration cont'd

This immediately implies an explicit formula for Ψ_λ:

$$\Psi_{(a,b,c)} \propto \sum_{\text{lozenge tiling of } PP(a,b,c)} \prod_{\substack{\text{lozenges } (i,k) \\ \text{of type } BC}} (1 - t^2 z_i/z_{k+a+2b+c}) \prod_{\substack{\text{lozenges } (i,j) \\ \text{of type } B}} (1 - t\, z_i/z_{j+a+b}) \prod_{\substack{\text{lozenges } (j,k) \\ \text{of type } C}} (1 - t\, z_{j+a+b}/z_{k+a+2b+c})$$

In particular, in the homogeneous limit $z_i = 1$, we immediately recover, noting $1 - t^2 = (1-t)(1+t) = -q^{-1/2}(1-t)\tau$,

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PDF's conjecture

The same strategy works for other series of examples. In fact, we recover this way more than the Razumov–Stroganov correspondence; we get a proof (for various series of examples) of

Conjecture (Di Francesco, '06)

For every link pattern π, Ψ_π can be decomposed as a sum of products of the form

$$\Psi_\pi = \sum_{f \in FPL_\pi} \prod_{a=1}^{n(n-1)} \left(q^{\alpha_{f,a}} z_{j_{f,a}} - q^{-\alpha_{f,a}} z_{i_{f,a}} \right)$$

where $\alpha_{f,a} \in \{1,2\}$, and the indexing set FPL_π is the set of FPLs with connectivity π.

which itself implies positivity of coefficients of $\Psi_{\pi|homogeneous}(\tau)$.