Schubert puzzles and quantum integrability

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Joint work with A. Knutson

Schubert calculus and quantum integrability

- It remains an open problem to provide a (positive, efficient) solution to Schubert calculus, i.e., to give a combinatorial formula for the structure constants of the cohomology of general flag varieties.
- My interest is in applying methods from quantum integrable systems (QIS) to solve this problem.
- We are also interested in generalizations (equivariance; quantum; more general cohomology theories, i.e., K-theory or elliptic cohomology).
- Thanks to the work of Nekrasov+Shatashvili, {Maulik,Aganagic}+Okounkov, Rimányi+Tarasov+Varchenko, etc (and related work in geometric representation theory), we know how to describe such cohomology spaces in terms of a QIS.

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The ring structure

- The ring structure also appears naturally, as the algebra of multiplication operators is the Bethe algebra (commutative subalgebra of the Yang–Baxter algebra) of the QIS.
- However, this does not obviously help with the calculation of structure constants.
  $\to$ Another idea is required to use quantum integrable methods for that.
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Geometric interpretation

- Start with a configuration space X, e.g., $X = Gr$.
- Pick an appropriate basis of $H^*_{\binom{T}{loc}}(X)$: the Schubert basis $(S^\lambda = [X^\lambda])$
- Take the tensor product of two such classes:

$$\begin{array}{ccc} H^*(X \times X) & \xrightarrow{\cong} & H^*(X) \otimes H^*(X) \\ [s^\lambda \times s^\mu] & \mapsto & S^\lambda \otimes S^\mu \end{array}$$

- Restrict to the diagonal: $\Delta : X \hookrightarrow X \times X$

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- Expand in the original basis.

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- ~~Restrict to the diagonal~~ Symplectic reduction: $X \cong \mu^{-1}(\eta)/U$

$$\begin{array}{ccccc} H^*(Y) & \xrightarrow{i^*} & H^*(\mu^{-1}(\eta)) & \xrightarrow{p_*} & H^*(X) \\ S^{\lambda+\mu} & & \longmapsto & & (\#)\ S^\lambda\, S^\mu \end{array}$$

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Case of Grassmannians

Pick

$$X = T^*Gr(\cdot, n) = \bigsqcup_{k=0}^{n} T^*Gr(k, n)$$

$T = (\mathbb{C}^\times)^n \times \mathbb{C}^\times$ acts on X. We index fixed points with binary strings in $\{0,1\}^n$ (with k 0s, $n-k$ 1s). $H_T^*(\cdot) = \mathbb{Z}[\hbar, x_1, \ldots, x_n]$.

Using the stable envelope construction [Maulik, Okounkov] with a cocharacter in the positive chamber, we define classes $S^\lambda \in H_T^*(X)$, $\lambda \in \{0,1\}^n$, which form a basis of $H^*_{T,loc}(X)$.

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Choice of Y

We choose Y to be T^* of the two-step flag variety:

$$Y = T^* Gr(\cdot, \cdot, 2n)$$

Its $(\mathbb{C}^\times)^{2n} \times \mathbb{C}^\times$ fixed points are indexed by strings in $\{0,1,2\}^{2n}$. We define analogously the stable basis $(S^\nu)_{\nu \in \{0,1,2\}^{2n}}$.

Given $\lambda, \mu \in \{0,1\}^n$, define

$$\lambda + \mu := \{\lambda_1 + 1, \ldots, \lambda_n + 1, \mu_1, \ldots, \mu_n\} \in \{0,1,2\}^{2n}$$

(corresponding to two of the three ways that a two-step flag variety is really one-step)

Note

$$S_{\lambda+\mu}|_\rho = \begin{cases} (\#)\ S^\lambda|_\sigma\, S^\mu|_\tau & \rho = \sigma + \tau \\ 0 & \text{else} \end{cases}$$

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Symplectic reduction

There is a symplectic reduction from $Y \supset T^* Gr(k, \ell, m; 2n)$ to $T^* Gr(k', \ell', m'; n)$ with

$$k' = \frac{\ell + m - k}{2} \qquad \ell' = \frac{k + m - \ell}{2} \qquad m' = \frac{k + \ell - m}{2}$$

by the unipotent group $\left(\begin{smallmatrix} 1 & \star \\ 0 & 1 \end{smallmatrix}\right)$ and moment map value $\eta = \left(\begin{smallmatrix} \cdot & 1 \\ \cdot & \cdot \end{smallmatrix}\right)$.

When $\ell = n$, $\ell' = 0$ and $T^* Gr(k', \ell', m'; n) \cong T^* Gr(k, n) \subset X$.

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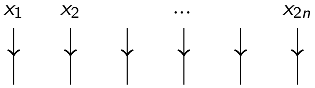
QIS reformulation

$$H_T^*(Y) \cong \mathbb{Z}[\hbar] \otimes \bigotimes_{i=1}^{2n} \mathbb{Z}[x_i]^3 \sim \text{Hilbert space of spin chain}$$



[ZJ, 2015] Symplectic reduction (of the kind we need) is given by fusion:

$$\cdots \otimes V(x) \otimes V(x' = x + \hbar) \otimes \cdots \to \cdots \otimes \bar{V}(x + \hbar/2) \otimes \cdots$$

Here we need $x_{i+n} = x_i + \hbar$, $i = 1, \ldots, n$.

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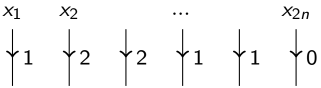

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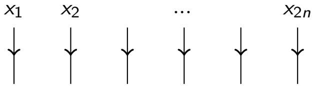

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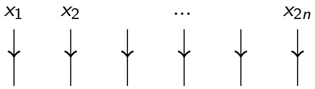

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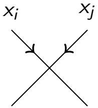
Before we can apply fusion, we need to reorder the factors of the tensor product → *R*-matrix: 

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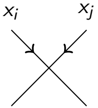
Before we can apply fusion, we need to reorder the factors of the tensor product → *R*-matrix: 

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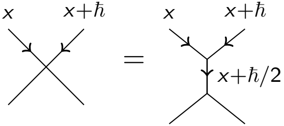

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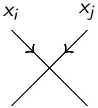
Before we can apply fusion, we need to reorder the factors of the tensor product → *R*-matrix: 

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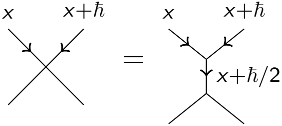

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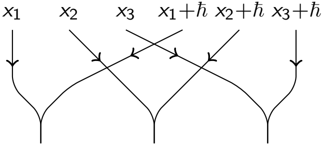

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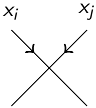
Before we can apply fusion, we need to reorder the factors of the tensor product $\to$ *R*-matrix: 

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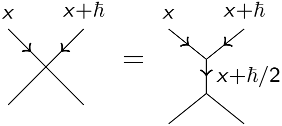

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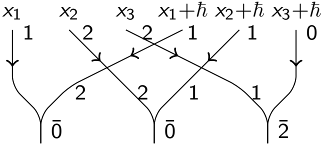
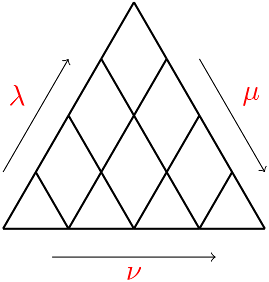


Puzzles

Use dual graphical notations, shift cyclically indices and write in binary:

$$c_{\nu}^{\lambda,\mu} = \left(\sum_{\substack{\text{edge labellings} \\ \text{with } 0,1,10}} \right)$$



$$\left(\prod_{\substack{\text{triangles,} \\ \text{lozenges}(i,j)}} \text{weight}(x_i - x_j) \right)$$

Allowed "nonequivariant" ($x_i = 0 \Rightarrow H_{\mathbb{C}^\times}(X)$) triangles:

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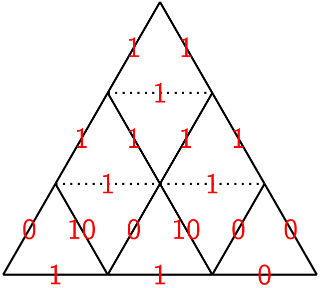

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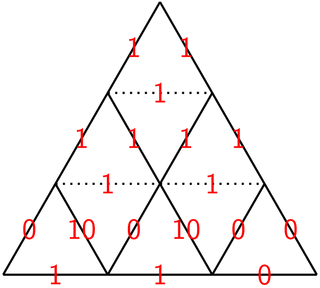

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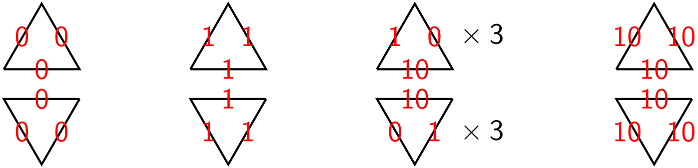

Final formula

Taking into account the factors, we can write

$$\tilde{S}^\lambda \tilde{S}^\mu = \sum_\nu c_\nu^{\lambda,\mu} \tilde{S}^\nu$$

where

$$\tilde{S}^\lambda := \frac{S^\lambda}{A_0} \in H^0_{T,loc}(X)$$

and A_0 is the class of the base Gr of T^*Gr.

$\tilde{S}^\lambda \sim$ Segre Schwartz–MacPherson (SSM) class.

Nonequivariantly,

$$c_\nu^{\lambda,\mu} = (-1)^{|\lambda|+|\mu|+|\nu|} \chi_{\mathrm{EP}}(X_o^\lambda \cap X_o^\mu \cap X_o^{\bar{\nu}})$$

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Schubert limit

Take $\hbar \to \infty$:

$$\tilde{S}^\lambda = S^\lambda A_0^{-1} = \hbar^{-|\lambda|} S_0^\lambda + \cdots$$

where S_0^λ is the corresponding Schubert class, and $|\lambda| = \#\{i < j : \lambda_i > \lambda_j\}$ is the inversion number of λ.

Renormalizing the weights of puzzles by appropriate powers of $\hbar$, we find the following shorter list of puzzle pieces, with simplified weights:

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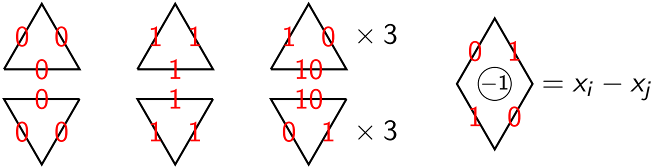

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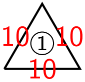
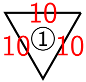
- In particular, in nonequivariant K-theory, triangles 

, 

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Partial flag varieties

We now consider the case of d-step flag varieties:

$$X = T^*\mathcal{F}_d, \qquad \mathcal{F}_d = \{0 \le V_1 \le \cdots \le V_d \le \mathbb{C}^n\}$$

which are Nakajima quiver varieties of type A_d.

QIS and root systems

$X \leftrightarrow$ model ❶, $Y \leftrightarrow$ model ❷.

	model ❶	dim rep ❶	model ❷	dim rep ❷
$d = 1$	A_1	2		
$d = 2$	A_2	3		
$d = 3$	A_3	4		
$d = 4$	A_4	5		
$d \geq 5$	A_d	$d + 1$		

NB. Can be generalized to any symmetric tensor power of the defining representation of A_d.

QIS and root systems

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$d = 1$	A_1	2	A_2	
$d = 2$	A_2	3	D_4	
$d = 3$	A_3	4	E_6	
$d = 4$	A_4	5	E_8	
$d \geq 5$	A_d	$d + 1$	Kac–Moody?	

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$d = 2$	A_2	3	D_4	8
$d = 3$	A_3	4	E_6	27
$d = 4$	A_4	5	E_8	$248 + 1$
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Puzzles and Schubert limit

One can again formulate the structure constants of $H_T^*(X)$ (or $K_T(X)$) in the SSM basis in terms of a triangular partition function of model 2 ("puzzle rule").

One can then take the limit $\hbar \to \infty$ (or $q^{\pm} \to 0$) to obtain a puzzle rule for ordinary Schubert calculus in partial flag varieties.

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In general, a (nonequivariant) puzzle is an assignment of labels from a certain set L_d to each edge of a triangle inside the triangular lattice of the plane, such that $\{0, 1, \ldots, d\} \subset L_d$, and the boundary edges are labelled with $\{0, 1, \ldots, d\}$ only.

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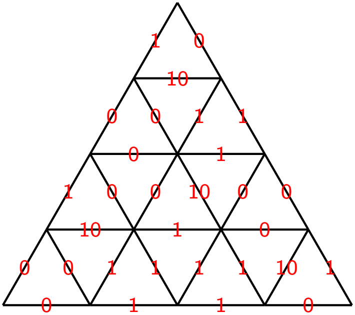

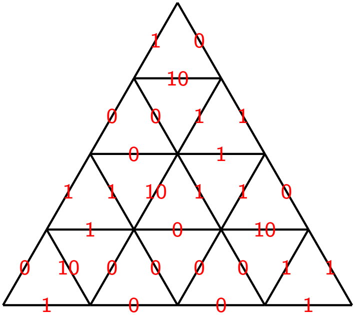

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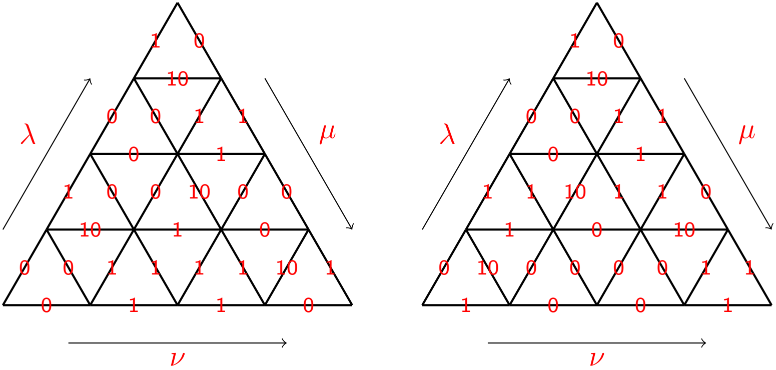
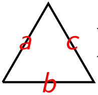
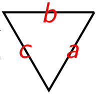


Fugacities

Assign a "fugacity" to each elementary triangle:

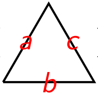
- In $H(\mathcal{F}_{d\leq 3})$,

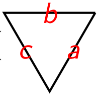
fug(

) = fug(

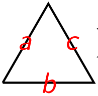
) $\in \{0, 1\}$

- In $K(\mathcal{F}_{d\leq 3})$,

fug(

), fug(

) $\in \{-1, 0, 1\}$

- In H or $K(\mathcal{F}_4)$, fug(

) $\in \mathbb{Q}\ldots$

The formula

The output of this construction is L_d and fugacities such that structure constants of $H/K(\mathcal{F}_d)$ are given by

$$c_{\lambda,\mu}^{\nu} = \sum_{\substack{\text{puzzles} \\ \text{sides } \lambda,\mu,\nu}} \prod_{\substack{\text{elementary} \\ \text{triangles}}} \text{fug(triangle)}$$

In particular, in $H(\mathcal{F}_{d\leq 3})$,

$$c_{\lambda,\mu}^{\nu} = \#\{H\text{-admissible puzzles with sides } \lambda,\mu,\nu\}$$

and in $K(\mathcal{F}_{d\leq 3})$,

$$c_{\lambda,\mu}^{\nu} = (-1)^{|\lambda|+|\mu|-|\nu|}\#\{K\text{-admissible puzzles with sides } \lambda,\mu,\nu\}$$

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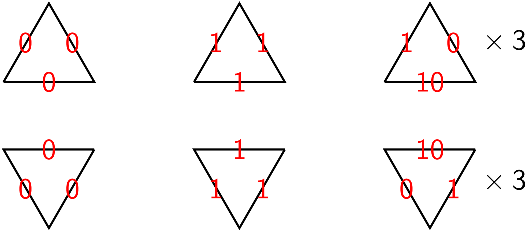
$H(\mathcal{F}_1) = H(Gr)$

$L_1 = \{0, 1, 10\}$

H-admissible triangles:



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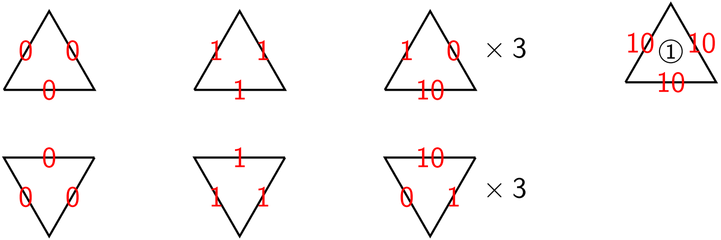

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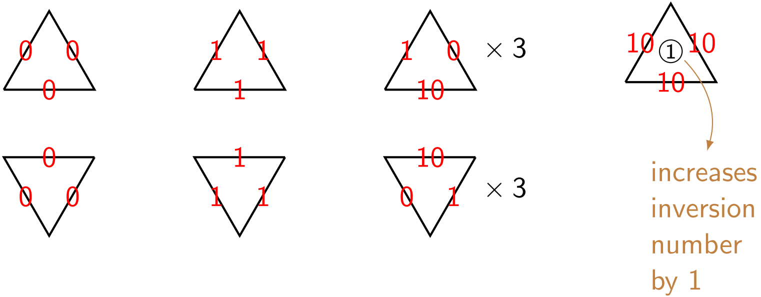

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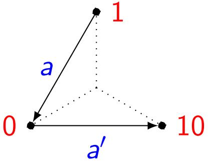

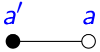

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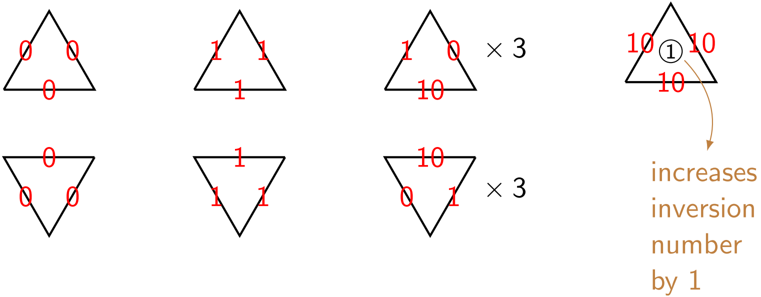

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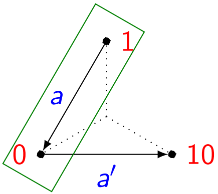

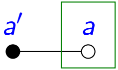

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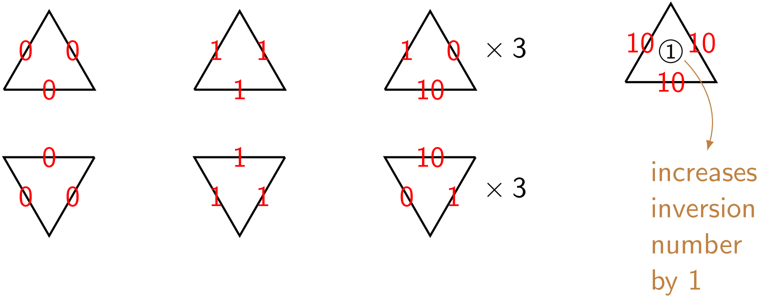

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×3 (triality)

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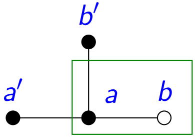

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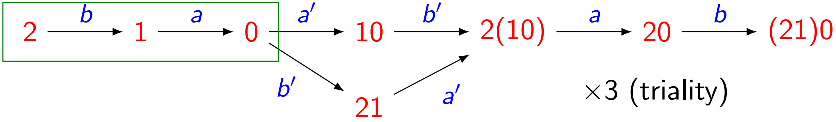

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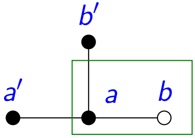

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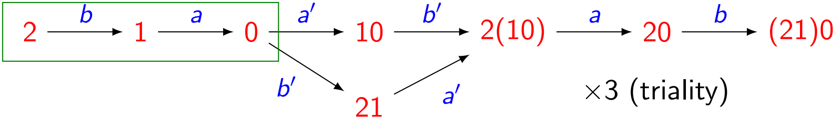

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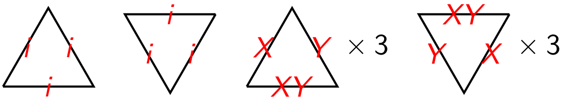

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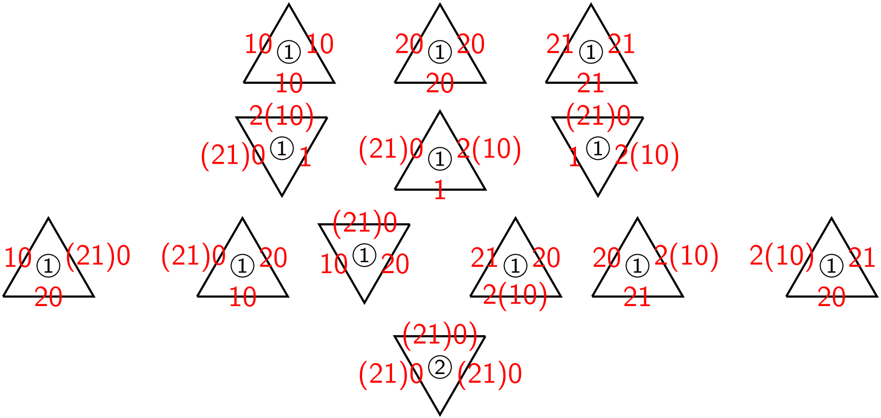

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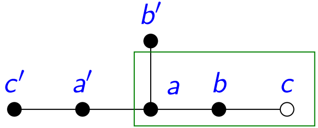

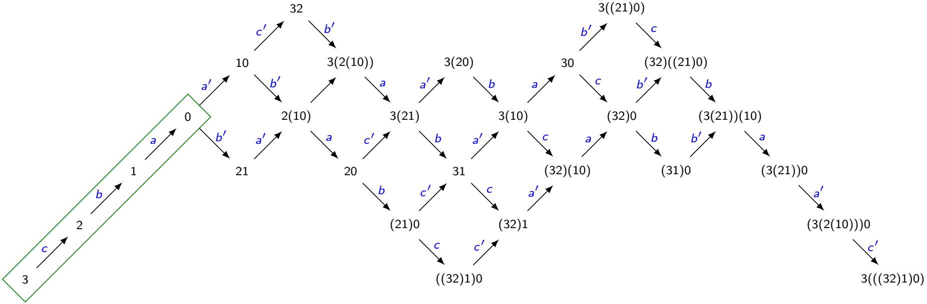

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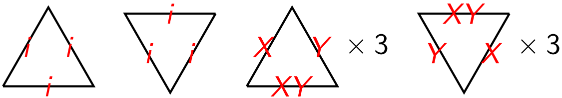

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This "simple" structure fails at $d = 4$:

- The required representation of E_8 is not minuscule ($248 + 1$-dimensional, with a $8 + 1$-dimensional zero weight space).
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$\times 3$.
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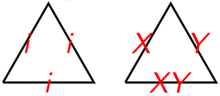
- There is no longer a labelling of the weights using multinumbers, such that H-admissible triangles are of the form 

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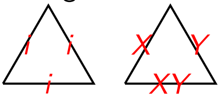
- There is no longer a labelling of the weights using multinumbers, such that H-admissible triangles are of the form 

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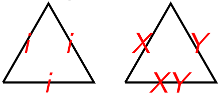
- There is no longer a labelling of the weights using multinumbers, such that H-admissible triangles are of the form  $\times 3$.

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