Quantum integrability and Combinatorics: Razumov–Stroganov type conjectures 2

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Outline of the lecture

1. Quantum Knizhnik–Zamolodchikov equation for Temperley–Lieb
   - Temperley–Lieb algebra
   - qKZ equation
   - Relation to loop model
2. Integral formulae and applications
   - Integral formulae for qKZ solutions
   - Relation to plane partitions
3. Orbital varieties

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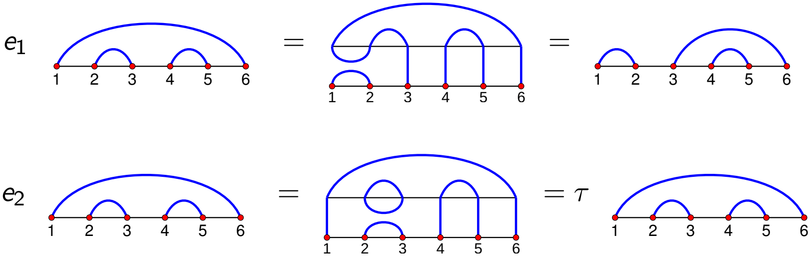
The Temperley–Lieb algebra $\mathrm{TL}_L(\tau)$ (a quotient of the Hecke algebra) is defined by generators e_i, $i = 1, \ldots, L-1$, and relations

$$e_i^2 = \tau e_i \qquad e_i e_{i\pm1} e_i = e_i \qquad e_i e_j = e_j e_i \qquad |i-j| > 1$$

Define the action of Temperley–Lieb generators e_i on link patterns: $(L = 2n)$



where the weight of a closed loop is τ.

R-matrix

Set $\tau = -q - 1/q$, and define the R-matrix:

$$\check{R}_i(u) = \frac{(q\,u - q^{-1})I + (u-1)e_i}{q - q^{-1}u}$$

It satisfies the Yang–Baxter equation:

$$\check{R}_i(u)\check{R}_{i+1}(uv)\check{R}_i(v) = \check{R}_{i+1}(v)\check{R}_i(uv)\check{R}_{i+1}(u)$$

and the unitarity equation:

$$\check{R}_i(u)\check{R}_i(1/u) = I$$

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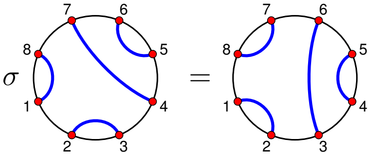
Smirnov's qKZ system

Consider the following system of equations for Ψ, a vector-valued polynomial in $z_1, \ldots, z_L, q, q^{-1}$: $(i = 1, \ldots, L-1)$

$$\check{R}_i(z_{i+1}/z_i)\Psi(z_1, \ldots, z_L) = \Psi(z_1, \ldots, z_{i+1}, z_i, \ldots, z_L) \quad (1)$$

$$\sigma^{-1}\Psi(z_1, \ldots, z_L) = c\,\Psi(z_2, \ldots, z_L, s\, z_1) \quad (2)$$

where σ rotates link patterns:



Level 1 Polynomial solution of qKZ

Fact: in size $L = 2n$, for $s = q^6$ (level 1), there exists a polynomial solution of degree $n(n-1)$, unique up to normalization.

Example ($2n = 4$)

Ψ $= (q\, z_1 - q^{-1} z_2)(q\, z_3 - q^{-1} z_4)$

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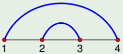
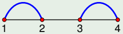
Ψ 

$$= (q\,z_1 - q^{-1} z_2)(q\,z_3 - q^{-1} z_4)$$

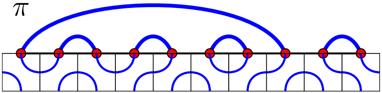
Ψ 

$$= (q\,z_2 - q^{-1} z_3)(q^{-2} z_4 - q^2 z_1)$$

Transfer matrix

Define the transfer matrix $T(t; z_1, \ldots, z_{2n})$ (for generic q)

$$T|\pi\rangle = \sum_{\text{plaquettes}} (\textit{weight})$$



$$= \sum_{\rho} T_{\rho\pi}|\rho\rangle$$

with weights $(q\, z_i - q^{-1} t)/(q\, t - q^{-1} z_i)$ for  and $(z_i - t)/(q\, t - q^{-1} z_i)$ for , and a weight $\tau = -q - 1/q$ for closed loops.

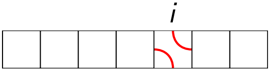
qKZ equation à la Frenkel–Reshetikhin

The actual qKZ equation is a consequence of (1) and (2):

$$\Psi(z_1, \ldots, s\, z_i, \ldots, z_L) = S_i(z_1, \ldots, z_{2n}) \Psi(z_1, \ldots, z_i, \ldots, z_L)$$

$(i = 1, \ldots, L-1)$ where

$$S_i(z_1, \ldots, z_{2n}) = T(t = z_i; z_1, \ldots, z_{2n})$$

$=$ 

Remark: qKZ is a system of compatible difference equations, in the same way that KZ is a system of compatible differential equations.

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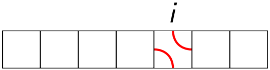
$=$ 

Remark: qKZ is a system of compatible difference equations, in the same way that KZ is a system of compatible differential equations.

Special point $q^3 = 1$

Assume $q = e^{\pm 2i\pi/3}$. Then $s = 1$ and Ψ is eigenvector of the S_i; by a Lagrange interpolation argument, it is eigenvector of all $T(t)$:

$$T(t; z_1, \ldots, z_{2n})\Psi(z_1, \ldots, z_{2n}) = \Psi(z_1, \ldots, z_{2n})$$

Thus, Ψ is the (unnormalized) equilibrium distribution of the inhomogeneous Markov process. The homogeneous case is recovered when $z_i = 1$.

Homogeneous limit for generic q

A question remains: what is the combinatorial meaning of the level 1 polynomial solution of qKZ for generic q? In particular, what can one say about the homogeneous limit $z_i = 1$?

Example ($2n = 4$)

Ψ

$|_{homogeneous} = 1$

Ψ

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where $\tau = -q - q^{-1}$.

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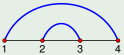
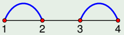
Ψ  $\Big|_{homogeneous} = 1$

Ψ  $\Big|_{homogeneous} = \tau$

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In connection with six-vertex model and XXZ spin chain, known integral formulae: $(1 \le a_1 < \cdots < a_n \le N)$

$$\Psi_{a_1,\ldots,a_n} = \prod_{1\le i<j\le 2n} (q\, z_i - q^{-1} z_j) \oint \cdots \oint \prod_{\ell=1}^{n} \frac{\mathrm{d}w_\ell}{2\pi i} \frac{\prod_{1\le \ell<m\le n}(w_m - w_\ell)(q\, w_\ell - q^{-1} w_m)}{\prod_{\ell=1}^{n} \prod_{1\le i\le a_\ell}(w_\ell - z_i) \prod_{a_\ell<i\le 2n}(q\, w_\ell - q^{-1} z_i)}$$

where the contours catch poles at $w_\ell = z_i$.

These correspond to a different basis:

$$\Psi_{a_1,\ldots,a_n} = \sum_{\pi} c_{\pi;a_1,\ldots,a_n} \Psi_\pi$$

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Integral formula for the sum

However, similar formulae can be written for various partial sums of components in the loop basis. In particular,

$$\sum_{\pi} \Psi_{\pi} = \prod_{1\le i<j\le 2n} (q\, z_i - q^{-1} z_j) \oint \cdots \oint \prod_{\ell=1}^{n} \frac{\mathrm{d}w_\ell (q\, w_\ell - z_{2\ell-1})}{2\pi i} \frac{\prod_{1\le \ell<m\le n} (w_m - w_\ell)(q\, w_\ell - q^{-1} w_m)}{\prod_{\ell=1}^{n} \prod_{1\le i\le 2\ell-1} (w_\ell - z_i) \prod_{2\ell-1\le i\le 2n} (q\, w_\ell - q^{-1} z_i)}$$

Homogeneous limit? Usual trick (cf matrix models): homographic transformation.

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Integral formula for the sum: homogeneous limit

Change variables: $w_\ell = \frac{1-q^{-1}u_\ell}{1-q\,u_\ell}$.

$$\sum_\pi \Psi_\pi\big|_{homogeneous} = \oint\cdots\oint\prod_{\ell=1}^{n}\frac{\mathrm{d}u_\ell(1+u_\ell)}{2\pi i\,u_\ell^{2\ell-1}} \prod_{1\le\ell<m\le n}(u_m-u_\ell)(1+u_\ell u_m+\tau u_m)$$

Antisymmetrisation of the integral formula

Antisymmetrize $\prod_{i=1}^n u_\ell^{-2\ell+2} \prod_{1\le\ell<m\le n}(1+u_\ell u_m+\tau u_m)$:

$$\prod_{1\le\ell\le m\le n}(1-u_\ell u_m)\prod_{i=1}^n u_\ell^{-2\ell+2}\prod_{1\le\ell<m\le n}(1+u_\ell u_m+\tau u_m)|_{anti,-}$$

$$=\prod_{1\le\ell<m\le n}(u_m^{-1}-u_\ell^{-1})(\tau+u_\ell^{-1}+u_m^{-1})$$

$$=\prod_{\ell=1}^n u_\ell^{-2\ell+2}(1+\tau u_\ell)^{\ell-1}|_{anti}$$

[Zeilberger '07]

Therefore,

$$\oint \cdots \oint \prod_{\ell=1}^{n} \frac{\mathrm{d}u_\ell(1+u_\ell)}{2\pi i\, u_\ell^{2\ell-1}} \prod_{1\le \ell<m\le n} (u_m-u_\ell)(1+u_\ell u_m+\tau u_m)$$

$$= \oint \cdots \oint \prod_{\ell=1}^{n} \frac{\mathrm{d}u_\ell(1+u_\ell)}{2\pi i\, u_\ell^{2\ell-1}} \frac{\prod_{1\le \ell<m\le n}(u_m-u_\ell)(1+\tau u_m)}{\prod_{1\le \ell\le m\le n}(1-u_\ell u_m)}$$

Using the classical identity
$\prod_{\ell=1}^{n} \frac{1}{1-u_\ell} \prod_{1\le \ell<m\le n} \frac{1}{1-u_\ell u_m} = \sum_{0\le r_0<r_2<\cdots<r_{n-1}} \frac{\det\left(u_i^{r_j}\right)}{\prod_{1\le \ell<m\le n}(u_\ell-u_m)}$,
we find:

$$\sum_\pi \Psi_\pi = \sum_{0\le r_1<r_2<\cdots<r_n} \tau^{n(n-1)-\sum_j r_j} \det\left(\binom{2i-r_j}{i}\right)_{1\le i\le n, 0\le j\le n-1}$$

Therefore,

$$\oint \cdots \oint \prod_{\ell=1}^{n} \frac{\mathrm{d}u_\ell (1+u_\ell)}{2\pi i\, u_\ell^{2\ell-1}} \prod_{1\le \ell<m\le n} (u_m - u_\ell)(1+u_\ell u_m + \tau u_m)$$
$$= \oint \cdots \oint \prod_{\ell=1}^{n} \frac{\mathrm{d}u_\ell (1+u_\ell)}{2\pi i\, u_\ell^{2\ell-1}} \frac{\prod_{1\le \ell<m\le n}(u_m-u_\ell)(1+\tau u_m)}{\prod_{1\le \ell\le m\le n}(1-u_\ell u_m)}$$

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$$\prod_{\ell=1}^{n} \frac{1}{1-u_\ell} \prod_{1\le \ell<m\le n} \frac{1}{1-u_\ell u_m} = \sum_{0\le r_0<r_2<\cdots<r_{n-1}} \frac{\det\left(u_i^{r_j}\right)}{\prod_{1\le\ell<m\le n}(u_\ell - u_m)},$$

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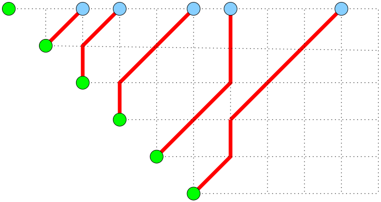
$$\boxed{\sum_\pi \Psi_\pi = \sum_{0\le r_1<r_2<\cdots<r_n} \tau^{n(n-1)-\sum_j r_j} \det\left(\binom{2i-r_j}{i}\right)_{1\le i\le n, 0\le j\le n-1}}$$

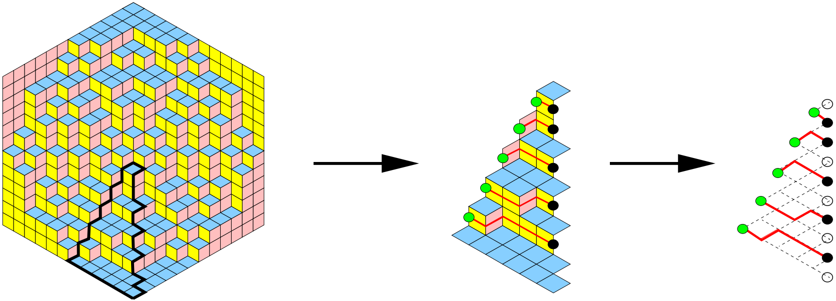
LGV formula for non-intersecting paths

As a special case of the Lindström–Gessel–Viennot formula,
$\det\left(\binom{2i-r_j}{i}\right)_{1\le i\le n, 0\le j\le n-1}$ = # non-intersecting lattice paths from $(i,-i)$ to $(r_j,0)$ with moves $(1,0)$ and $(1,1)$, so that

$$\sum_\pi \Psi_\pi = \sum_{\mathrm{NILPs}} \tau^{\#\ up\ steps}$$



Totally Symmetric Self-Complementary Plane Partitions



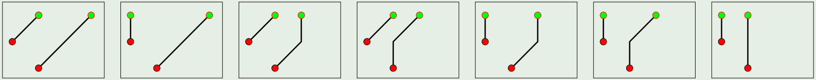
TSSCPP **NILP**

$$\sum_{\pi} \Psi_\pi \big|_{homogeneous} = \sum_{\text{TSSCPPs}} \tau^{\#\ pink\ \ lozenges}$$

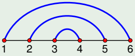
Remark: #TSSCPPs = #ASMs = A_n. (but no known bijection!)

Example ($2n = 6$)

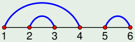
There are $A_3 = 7$ TSSCPPs:



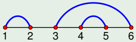
$\mathbf{1}$ τ τ τ $\tau^{\mathbf{2}}$ $\tau^{\mathbf{2}}$ $\tau^{\mathbf{3}}$

Ψ 

$\Big|_{homogeneous} = 1$

Ψ 

$\Big|_{homogeneous} = \tau^2$

Ψ 

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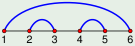
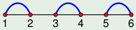
Ψ 

$\Big|_{homogeneous} = 2\tau$

Ψ 

$\Big|_{homogeneous} = \tau^3 + \tau$

so that $\sum_\pi \Psi_\pi \big|_{homogeneous} = 1 + 3\tau + 2\tau^2 + \tau^3$.

Another combinatorial point: $q = -1$

Consider $q = -1$, i.e. a weight of $\tau = 2$ per loop.

Example ($2n = 6$)

$$\Psi|_{\tau=2} = (1, 4, 4, 4, 10) \qquad \sum_{\pi} \Psi_\pi|_{\tau=2} = 23$$

$$\sum_{\pi} \Psi_\pi|_{\tau=2} = \int_{T^2=0} \mathrm{d}\mu(T) e^{-\pi \operatorname{tr} TT^\dagger}$$

where T is complex upper triangular.

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An algebro-geometric interpretation

The integral $\int_{T^2=0} \mathrm{d}\mu(T) e^{-\pi \mathrm{tr}\, TT^\dagger}$ computes the degree of $V = \{T^2 = 0,\ \ T \textit{ upper triangular}\}$, i.e. the generic number of intersections with a linear subspace of complementary dimension.

V is reducible: its components (orbital varieties) V_π are naturally indexed by link patterns π, in such a way that

$$\deg V_\pi = \Psi_\pi|_{\tau=2}$$

Justification: equivariant cohomology interpretation of the rational qKZ equation. . .

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