Quantum integrability and Combinatorics: Razumov–Stroganov type conjectures

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Outline of the lectures

1. The Temperley–Lieb model of loops
   - Definition of the model
   - Relation to Markov process on link patterns
   - Perron–Frobenius eigenvector
   - Some observations
2. Fully Packed Loops and Razumov–Stroganov conjecture
   - Definition of FPLs
   - Classes of FPLs
   - Razumov–Stroganov conjecture
3. Introduction of inhomogeneities into the loop model
   - Defintion of the inhomogeneous model
   - Properties of the inhomogeneous eigenvector

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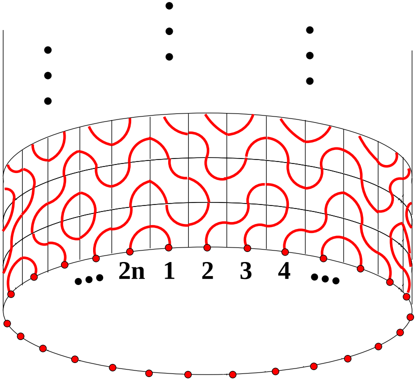
Consider the following probabilistic model. Fill some two-dimensional surface with boundary with plaquettes:

 with probability p,  with probability $1-p$. $(0 < p < 1)$



Case of the half-infinite cylinder geometry ("periodic boundary conditions")

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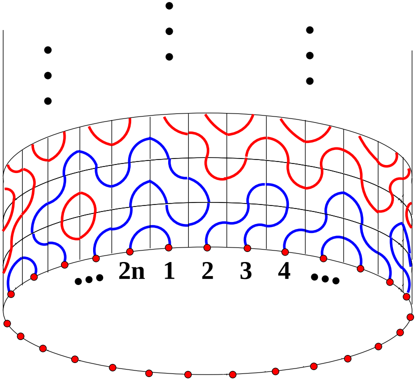


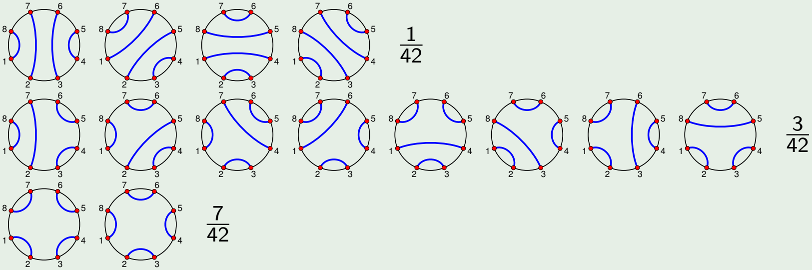
Case of the half-infinite cylinder geometry ("periodic boundary conditions")

Probability law of the connectivity of the external vertices?

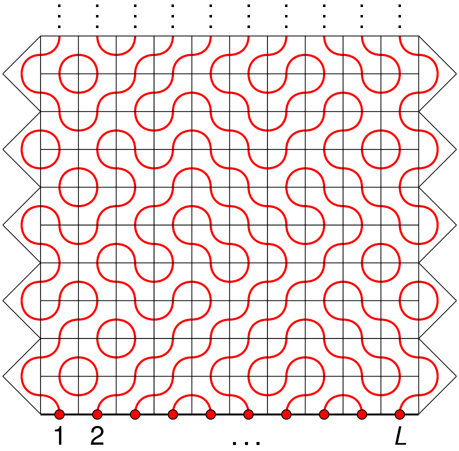
The connectivity of the external vertices can be encoded into a link pattern = a planar pairing of $2n$ points on a circle.

Example

In size $L = 2n = 8$,



Another geometry: the strip ("closed boundary conditions")



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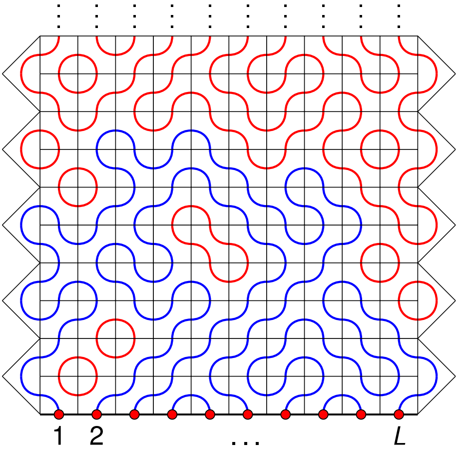

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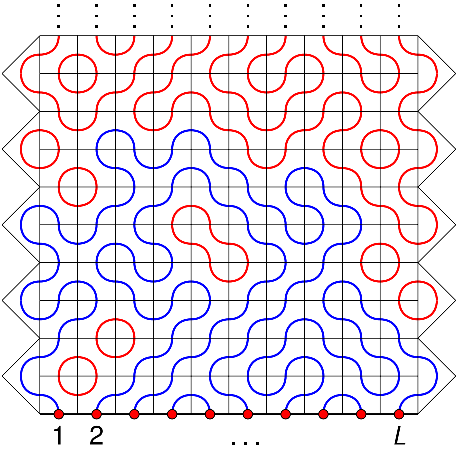
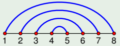
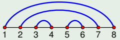
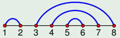
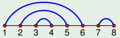
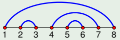
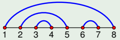
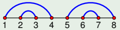
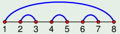
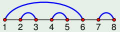
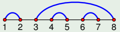
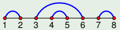
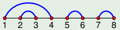
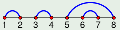
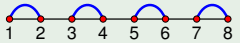


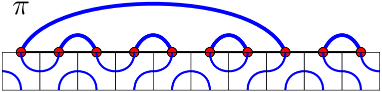
Example ($L = 8$)

	$\frac{1}{646}$		$\frac{6}{646}$
	$\frac{14}{646}$		$\frac{14}{646}$
	$\frac{14}{646}$		$\frac{14}{646}$
	$\frac{30}{646}$		$\frac{50}{646}$
	$\frac{56}{646}$		$\frac{56}{646}$
	$\frac{71}{646}$		$\frac{75}{646}$
	$\frac{75}{646}$		$\frac{170}{646}$

Define the matrix T corresponding to the effect of the insertion of one row (or two rows) of plaquettes on the link patterns:

$$T|\pi\rangle = \sum_{\text{plaquettes}} p^{\#}(1-p)^{\#}$$

π



$$= \sum_{\rho} T_{\rho\pi}|\rho\rangle$$

Problem reformulated as a stochastic process on link patterns.

In particular, the probabilities P_π form a vector $|P\rangle = \sum_\pi P_\pi|\pi\rangle$ which is the equilibrium distribution of the process:

$$|P\rangle = \lim_{k\to\infty} T^k|\alpha\rangle$$

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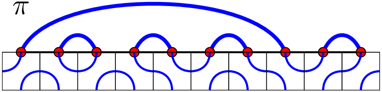
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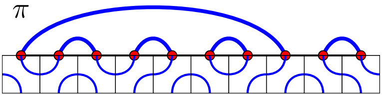

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Theorem (Perron 1907, Frobenius 1912)

Let A be a matrix with non-negative entries. Then

- *There is a real eigenvalue ρ of A such that any other eigenvalue λ satisfies $|\lambda| \leq \rho$.*
- *There is an eigenvector associated to ρ which has non-negative entries.*

Assume furthermore that A is primitive, i.e. the entries of A^k are positive for some k. Then

- *There is a real eigenvalue ρ of A such that any other eigenvalue λ satisfies $|\lambda| < \rho$.*
- *The eigenspace associated to ρ is one-dimensional.*
- *The eigenvector associated to ρ can be chosen to have positive entries, and it is the only eigenvector with non-negative entries.*

Application to T

As any Markov matrix, T possesses the following properties:

- Its entries are non-negative.
- It has the left eigenvector $\langle 1| := (1,\ldots,1)$ with eigenvalue 1, expressing the conservation of probability: $\langle 1|T = \langle 1|$.

Furthermore it also satisfies:

- T is primitive since T^n has positive entries:

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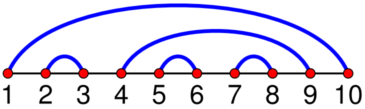

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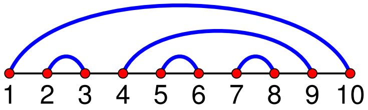

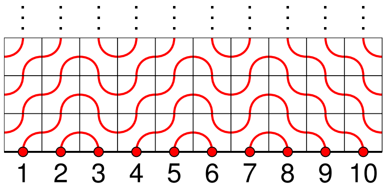

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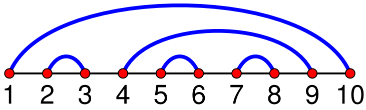

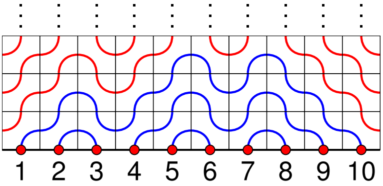

Application to T cont'd

We conclude that T possesses a (right) eigenvector Ψ which is given up to normalization by the equation $T|\Psi\rangle = |\Psi\rangle$.

Remark: one could set $\langle 1|\Psi\rangle = 1$; however it is convenient to choose a different normalization for Ψ.

In particular as $k \to \infty$, contributions of other eigenvalues decay exponentially and

$$T^\infty := \lim_{k\to\infty} T^k = \frac{|\Psi\rangle\langle 1|}{\langle 1|\Psi\rangle}$$

Returning to the vector of probabilities in the original problem, we find:

$$|P\rangle = T^\infty|\alpha\rangle = \frac{|\Psi\rangle}{\langle 1|\Psi\rangle}$$

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Conjectures [de Gier, Nienhuis '01]

Define

$$A_n = \frac{1!4!7!\cdots(3n-2)!}{n!(n+1)!(n+2)!\cdots(2n-1)!} = 1, 2, 7, 42, 429\ldots$$

Normalize Ψ so that the smallest components, with patterns of the type

, are set to 1. Then:

1. All components are Ψ are (positive) integers.
2. The largest components of Ψ correspond to patterns of the type

and are equal to A_{n-1}.
[PDF, PZJ + Zeilberger '07 *or* Razumov, Stroganov, PZJ '07]
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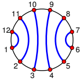
Normalize Ψ so that the smallest components, with patterns of the type 

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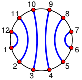
Normalize Ψ so that the smallest components, with patterns of the type 

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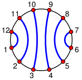
Normalize Ψ so that the smallest components, with patterns of the type 

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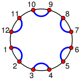
2. The largest components of Ψ correspond to patterns of the type 

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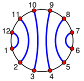
Normalize Ψ so that the smallest components, with patterns of the type 

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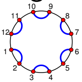
2. The largest components of Ψ correspond to patterns of the type 

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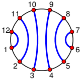
Normalize Ψ so that the smallest components, with patterns of the type 

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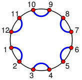
2. The largest components of Ψ correspond to patterns of the type 

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What does this loop model have to do with ASMs? combinatorial intepretation of each component?

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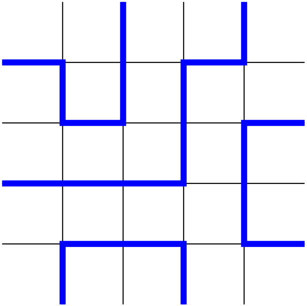

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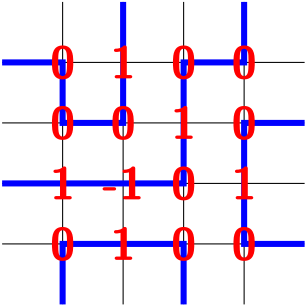

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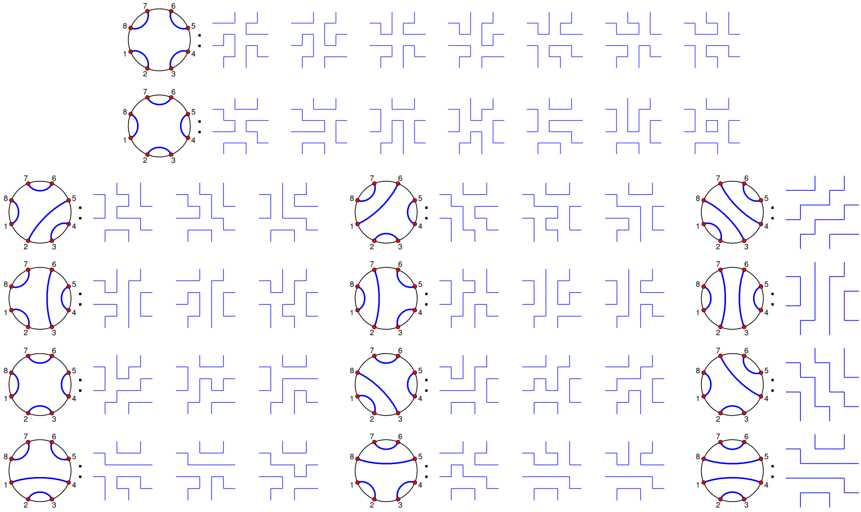
A Fully Packed Loop configuration (FPL) on a $n \times n$ square grid:

$$\begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{matrix}$$

Fact: FPLs are in bijection with ASMs.

It is natural to group FPLs by connectivity of their endpoints: cf



Conjecture [Razumov, Stroganov '01]

Denote by $A(\pi)$ the number of FPLs with connectivity described the link pattern π. This is exactly the (unnormalized) probability of pattern π in the model of loops with the geometry of the cylinder.

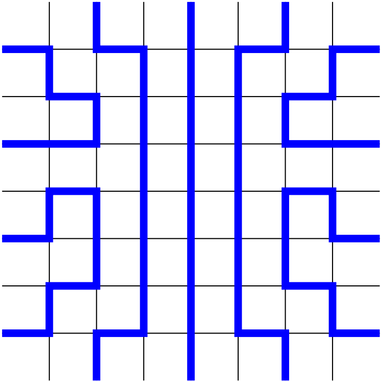
In other words $|\Psi\rangle = \sum_\pi A(\pi)|\pi\rangle$ is the (unnormalized) equilibrium distribution of the Markov process of loops:

$$T|\Psi\rangle = |\Psi\rangle$$

Remark: The RS conjecture implies observations 1 and 3 of de Gier, Nienhuis.

A variant of RS

Vertically Symmetric Fully Packed Loop configurations: (VSFPLs)



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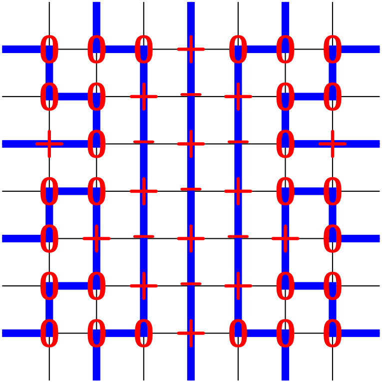

A variant of RS

Vertically Symmetric Fully Packed Loop configurations: (VSFPLs)

0	0	0	+	0	0	0
0	0	+	−	+	0	0
+	0	−	+	−	0	+
0	0	+	−	+	0	0
0	+	−	+	−	+	0
0	0	+	−	+	0	0
0	0	0	+	0	0	0

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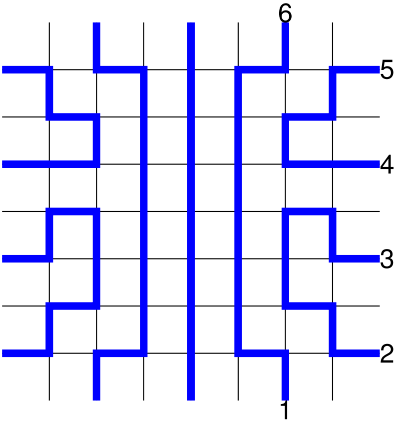

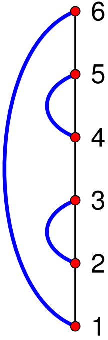

A variant of RS cont'd

Conjecture [Pierce, Rittenberg, de Gier, Nienhuis '02]: the number of VSFPLs of size $2n+1$ with connectivity π is the (unnormalized) probability of pattern π in the model of loops with the strip geometry of size $2n$.

In particular,

$$A^V_{2n+1} = \prod_{j=0}^{n-1} (3j+2) \frac{(2j+1)!(6j+3)!}{(4j+2)!(4j+3)!} = 1, 1, 3, 26, 646 \ldots$$

is the normalization of probabilities.

Example

A variant of RS cont'd

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Example

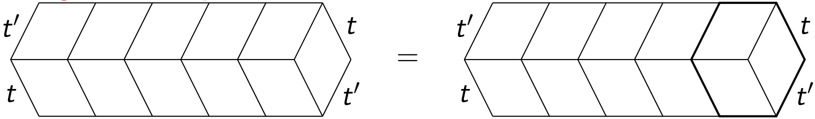
Consider the probabilistic model (on the cylinder) with probabilities p_i depending on the column $i = 1, \ldots, 2n$, and the corresponding transfer matrix:

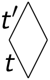
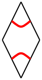
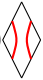
$T = \prod_{i=1}^{2n} \Big(p_i$  $+ (1 - p_i)$  $\Big)$

Parametrize the probabilities as $p_i = \frac{z_i - q\,t}{t - q\,z_i}$, $q = e^{2i\pi/3}$.
z_i are the spectral parameters.

Integrability

Yang–Baxter Equation:



where  $= (t - q\,t')$  $+ q^2(t - t')$  . YBE implies that

$$[T(t; z_1, \ldots, z_{2n}), T(t'; z_1, \ldots, z_{2n})] = 0$$

Thus, the equilibrium distribution eigenvector given by

$$T(t; z_1, \ldots, z_{2n})|\Psi(z_1, \ldots, z_{2n})\rangle = |\Psi(z_1, \ldots, z_{2n})\rangle$$

only depends on the z_i. (in particular, independent of p in homog.)

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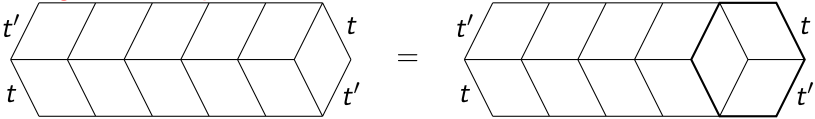
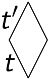
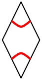
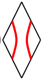


where  $= (t - q\, t')$  $+ q^2(t - t')$  . YBE implies that

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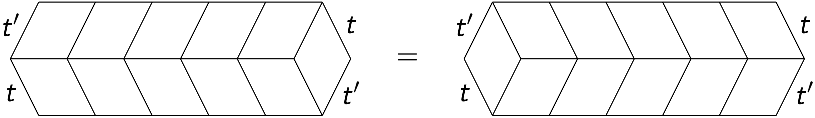
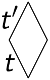
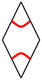
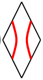


where 

$= (t - q\,t')$  $+ q^2(t - t')$  . YBE implies that

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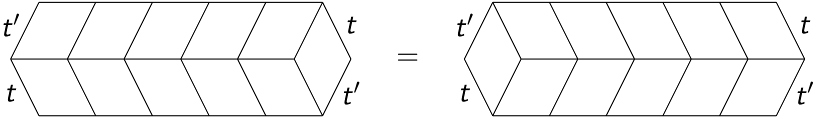
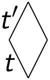
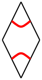
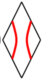


where 

$= (t - q\,t')$  $+ q^2(t - t')$  . YBE implies that

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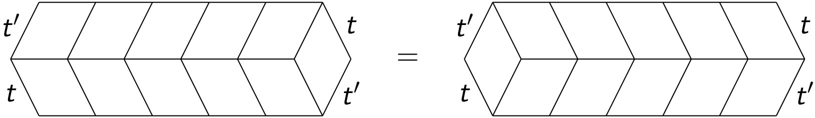
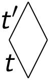
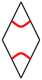
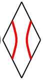


where 

$= (t - q\,t')$  $+ q^2(t - t')$  . YBE implies that

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  $|\Psi(z_1,\ldots,z_{2n})\rangle$ can be normalized in such a way that its components are homogenous polynomials of total degree $n(n-1)$ and of partial degree at most $n-1$ in each z_i.
- Factorization and symmetry.
  The components possess various linear factors and properties of symmetry by exchange of variables.
  In particular, their sum is a symmetric polynomial of all z_i.
- Recursion relations.
  Components of $|\Psi(z_1,\ldots,z_{2n})\rangle$ satisfy linear recursion relations; their sum is entirely determined by these:

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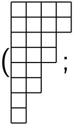

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