02/2005

Combinatorics of the Brauer Loop scheme

P. Zinn-Justin

- The Brauer Loop scheme

◇ The deformed matrix algebra; definition of the scheme

◇ Torus action and Equivariant Cohomology

- The $O(1)$ Brauer Loop model

◇ Definition

◇ Transfer Matrix and Perron–Frobenius eigenvector

◇ Multi-parameter generalization

- Proof of equivalence: Schubert calculus vs Yang–Baxter equation

- Two applications

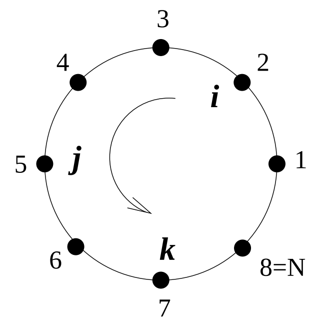
P. Di Francesco, P. Zinn-Justin, *Inhomogeneous model of crossing loops...*, `math-ph/0412031`.
A. Knutson, P. Zinn-Justin, *A scheme related to the Brauer loop model*, `math.AG/0503224`.

Deformed matrix product

For P, Q two $N \times N$ matrices define the product $P \bullet Q$:

$$(P \bullet Q)_{ik} = \sum_{j:\,(i \le j \le k)\ cyc} P_{ij} Q_{jk} \qquad i, k = 1, \ldots, N$$

where $(i \le j \le k)\ cyc$ means that i, j, k are in cyclic order: (and $i = k \Rightarrow i = j = k$)



$(M_N(\mathbb{C}), \bullet, +)$ associative algebra. A matrix is invertible iff its diagonal elements are non-zero.

Alternate definition: ("interpolation" between usual and deformed product)

if $R_N(\mathbb{C})$ is the subspace of upper triangular matrices and

$$R_N(\mathbb{C}[t]) = R_N(\mathbb{C}) \oplus t M_N(\mathbb{C}) \oplus t^2 M_N(\mathbb{C}) \oplus \cdots$$

then our algebra is isomorphic to $R_N(\mathbb{C}[t])/t R_N(\mathbb{C}[t])$: $M \mapsto M_{\le} + t M_{>}$.

The affine scheme E

Define in the space $M_N^0(\mathbb{C})$ of matrices with zero diagonals:

$$E := \{\, M \in M_N^0(\mathbb{C}) : M \bullet M = 0 \,\}$$

Explicitly, the equations defining the scheme E read:

$$\sum_{j:(i\le j\le k)\ cyc} M_{ij} M_{jk} = 0 \qquad \forall i, k$$

Q1: what are the components of E? what is their dimension?

Experimental answer: to simplify, in what follows we assume N even ($N = 2n$). Then

1) E is equidimensional:

$$E = \bigcup_{\pi} E_\pi$$

with $\dim E_\pi = N^2/2$.

2) E, and each of its components, are generically reduced.

(examples in two slides...)

Torus action and equivariant cohomology

Action of $T = (\mathbb{C}^{N+1}, +)$ on $M_N(\mathbb{C})$:

$$(a, z_1, \ldots, z_N) : M_{ij} \mapsto \mathrm{e}^{a + z_i - z_j} M_{ij}$$

Equivariant cohomology $H_T^*(M_N(\mathbb{C})) \subset \mathbb{C}[a, z_1, \ldots, z_N]$.

Algebraic substitute: **multidegree** $\operatorname{mdeg}_W X$ of a T-invariant scheme $X \subset W$ defined by

(1) If $X = W = \{0\}$ then $\operatorname{mdeg}_W X = 1$.

(2) If X has top-dimensional components X_i with multiplicity m_i, $\operatorname{mdeg}_W X = \sum_i m_i \operatorname{mdeg}_W X_i$.

(3) If X is a variety and H is a T-invariant hyperplane in W,

- (a) If $X \not\subset H$, then $\operatorname{mdeg}_W X = \operatorname{mdeg}_H (X \cap H)$.
- (b) If $X \subset H$, then $\operatorname{mdeg}_W X = (\operatorname{mdeg}_H X) \cdot [W/H]_T$.

Here $W = M_N^0(\mathbb{C})$, $[M_{ij}]_T = a + z_i - z_j$.

Remark 1: $\operatorname{mdeg}_W X$ is a homogeneous polynomial, of degree the codimension of X in W.

Remark 2: $\operatorname{mdeg} X|_{a=1, z_i=0} = \deg X$.

Multidegree of E_π

The action of T preserves E and its components E_π.

(note that the action is $M \mapsto \mathrm{e}^a\, \mathrm{e}^Z \bullet M \bullet \mathrm{e}^{-Z}$ where $\mathrm{e}^Z = \mathrm{diag}(\mathrm{e}^{z_1}, \ldots, \mathrm{e}^{z_N})$)

Q2: what is $\mathrm{mdeg}\, E_\pi$? ($\deg E_\pi$?)

Example 1: $N = 4$. Three components:

⋆ One component of degree 1:

$$E_1 = \left\{ M = \begin{pmatrix} 0 & 0 & m_{13} & m_{14} \\ m_{21} & 0 & 0 & m_{24} \\ m_{31} & m_{32} & 0 & 0 \\ 0 & m_{42} & m_{43} & 0 \end{pmatrix} \right\}$$

⋆ Two components of degree 3:

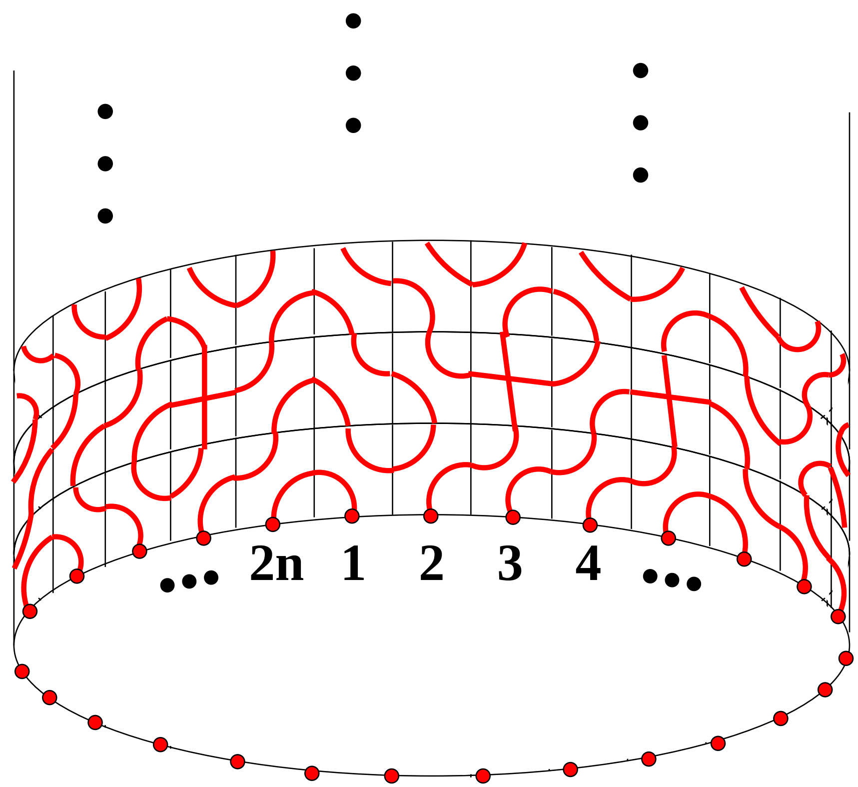
$$E_2 = \left\{ M = \begin{pmatrix} 0 & m_{12} & m_{13} & m_{14} \\ m_{21} & 0 & 0 & m_{24} \\ m_{31} & m_{32} & 0 & m_{34} \\ 0 & m_{42} & m_{43} & 0 \end{pmatrix} \quad \begin{array}{r} m_{12}m_{24} + m_{13}m_{34} = 0 \\ m_{31}m_{12} + m_{34}m_{42} = 0 \\ m_{13}m_{31} - m_{24}m_{42} = 0 \end{array} \right\}$$

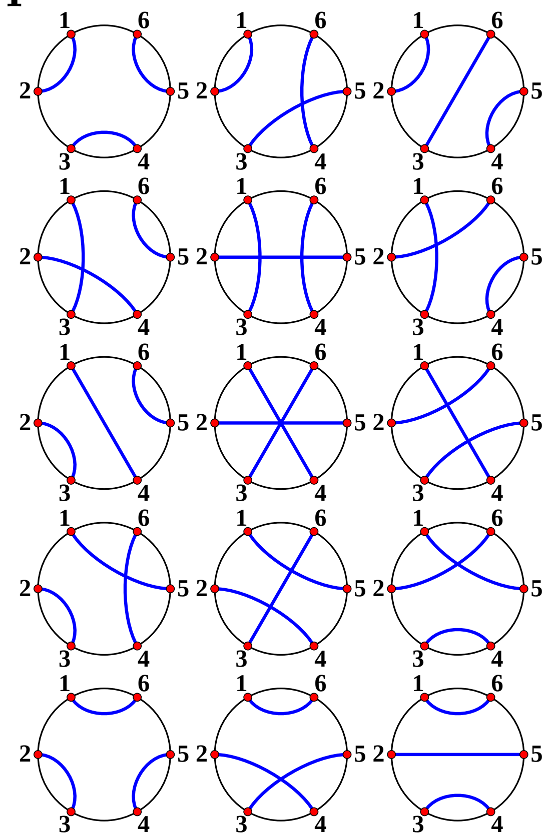
$$E_3 = S(E_2)$$

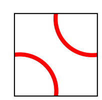
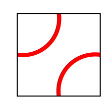
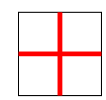
where S is the cycling automorphism $M_{ij} \mapsto M_{i+1\, j+1}$. $\Rightarrow \deg E = 7$.

Example 2: $N = 6$: $(\deg E_\pi) = (1, 3, 3, 3, 13, 13, 13, 13, 13, 13, 31, 31, 31, 63, 63)$. $\deg E = 307$.

Brauer model of loops





Probability that external vertex i is connected to vertex j? (proba:  $=$  $= 4/9$,  $= 1/9$)

$\rightarrow$ vector $\left|\Psi_n\right\rangle$, whose components are indexed by **crossing link patterns**, satisfying

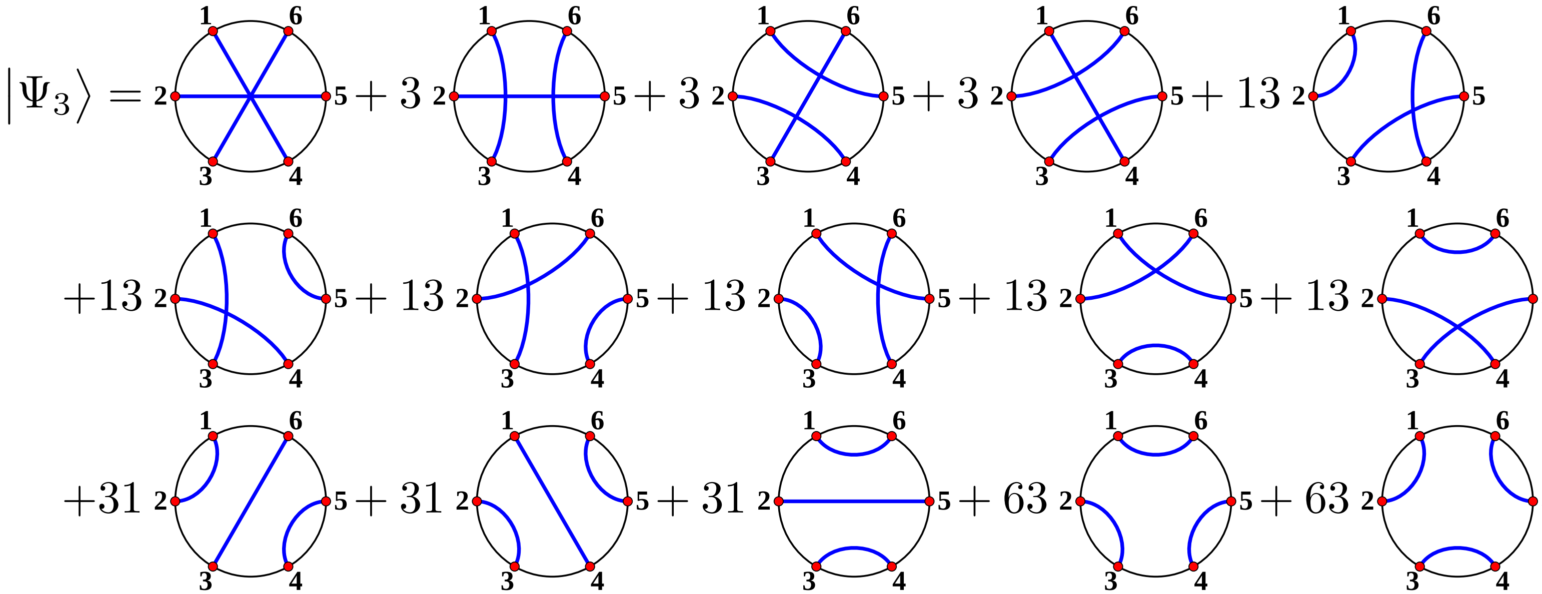
$$T_n \left|\Psi_n\right\rangle = \left|\Psi_n\right\rangle$$

where T_n is the **transfer matrix** that adds a row to the semi-infinite cylinder.

Brauer model of loops cont'd

NB: π = crossing link pattern, or chord diagram, or Brauer diagram, or fixed-point free involution.

Example: for $n = 3$ ($N = 2n = 6$), up to normalization, $\left|\Psi_3\right\rangle$ reads



Conjectures [de Gier, Nienhuis oct '04] (now theorem [PDF, PZJ dec '04])

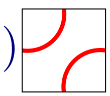
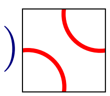
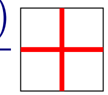
(1) The components can be chosen to be integers, the smallest being 1.

(2) *Some* of the components are degrees of algebraic varieties.

→ What about multidegrees?

Inhomogeneous Brauer model of loops [PDF, PZJ '04]

Introduce local probabilities dependent on the column i via a parameter z_i respecting **integrability** of the model (i.e. satisfying Yang–Baxter equation).

$$T_n(t|z_1,\dots,z_{2n}) = \prod_{i=1}^{2n} \Big((t-z_i)$$

$$+ (1-t+z_i)$$

$$+ \frac{(t-z_i)(1-t+z_i)}{2}$$

$$\Big)$$

$$T_n(t;z_1\dots,z_{2n})\,\big|\Psi_n(z_1,\dots,z_{2n})\big\rangle = \big|\Psi_n(z_1,\dots,z_{2n})\big\rangle$$

⋆ *Polynomiality.*

The $\Psi_\pi(z_1,\dots,z_{2n})$ can be chosen to be coprime polynomials; they are then of total degree $2n(n-1)$ and of partial degree at most $2(n-1)$ in each z_i, with integer coefficients.

⋆ *Factorization, Recursion relations*... $\longrightarrow$ entirely fixed (see next slides)

⋆ *Sum rule.*

$$\sum_\pi \Psi_\pi(z_1,\dots,z_{2n}) = \mathrm{Pf}\left(\frac{z_i-z_j}{1-(z_i-z_j)^2}\right)_{1\le i,j\le 2n} \times \prod_{1\le i<j\le 2n} \frac{1-(z_i-z_j)^2}{z_i-z_j}$$

General relation scheme ↔ statistical model

Conjecture [PZJ]: There is a natural way to index irreducible components E_π of E with crossing link patterns π of size $N = 2n$, in such a way that their multidegrees are the homogeneized components of the eigenvector of the inhomogeneous Brauer loop model:

$$\mathrm{mdeg}\, E_\pi|_{a=1} = \Psi_\pi(z_1, \ldots, z_{2n})$$

In particular, the sum $\sum_\pi \Psi_\pi(z_1, \ldots, z_{2n})$ is the multidegree of E itself.

($\mathrm{mdeg}\, E_\pi = \Psi_\pi$ proved in the de Gier–Nienhuis case in [PDF, PZJ '04]; full proof in [AK, PZJ])

Remark: when all $z_i = 0$, the lhs is the $\deg E_\pi$ and the rhs the entry of the homogeneous model.

Idea of proof:

⋆ Apply **Schubert calculus** type arguments to the E_π in order to show that their multidegree satisfies recursion relations (wrt the number of crossings of π).

⋆ Use **Yang–Baxter** and related equations to show that the Ψ_π satisfy the very same relations.

Definition of the E_π

Define $s_i(M) := \sum_{j=1}^N M_{ij}M_{ji}$ for $M \in E = \{M \bullet M = 0\}$.

Two simple lemmas:

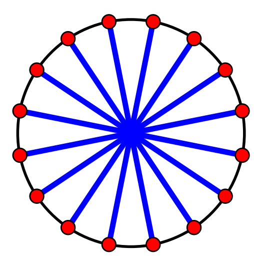
(1) E (and therefore each E_π) is stable by $\bullet$-conjugation by any invertible matrix.

(2) $s_i(M) = s_i(P \bullet M \bullet P^{-1})$ for all i, $M \in E$, P invertible.

Motivates the following two equivalent definitions:

$$E_\pi = \overline{\bigcup_{t \text{ diag}} Orb(\pi t)} = \overline{\{\, P \bullet \pi t \bullet P^{-1}, t \text{ diag}, P \text{ inv} \,\}} \qquad (\pi \equiv \text{the matrix of involution } \pi)$$

$$= \overline{\{\, M \in E : s_i(M) = s_j(M) \text{ if and only if } j \in \{i, \pi(i)\} \,\}}$$

Special case: "trivial" component. $\pi_0 =$  , $E_{\pi_0} = \begin{pmatrix} 0 & \cdots & 0 & \star & \cdots & \star \\ \star & 0 & \cdots & 0 & \star & \cdots \\ & \ddots & \ddots & & \ddots & \ddots \\ \star & \cdots & \star & 0 & \cdots & 0 \\ \ddots & \ddots & & \ddots & \ddots & \\ \cdots & 0 & \star & \cdots & \star & 0 \end{pmatrix}$

$$\text{mdeg}\, E_{\pi_0} = \prod_{\substack{1 \le i < j \le 2n \\ j-i<n}} (a + z_i - z_j) \prod_{\substack{1 \le i < j \le 2n \\ j-i>n}} (a + z_j - z_i)$$

Schubert calculus

⋆ Define $L_i = \{$invertible matrices with off-diagonal elements at $(i, i+1)$, $(i+1, i)\}$,

$B_i = \{$invertible matrices with off-diagonal elements at $(i+1, i)\}$ and

$$-\partial_i : L_i \times_{B_i} M_N(\mathbb{C}) \to M_N(\mathbb{C})$$

$$(P, M) \to PMP^{-1} \qquad \text{(ordinary product)}$$

If $-\partial_{i|L_i \times_{B_i} X}$ generically one-to-one, then

$$\text{mdeg}(-\partial_i)X = -\partial_i \text{mdeg}\, X$$

where $\partial_i = \frac{1}{z_{i+1} - z_i}(\tau_i - 1)$ and $\tau_i F(z_i, z_{i+1}) = F(z_{i+1}, z_i)$.

⋆ Assume π has no arch between i and $i+1$. Define $E_\pi^{\{i,i+1\}} = \{M \in E_\pi : M_{i+1,i} = 0\}$. Note

$$\text{mdeg}\, E_\pi^{\{i,i+1\}} = (a + z_{i+1} - z_i)\text{mdeg}\, E_\pi$$

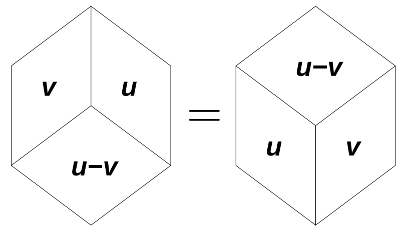
Apply $-\partial_i$ to $E_\pi^{\{i,i+1\}}$, then impose $(M \bullet M)_{i+1\, i} = 0$:

$$-(2a + z_{i+1} - z_i)\partial_i \text{mdeg}\, E_\pi^{\{i,i+1\}} = \text{mdeg}\, E_\pi^{\{i,i+1\}} + \text{mdeg}\, E_{f_i \cdot \pi}^{\{i,i+1\}}$$

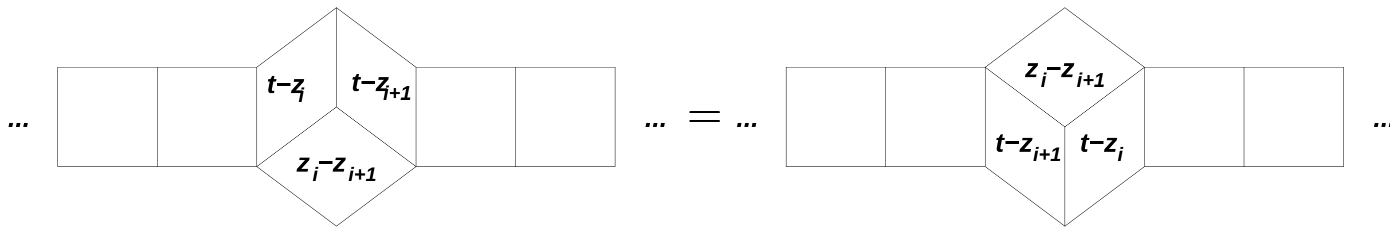
Final formula:

$$\text{mdeg}\, E_{f_i \cdot \pi} = -(a + z_i - z_{i+1})(2a\partial_i + \tau_i)\left(\frac{\text{mdeg}\, E_\pi}{a + z_i - z_{i+1}}\right)$$

Yang–Baxter equation and intertwining relations



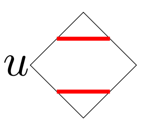
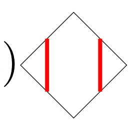
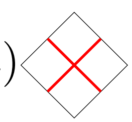
Applied to the transfer matrix:



or more explicitly

$$\check{R}_i(z_i - z_{i+1}) T_n(t; z_1, \ldots, z_i, z_{i+1}, \ldots, z_{2n}) = T_n(t; z_1, \ldots, z_{i+1}, z_i, \ldots, z_{2n}) \check{R}_i(z_i - z_{i+1})$$

where

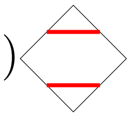
$\check{R}(u) =$ u  $+ (1-u)$  $+ \frac{1}{2}u(1-u)$ 

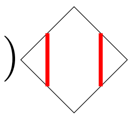
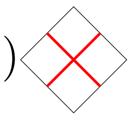
over $(1-u/2)(1+u)$

The intertwining relation implies (NB: fixing the normalization is non-trivial!)

$$\left|\Psi_n(\ldots, z_{i+1}, z_i, \ldots)\right\rangle = \check{R}(z_i - z_{i+1}) \left|\Psi_n(\ldots, z_i, z_{i+1}, \ldots)\right\rangle$$

Recursion relations for the components

$$\left(1+\frac{1}{2}(z_{i+1}-z_i)\right)(1+z_i-z_{i+1})\ \tau_i \left|\Psi_n\right\rangle = \Big((z_i-z_{i+1})$$ 

$$+\,(1+z_{i+1}-z_i)$$  $$+\,\frac{1}{2}(z_i-z_{i+1})(1+z_{i+1}-z_i)$$  $$\Big)\left|\Psi_n\right\rangle$$

(1) One can show that the only value of Ψ_{π_0} compatible with these equations is

$$\Psi_{\pi_0} = \prod_{\substack{1\le i<j\le 2n \\ j-i<n}} (1+z_i-z_j) \prod_{\substack{1\le i<j\le 2n \\ j-i>n}} (1+z_j-z_i)$$

(2) If π is such that there is no arch between i and $i+1$, the previous formula simplifies to

$$\left(1+\frac{1}{2}(z_{i+1}-z_i)\right)(1+z_i-z_{i+1})\Psi_\pi(z_1,\ldots,z_{i+1},z_i,\ldots,z_{2n})$$
$$= (1+z_{i+1}-z_i)\Psi_\pi(z_1,\ldots,z_{2n}) + \frac{1}{2}(z_i-z_{i+1})(1+z_{i+1}-z_i)\Psi_{f_i\cdot\pi}(z_1,\ldots,z_{2n})$$

which is equivalent to the recursion relation for the multidegrees:

$$\Psi_{f_i\cdot\pi} = (1+z_i-z_{i+1})(2\,\partial_i-\tau_i)\left(\frac{\Psi_\pi}{1+z_i-z_{i+1}}\right)$$

Application 1: (multi)degree of E and of $\{M^2=0\}$

Theorem [AK, PZJ using PDF, PZJ]: the multidegree of E is

$$\operatorname{mdeg} E = \operatorname{Pf}\left(\frac{z_i - z_j}{a^2-(z_i-z_j)^2}\right)_{1\le i,j\le 2n} \times \prod_{1\le i<j\le 2n} \frac{a^2-(z_i-z_j)^2}{z_i-z_j}$$

Its degree is

$$\deg E = \det\left[\binom{2i+2j+1}{2i}\right]_{0\le i,j\le n-1} = 1, 7, 307, \ldots$$

When $n\to\infty$, $\log\deg E \sim n^2 \times 2\log(\pi/2)$.

Remark:

$$\operatorname{mdeg}\{\, M\in M(N,\mathbb{C}) : M^2 = 0\,\} = \operatorname{mdeg}\{\, M\in M(N,\mathbb{C}) : M\bullet M = 0\,\} = 2^n \operatorname{mdeg} E$$

Proof [AK, PZJ?]: use localization; or equivalently, the integral formula

$$\operatorname{mdeg}\{\, M\in M(N,\mathbb{C}) : M^2 = 0\,\} = \frac{\int_{M^2=0} \mathrm{d}\mu(M)\, \mathrm{e}^{-\pi\sum_{i,j}|M_{ij}|^2(a+z_i-z_j)}}{\int_{M\in M_n(\mathbb{C})} \mathrm{d}\mu_0(M)\, \mathrm{e}^{-\pi\sum_{i,j}|M_{ij}|^2(a+z_i-z_j)}}$$

and apply Harish Chandra–Itzkyson–Zuber integral / Duistermaat–Heckman theorem.

(The multidegree is preserved by deformation of the product)

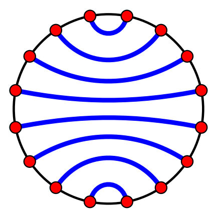
Application 2: (multi)degree of the commuting variety

Define the **commuting variety** to be the scheme

$$C = \{ (X,Y) \in M_n(\mathbb{C})^2 : XY = YX \}$$

It is a classical difficult problem to compute the degree of C. (previously known up to $n = 4$ only)

Observation [A. Knutson '03]: there is a Gröbner degeneration from $C \times V$ to E_π where $N = 2n$

and $\pi =$  .

In particular, $\deg C = \deg E_\pi = 1,\ 3,\ 31,\ 1145,$

[dG, N] $154881,\ 77899563,\ 147226330175,\ 1053765855157617,$

[PZJ] $28736455088578690945,\ 3000127124463666294963283,\ 1203831304687539089648950490463,$

$\ldots$

$$\log\deg C \sim n^2 \times \log 2 \qquad n \to \infty$$