Schubert puzzles as exactly solvable models

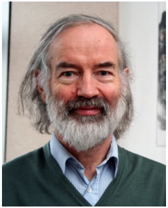
P. Zinn-Justin

The University of Melbourne

- A. Knutson, P. Zinn-Justin, Schubert puzzles and integrability I: invariant trilinear forms, arXiv:1706.10019
- A. Knutson, P. Zinn-Justin, Schubert puzzles and integrability II: multiplying motivic Segre classes, arXiv:2102.00563
- A. Knutson, P. Zinn-Justin, Schubert puzzles and integrability III: separated descents, arXiv:2306.13855
- P. Zinn-Justin, The *CotangentSchubert* Macaulay2 package

Introduction

Schubert polynomials were introduced by Lascoux and Schützenberger to represent cohomology classes of Schubert cycles in flag varieties.



Alain Lascoux
(1944–2013)

Marcel-Paul Schützenberger
(1920–1996)

Definition

Given $\sigma \in \mathcal{S}_\infty = \bigcup_{n\geq 1} \mathcal{S}_n$, define $\mathfrak{S}^\sigma \in R := \mathbb{Z}[x_1, x_2, \ldots]$ inductively by

$$\mathfrak{S}^\sigma = \partial_i \, \mathfrak{S}^{\sigma s_i} \qquad \text{for } \sigma(i) < \sigma(i+1)$$

$$\mathfrak{S}^{n\ldots 21} = \prod_{i=1}^{n} x_i^{n-i}$$

where s_i is the elementary transposition $i \leftrightarrow i+1$ and ∂_i is the corresponding divided difference operator

$$\partial_i f := \frac{f - f|_{x_i \leftrightarrow x_{i+1}}}{x_i - x_{i+1}} \qquad f \in R$$

Example: $n = 3$

$$\mathfrak{S}^{321} = x_1^2 x_2$$

∂_1 ↙ ↘ ∂_2

$$\mathfrak{S}^{231} = x_1 x_2 \qquad \mathfrak{S}^{312} = x_1^2$$

∂_2 ↓ ↓ ∂_1

$$\mathfrak{S}^{213} = x_1 \qquad \mathfrak{S}^{132} = x_1 + x_2$$

∂_1 ↘ ↙ ∂_2

$$\mathfrak{S}^{123} = 1$$

Example: $n = 3$

$$\mathfrak{S}^{3|2|1} = x_1^2 x_2$$

∂_1 ↙ ↘ ∂_2

$$\mathfrak{S}^{23|1} = x_1 x_2 \qquad \mathfrak{S}^{3|12} = x_1^2$$

∂_2 ↓ ↓ ∂_1

$$\mathfrak{S}^{2|13} = x_1 \qquad \mathfrak{S}^{13|2} = x_1 + x_2$$

∂_1 ↘ ↙ ∂_2

$$\mathfrak{S}^{123} = 1$$

Descents

Define the descent set of a permutation

$$D(\sigma) := \{i \in \mathbb{Z}_{>0} : \sigma_i > \sigma_{i+1}\} \qquad \sigma \in \mathcal{S}_\infty$$

Then it is obvious from their definition that Schubert polynomials are symmetric in variables between two descents, and do not depend on variables after the last descent.

Important example: Grassmannian permutations. If σ has no descent outside $k \in \mathbb{Z}_{>0}$, then $\mathfrak{S}_\sigma$ is a symmetric polynomial in $x_1, \ldots, x_k$ – in fact, a Schur polynomial.

Descents

Define the descent set of a permutation

$$D(\sigma) := \{i \in \mathbb{Z}_{>0} : \sigma_i > \sigma_{i+1}\} \qquad \sigma \in \mathcal{S}_\infty$$

Then it is obvious from their definition that Schubert polynomials are symmetric in variables between two descents, and do not depend on variables after the last descent.

Important example: Grassmannian permutations. If σ has no descent outside $k \in \mathbb{Z}_{>0}$, then $\mathfrak{S}_\sigma$ is a symmetric polynomial in $x_1, \ldots, x_k$ – in fact, a Schur polynomial.

The structure constants

The $\{\mathfrak{S}^\sigma,\ \sigma \in \mathcal{S}_\infty\}$ form a basis of $R \Rightarrow$ one can expand products:

$$\mathfrak{S}^\pi \mathfrak{S}^\rho = \sum_{\sigma \in \mathcal{S}_\infty} c_\sigma^{\pi\rho} \mathfrak{S}^\sigma \qquad \pi, \rho \in \mathcal{S}_\infty$$

It is well-known that $c_\sigma^{\pi\rho} \in \mathbb{Z}_{\geq 0}$.

Each $\langle \mathfrak{S}^\sigma,\ D(\sigma) \subseteq \mathcal{D} \rangle$ is a subring, i.e.,

$$c_\sigma^{\pi\rho} \neq 0 \quad \Rightarrow \quad D(\sigma) \subseteq D(\pi) \cup D(\rho)$$

For example, within the subring $\langle \mathfrak{S}^\sigma,\ D(\sigma) \subseteq \{k\} \rangle$, the $c_\sigma^{\pi\rho}$ are the famous Littlewood–Richardson coefficients, for which numerous combinatorial formulae are known.

More generally, we'll be interested in computing $c_\sigma^{\pi\rho}$ when we put various restrictions on $D(\pi)$ and $D(\rho)$.

The structure constants

The $\{\mathfrak{S}^\sigma,\ \sigma \in \mathcal{S}_\infty\}$ form a basis of $R \Rightarrow$ one can expand products:

$$\mathfrak{S}^\pi \mathfrak{S}^\rho = \sum_{\sigma \in \mathcal{S}_\infty} c_\sigma^{\pi\rho} \mathfrak{S}^\sigma \qquad \pi, \rho \in \mathcal{S}_\infty$$

It is well-known that $c_\sigma^{\pi\rho} \in \mathbb{Z}_{\geq 0}$.

Each $\langle \mathfrak{S}^\sigma,\ D(\sigma) \subseteq \mathcal{D} \rangle$ is a subring, i.e.,

$$c_\sigma^{\pi\rho} \neq 0 \quad \Rightarrow \quad D(\sigma) \subseteq D(\pi) \cup D(\rho)$$

For example, within the subring $\langle \mathfrak{S}^\sigma,\ D(\sigma) \subseteq \{k\} \rangle$, the $c_\sigma^{\pi\rho}$ are the famous Littlewood–Richardson coefficients, for which numerous combinatorial formulae are known.

More generally, we'll be interested in computing $c_\sigma^{\pi\rho}$ when we put various restrictions on $D(\pi)$ and $D(\rho)$.

The structure constants

The $\{\mathfrak{S}^\sigma,\ \sigma \in \mathcal{S}_\infty\}$ form a basis of $R \Rightarrow$ one can expand products:

$$\mathfrak{S}^\pi \mathfrak{S}^\rho = \sum_{\sigma \in \mathcal{S}_\infty} c_\sigma^{\pi\rho} \mathfrak{S}^\sigma \qquad \pi, \rho \in \mathcal{S}_\infty$$

It is well-known that $c_\sigma^{\pi\rho} \in \mathbb{Z}_{\geq 0}$.

Each $\langle \mathfrak{S}^\sigma,\ D(\sigma) \subseteq \mathcal{D} \rangle$ is a subring, i.e.,

$$c_\sigma^{\pi\rho} \neq 0 \quad \Rightarrow \quad D(\sigma) \subseteq D(\pi) \cup D(\rho)$$

For example, within the subring $\langle \mathfrak{S}^\sigma,\ D(\sigma) \subseteq \{k\} \rangle$, the $c_\sigma^{\pi\rho}$ are the famous Littlewood–Richardson coefficients, for which numerous combinatorial formulae are known.

More generally, we'll be interested in computing $c_\sigma^{\pi\rho}$ when we put various restrictions on $D(\pi)$ and $D(\rho)$.

The structure constants

The $\{\mathfrak{S}^\sigma,\ \sigma\in\mathcal{S}_\infty\}$ form a basis of $R \Rightarrow$ one can expand products:

$$\mathfrak{S}^\pi\mathfrak{S}^\rho = \sum_{\sigma\in\mathcal{S}_\infty} c_\sigma^{\pi\rho}\mathfrak{S}^\sigma \qquad \pi,\rho\in\mathcal{S}_\infty$$

It is well-known that $c_\sigma^{\pi\rho}\in\mathbb{Z}_{\geq 0}$.

Each $\langle\mathfrak{S}^\sigma,\ D(\sigma)\subseteq\mathcal{D}\rangle$ is a subring, i.e.,

$$c_\sigma^{\pi\rho}\neq 0 \quad\Rightarrow\quad D(\sigma)\subseteq D(\pi)\cup D(\rho)$$

For example, within the subring $\langle\mathfrak{S}^\sigma,\ D(\sigma)\subseteq\{k\}\rangle$, the $c_\sigma^{\pi\rho}$ are the famous Littlewood–Richardson coefficients, for which numerous combinatorial formulae are known.

More generally, we'll be interested in computing $c_\sigma^{\pi\rho}$ when we put various restrictions on $D(\pi)$ and $D(\rho)$.

The structure constants

The $\{\mathfrak{S}^\sigma,\ \sigma \in \mathcal{S}_\infty\}$ form a basis of $R \Rightarrow$ one can expand products:

$$\mathfrak{S}^\pi \mathfrak{S}^\rho = \sum_{\sigma \in \mathcal{S}_\infty} c_\sigma^{\pi\rho} \mathfrak{S}^\sigma \qquad \pi, \rho \in \mathcal{S}_\infty$$

It is well-known that $c_\sigma^{\pi\rho} \in \mathbb{Z}_{\geq 0}$.

Each $\langle \mathfrak{S}^\sigma,\ D(\sigma) \subseteq \mathcal{D} \rangle$ is a subring, i.e.,

$$c_\sigma^{\pi\rho} \neq 0 \quad \Rightarrow \quad D(\sigma) \subseteq D(\pi) \cup D(\rho)$$

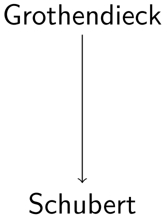
For example, within the subring $\langle \mathfrak{S}^\sigma,\ D(\sigma) \subseteq \{k\} \rangle$, the $c_\sigma^{\pi\rho}$ are the famous Littlewood–Richardson coefficients, for which numerous combinatorial formulae are known.

More generally, we'll be interested in computing $c_\sigma^{\pi\rho}$ when we put various restrictions on $D(\pi)$ and $D(\rho)$.

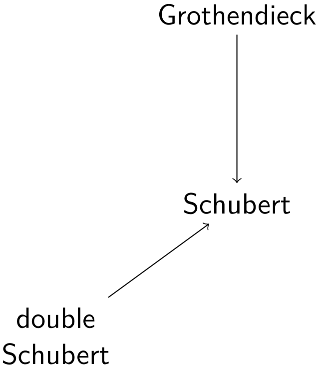
The generalisations [II]

Schubert

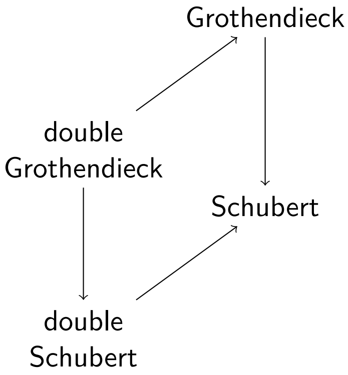
The generalisations [II]



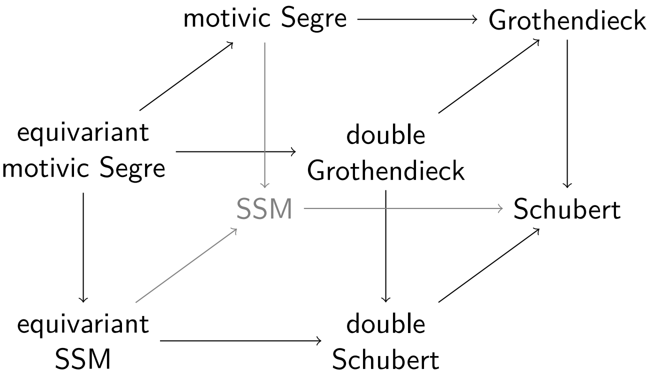
The generalisations [II]



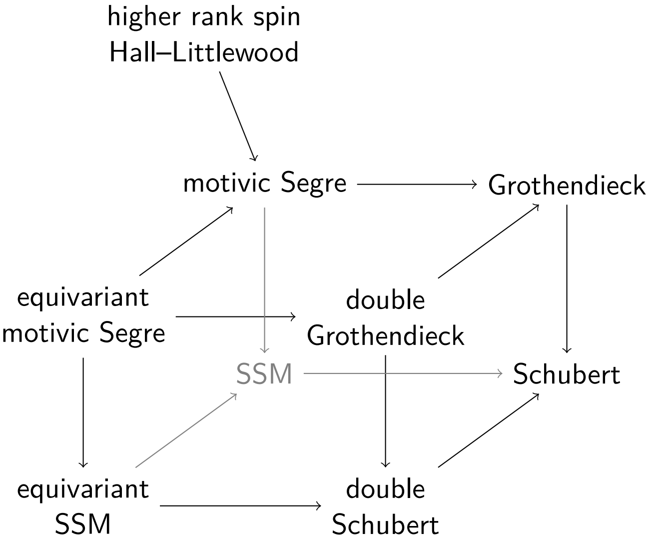
The generalisations [II]



The generalisations [II]



The generalisations [II]



Where does exactly solvability come into play?

- My interest is in applying methods from exactly solvable models (a.k.a. quantum integrable systems), an area of mathematical physics, to the study of such families of polynomials.
- Circa 2008–2014, I discovered that many families of (symmetric/not) polynomials can be expressed as partition functions of exactly solvable models (Schur, Schur-Q, LLT, Schubert, Grothendieck, . . .)
- In the case of Schubert/Grothendieck, this is closely related to the work of [Bergeron and Billey '93, Fomin and Kirillov '94].
- By now, there's a clear picture of a deep connection between these families of polynomials/rational functions, geometric representation theory and exact solvability [Nekrasov et al, Okounkov et al, Rimányi, Tarasov and Varchenko]

Where does exactly solvability come into play?

- My interest is in applying methods from exactly solvable models (a.k.a. quantum integrable systems), an area of mathematical physics, to the study of such families of polynomials.
- Circa 2008–2014, I discovered that many families of (symmetric/not) polynomials can be expressed as partition functions of exactly solvable models (Schur, Schur-Q, LLT, Schubert, Grothendieck, . . .)
- In the case of Schubert/Grothendieck, this is closely related to the work of [Bergeron and Billey '93, Fomin and Kirillov '94].
- By now, there's a clear picture of a deep connection between these families of polynomials/rational functions, geometric representation theory and exact solvability [Nekrasov et al, Okounkov et al, Rimányi, Tarasov and Varchenko]

Where does exactly solvability come into play?

- My interest is in applying methods from exactly solvable models (a.k.a. quantum integrable systems), an area of mathematical physics, to the study of such families of polynomials.
- Circa 2008–2014, I discovered that many families of (symmetric/not) polynomials can be expressed as partition functions of exactly solvable models (Schur, Schur-Q, LLT, Schubert, Grothendieck, . . .)
- In the case of Schubert/Grothendieck, this is closely related to the work of [Bergeron and Billey '93, Fomin and Kirillov '94].
- By now, there's a clear picture of a deep connection between these families of polynomials/rational functions, geometric representation theory and exact solvability [Nekrasov et al, Okounkov et al, Rimányi, Tarasov and Varchenko]

Where does exactly solvability come into play?

- My interest is in applying methods from exactly solvable models (a.k.a. quantum integrable systems), an area of mathematical physics, to the study of such families of polynomials.
- Circa 2008–2014, I discovered that many families of (symmetric/not) polynomials can be expressed as partition functions of exactly solvable models (Schur, Schur-Q, LLT, Schubert, Grothendieck, . . .)
- In the case of Schubert/Grothendieck, this is closely related to the work of [Bergeron and Billey '93, Fomin and Kirillov '94].
- By now, there's a clear picture of a deep connection between these families of polynomials/rational functions, geometric representation theory and exact solvability [Nekrasov et al, Okounkov et al, Rimányi, Tarasov and Varchenko]

Permutations and strings

We encode permutations using strings:

$$\sigma \quad 1\ 3\ 6 \mid 2\ 5 \mid 4\ 7$$

$$\omega \quad 0\ 0\ 0 \mid 1\ 1 \mid 2\ 2$$

$$\lambda \quad 0\ 1\ 0\ 2\ 1\ 0\ 2$$

There is freedom to add gratuitous nondescents, and to increase the size.

Conversely, given λ, σ is the inverse of its standardisation.

Permutations and strings

We encode permutations using strings:

alphabet is some arbitrary totally ordered set

σ	1	3	6 |	2	5 |	4	7
ω	0	0	0 |	1	1 |	2	2

λ 0 1 0 2 1 0 2

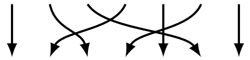
There is freedom to add gratuitous nondescents, and to increase the size.

Conversely, given λ, σ is the inverse of its standardisation.

Permutations and strings

We encode permutations using strings:

σ	1	3	6	2	5	4	7
ω	0	0	0	1	1	2	2



λ	0	1	0	2	1	0	2

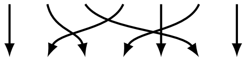
There is freedom to add gratuitous nondescents, and to increase the size.

Conversely, given λ, σ is the inverse of its standardisation.

Permutations and strings

We encode permutations using strings:

σ	1	3	6 |	2	5 |	4	7
ω	0	0	0 |	1	1 |	2	2



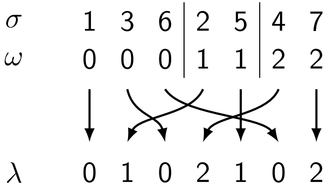
λ	0	1	0	2	1	0	2

There is freedom to add gratuitous nondescents, and to increase the size.

Conversely, given λ, σ is the inverse of its standardisation.

Permutations and strings

We encode permutations using strings:

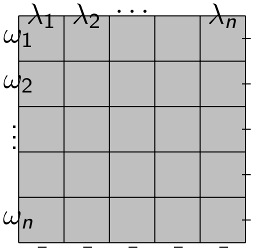


There is freedom to add gratuitous nondescents, and to increase the size.

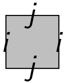
Conversely, given λ, σ is the inverse of its standardisation.

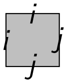
Schubert polynomials as an exactly solvable model

We have the following “partition function” identity, given any string λ corresponding to σ, with $\omega = sort(\lambda)$:

$\mathfrak{S}^\sigma =$ 

a sum over labellings of internal edges, such that each plaquette is

- 

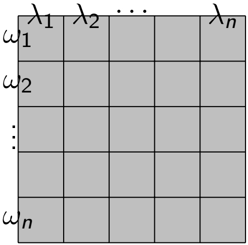
, $i < j$, each of which on the r^{th} row contributes a x_r.
- 

.

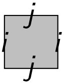
where $i, j \in \mathbb{Z}_{\geq 0} \cup \{\_\}$ with the convention $i < \_$ for all $i \in \mathbb{Z}_{\geq 0}$.

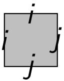
Schubert polynomials as an exactly solvable model

We have the following "partition function" identity, given any string λ corresponding to σ, with $\omega = sort(\lambda)$:

$\mathfrak{S}^\sigma =$ 

a sum over labellings of internal edges, such that each plaquette is

- 

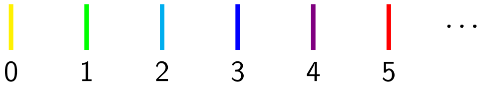
, $i < j$, each of which on the r^{th} row contributes a x_r.
- 

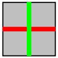
.

where $i, j \in \mathbb{Z}_{\geq 0} \cup \{\_\}$ with the convention $i < \_$ for all $i \in \mathbb{Z}_{\geq 0}$.

Schubert computation example

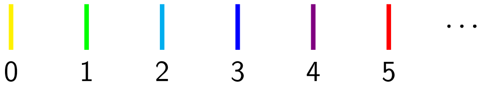
We represent the string digits as colours:

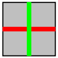


so that plaquettes look like  and .

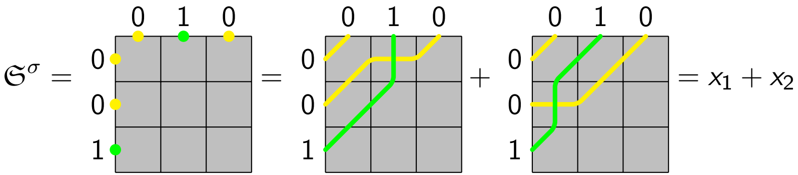
Schubert computation example

We represent the string digits as colours:



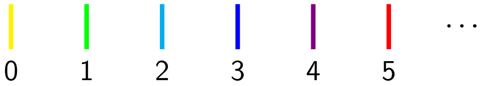
so that plaquettes look like  and .

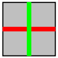
For example, if $\sigma = 132$, $\lambda = 010$ and



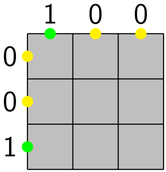
Schubert computation example

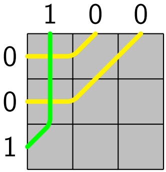
We represent the string digits as colours:



so that plaquettes look like  and .

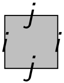
For example, if $\sigma = 231$, $\lambda = 100$ and

$\mathfrak{S}^\sigma =$ 

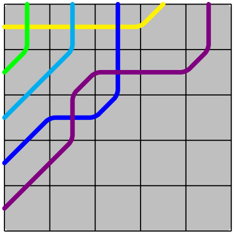
$=$ 

$= x_1 x_2$

Digression: Generic Pipe Dreams

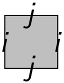
If one relaxes the constraint on i and j in the crossing 

, then one gets what we call Generic Pipe Dreams:

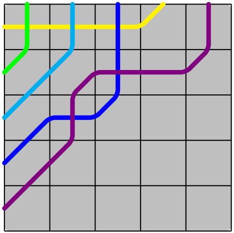


These are relevant to the computation of the motivic Segre class rational functions; in the limit to Grothendieck polynomials, one recovers either ordinary pipe dreams or bumpless pipe dreams [Lam Lee Shimozono '18, Weigandt '18].

Digression: Generic Pipe Dreams

If one relaxes the constraint on i and j in the crossing 

, then one gets what we call Generic Pipe Dreams:

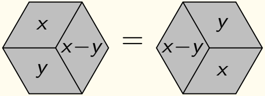


These are relevant to the computation of the motivic Segre class rational functions; in the limit to Grothendieck polynomials, one recovers either ordinary pipe dreams or bumpless pipe dreams [Lam Lee Shimozono '18, Weigandt '18].

The Yang–Baxter equation

The Yang–Baxter equation [Brézin and Zinn-Justin, '66; Yang, '67; Baxter, '70s] is the signature of exact solvability.

Lemma



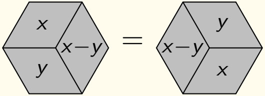
Apply it repeatedly to our partition function ($x = x_i, y = x_{i+1}$)

we obtain, for $\omega_i = \omega_{i+1}$ the symmetry under $x_i \leftrightarrow x_{i+1}$, or for $\omega_i < \omega_{i+1}$, the induction formula for Schubert polynomials.

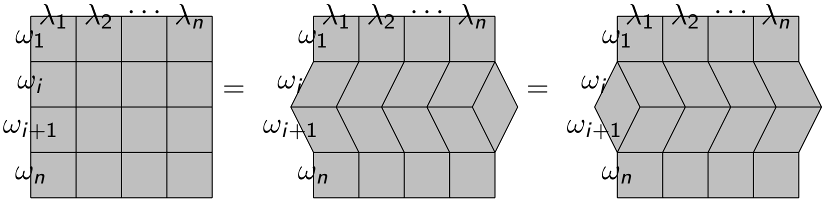
The Yang–Baxter equation

The Yang–Baxter equation [Brézin and Zinn-Justin, '66; Yang, '67 Baxter, '70s] is the signature of exact solvability.

Lemma



Apply it repeatedly to our partition function $(x = x_i, y = x_{i+1})$



we obtain, for $\omega_i = \omega_{i+1}$ the symmetry under $x_i \leftrightarrow x_{i+1}$, or for $\omega_i < \omega_{i+1}$, the induction formula for Schubert polynomials.

Structure constants as an exactly solvable model

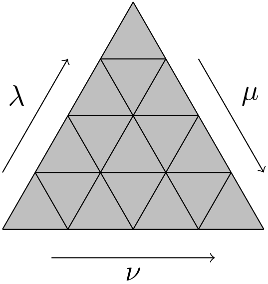
- This reformulation of Schubert polynomials as partition function does not obviously help with our goal, which is the computation of $c_{\sigma}^{\pi\rho}$. $\rightarrow$ Another idea is required to use exactly solvable methods for that.
- In 2008, I proposed to reinterpret Knutson–Tao puzzles as an exactly solvable model.

Structure constants as an exactly solvable model

- This reformulation of Schubert polynomials as partition function does not obviously help with our goal, which is the computation of $c_{\sigma}^{\pi\rho}$. $\to$ Another idea is required to use exactly solvable methods for that.
- In 2008, I proposed to reinterpret Knutson–Tao puzzles as an exactly solvable model.

Structure constants as an exactly solvable model

- This reformulation of Schubert polynomials as partition function does not obviously help with our goal, which is the computation of $c_\sigma^{\pi\rho}$. $\rightarrow$ Another idea is required to use exactly solvable methods for that.
- In 2008, I proposed to reinterpret Knutson–Tao puzzles as an exactly solvable model.

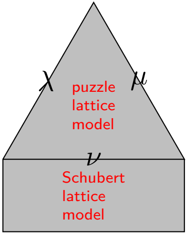
$c_\sigma^{\pi\rho} =$ 

$\lambda \mapsto \pi$, $\mu \mapsto \rho$, $\nu \mapsto \sigma$

The main theorem of I-II-III

For the purposes of this slide, we index $\mathfrak{S}$ and c with strings rather than permutations.

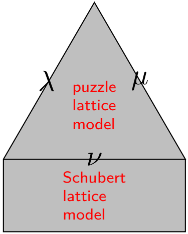
$$\sum_{\nu} c_{\nu}^{\lambda\mu} \mathfrak{S}^{\nu} =$$

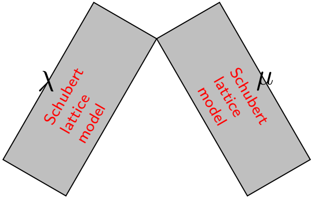


The main theorem of I-II-III

For the purposes of this slide, we index $\mathfrak{S}$ and c with strings rather than permutations.

$$\sum_\nu c_\nu^{\lambda\mu} \mathfrak{S}^\nu =$$



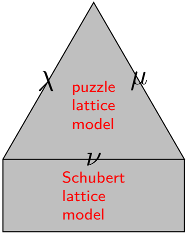


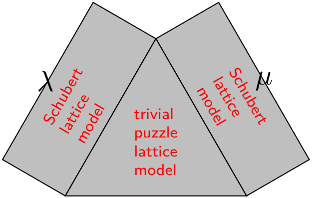
$$= \mathfrak{S}^\lambda \mathfrak{S}^\mu$$

The main theorem of I-II-III

For the purposes of this slide, we index $\mathfrak{S}$ and c with strings rather than permutations.

$$\sum_{\nu} c_{\nu}^{\lambda\mu}\mathfrak{S}^{\nu} =$$



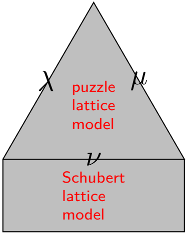


$$= \mathfrak{S}^{\lambda}\mathfrak{S}^{\mu}$$

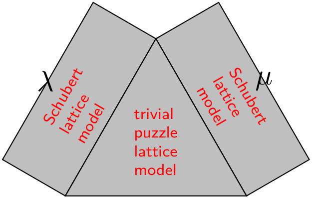
The main theorem of I-II-III

For the purposes of this slide, we index $\mathfrak{S}$ and c with strings rather than permutations.

$$\sum_{\nu} c_{\nu}^{\lambda\mu} \mathfrak{S}^{\nu} =$$



|| YBE-like moves

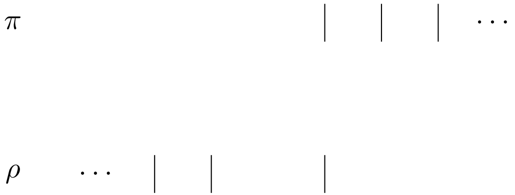


$$= \mathfrak{S}^{\lambda} \mathfrak{S}^{\mu}$$

Separated descents

We say that $\pi, \rho \in \mathcal{S}_\infty$ have **separated descents** if

$$\min D(\pi) \geq \max D(\rho)$$



Separated descents

We say that $\pi, \rho \in \mathcal{S}_\infty$ have **separated descents** if

$$\min D(\pi) \geq \max D(\rho)$$

seemingly gratuitous

$$\pi \qquad 2 \quad 4 \quad 5 \quad 7 \mid 6 \mid 3 \mid 1 \quad 8$$

$$\rho \qquad 4 \mid 3 \mid 1 \quad 5 \mid 2 \quad 6 \quad 7 \quad 8$$

Separated descents

We say that $\pi, \rho \in \mathcal{S}_\infty$ have separated descents if

$$\min D(\pi) \geq \max D(\rho)$$

$$\begin{array}{c|cccc|c|c|cc}
\pi & 2 & 4 & 5 & 7 & 6 & 3 & 1 & 8 \\
\omega_1 & _ & _ & _ & _ & 3 & 4 & 5 & 5
\end{array}$$

$$\begin{array}{c|c|c|cc|cccc}
\omega_2 & 0 & 1 & 2 & 2 & _ & _ & _ & _ \\
\rho & 4 & 3 & 1 & 5 & 2 & 6 & 7 & 8
\end{array}$$

Separated descents

We say that $\pi, \rho \in \mathcal{S}_\infty$ have separated descents if

$$\min D(\pi) \geq \max D(\rho)$$

π	2	4	5	7 |	6 |	3 |	1	8
ω_1	_	_	_	_	3	4	5	5
ω_3	0 |	1 |	2	2	3	4	5	5
ω_2	0	1	2	2	_	_	_	_
ρ	4 |	3 |	1	5 |	2	6	7	8

Separated descents

We say that $\pi, \rho \in \mathcal{S}_\infty$ have separated descents if

$$\min D(\pi) \geq \max D(\rho)$$

π	2	4	5	7	6	3	1	8
ω_1	_	_	_	_	3	4	5	5
ω_3	0	1	2	2	3	4	5	5
ω_2	0	1	2	2	_	_	_	_
ρ	4	3	1	5	2	6	7	8

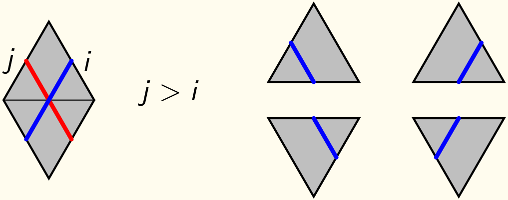
We use ω_1 to encode π, ω_2 to encode ρ, and ω_3 to encode σ:

$$\lambda = 5\,\_\,4\,\_\,\_\,3\,\_\,5 \qquad \mu = 2\,\_\,1\,0\,2\,\_\,\_\,\_$$

Separated descent rule

Theorem (A. Knutson, P. Z-J, '20, III)

Let π and ρ have separated descents. The coefficient of $\mathfrak{S}_\sigma$ in the expansion of $\mathfrak{S}_\pi \mathfrak{S}_\rho$ is the number of puzzles made of paths going SW/SE, such that no triangle is empty, and paths can only cross at horizontal edges, with the additional constraint:

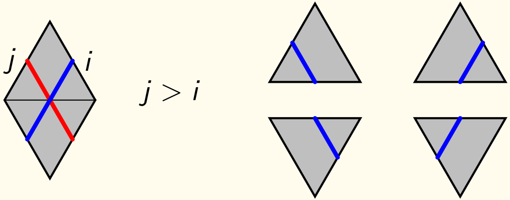


- The size n of the puzzle must be chosen so that $\pi, \rho, \sigma \in S_n$.
- See also [Huang, '21].

Separated descent rule

Theorem (A. Knutson, P. Z-J, '20, III)

Let π and ρ have separated descents. The coefficient of $\mathfrak{S}_\sigma$ in the expansion of $\mathfrak{S}_\pi \mathfrak{S}_\rho$ is the number of puzzles made of paths going SW/SE, such that no triangle is empty, and paths can only cross at horizontal edges, with the additional constraint:

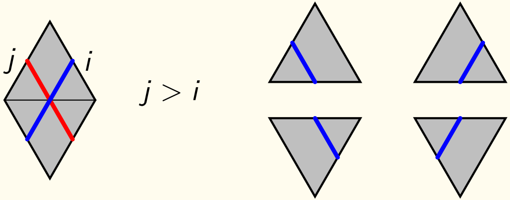


- The size n of the puzzle must be chosen so that $\pi, \rho, \sigma \in \mathcal{S}_n$.
- See also [Huang, '21].

Separated descent rule

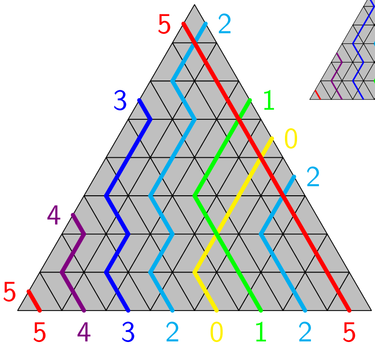
Theorem (A. Knutson, P. Z-J, '20, III)

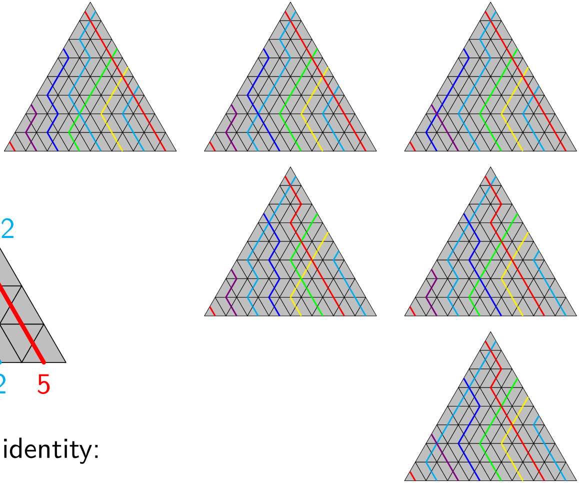
Let π and ρ have separated descents. The coefficient of $\mathfrak{S}_\sigma$ in the expansion of $\mathfrak{S}_\pi \mathfrak{S}_\rho$ is the number of puzzles made of paths going SW/SE, such that no triangle is empty, and paths can only cross at horizontal edges, with the additional constraint:



- The size n of the puzzle must be chosen so that $\pi, \rho, \sigma \in \mathcal{S}_n$.
- See also [Huang, '21].

Separated descent example



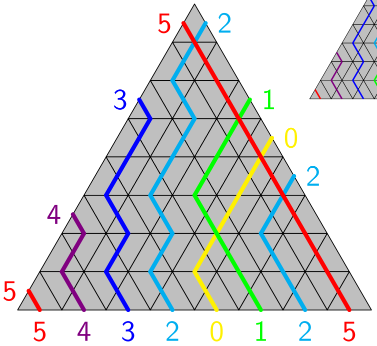


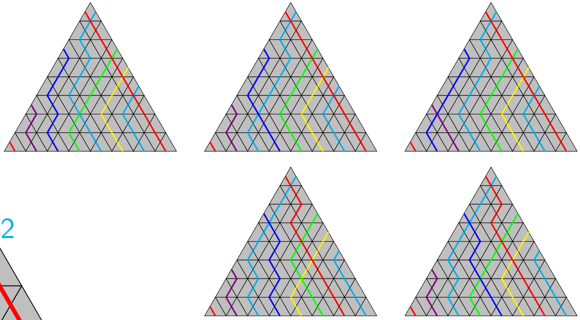
This leads to the following identity:

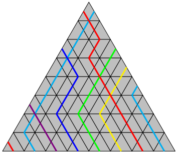
$$\mathfrak{S}^{24576318}\mathfrak{S}^{43152678} = \mathfrak{S}^{56473218} + \mathfrak{S}^{64573218}$$

$$+ \mathfrak{S}^{65374218} + \mathfrak{S}^{65472318} + \mathfrak{S}^{56384217} + \mathfrak{S}^{64385217} + \mathfrak{S}^{65284317}$$

Separated descent example







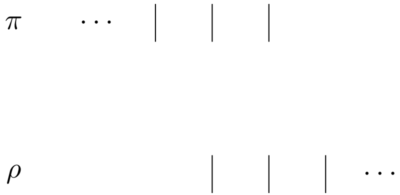
This leads to the following identity:

$$\mathfrak{S}^{2457|6|3|18}\mathfrak{S}^{4|3|15|2678} = \mathfrak{S}^{56|47|3|2|18} + \mathfrak{S}^{6|457|3|2|18}$$

$$+\mathfrak{S}^{6|5|37|4|2|18}+\mathfrak{S}^{6|5|47|23|18}+\mathfrak{S}^{56|38|4|2|17}+\mathfrak{S}^{6|4|38|5|2|17}+\mathfrak{S}^{6|5|28|4|3|17}$$

Almost separated descents

We say that $\pi, \rho \in \mathcal{S}_\infty$ have almost separated descents if the last two descents of π occur at (or before) the first two descents of ρ:



Almost separated descents

We say that $\pi, \rho \in \mathcal{S}_\infty$ have almost separated descents if the last two descents of π occur at (or before) the first two descents of ρ:

$$\pi \qquad 4 \mid 1 \,\vdots\, 3 \mid 2 \quad 5 \quad 6 \quad 7$$

$$\rho \qquad 2 \quad 5 \mid 4 \mid 3 \mid 1 \quad 6 \quad 7$$

Almost separated descents

We say that $\pi, \rho \in \mathcal{S}_\infty$ have almost separated descents if the last two descents of π occur at (or before) the first two descents of ρ:

π	4 |	1 ¦	3 |	2	5	6	7
ω_1	0 |	1 ¦	2 |	_	_	_	_
ω_2	_	_ |	2 |	3 |	4	4	4
ρ	2	5 |	4 |	3 |	1	6	7

Almost separated descents

We say that $\pi, \rho \in \mathcal{S}_\infty$ have almost separated descents if the last two descents of π occur at (or before) the first two descents of ρ:

π	4	1	3	2	5	6	7
ω_1	0	1	2	_	_	_	_
ω_3	0	1	_	3	4	4	4
ω_2	_	_	2	3	4	4	4
ρ	2	5	4	3	1	6	7

Almost separated descents

We say that $\pi, \rho \in \mathcal{S}_\infty$ have almost separated descents if the last two descents of π occur at (or before) the first two descents of ρ:

π	4	1	3	2	5	6	7
ω_1	0	1	2	_	_	_	_
ω_3	0	1	_	3	4	4	4
ω_2	_	_	2	3	4	4	4
ρ	2	5	4	3	1	6	7

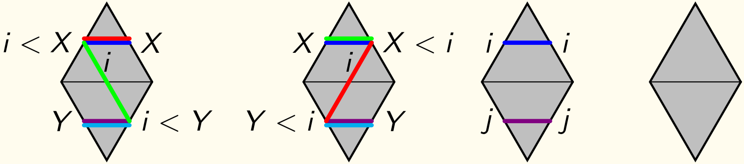
We use ω_1 to encode π, ω_2 to encode ρ, and ω_3 to encode σ:

$$\lambda = 1\,\_\,2\,0\,\_\,\_\,\_ \qquad \mu = 4\,\_\,3\,2\,\_\,4\,4$$

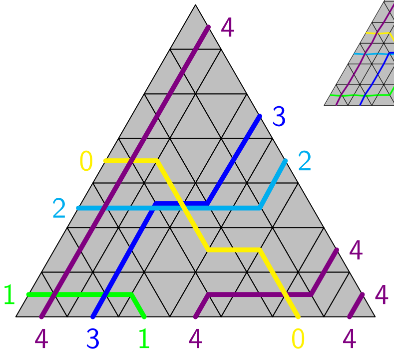
Almost separated descent rule

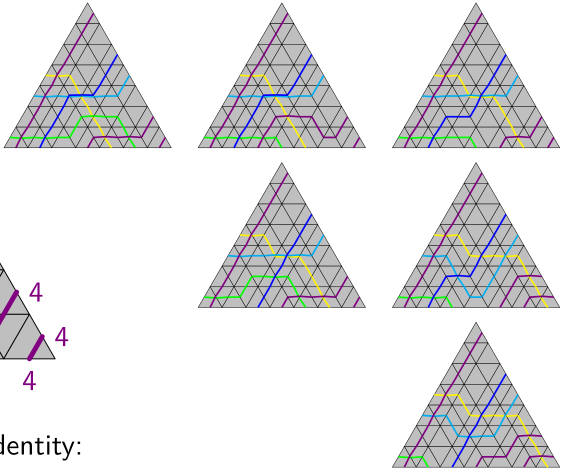
Theorem (A. Knutson, P. Z-J, 2023, III)

Let π and ρ have almost separated descents. The coefficient of $\mathfrak{S}_\sigma$ in the expansion of $\mathfrak{S}_\pi \mathfrak{S}_\rho$ is the number of puzzles made of paths going E/NE/SE, such that multiple paths of distinct colours can cross NW/NE edges (i.e., a subset $X \subseteq \{0, \ldots, d\}$), but at most one path deviates from the horizontal in any given triangle, with the further restriction on allowed triangles: (for the bottom row, use only top halves)



Almost separated descent example

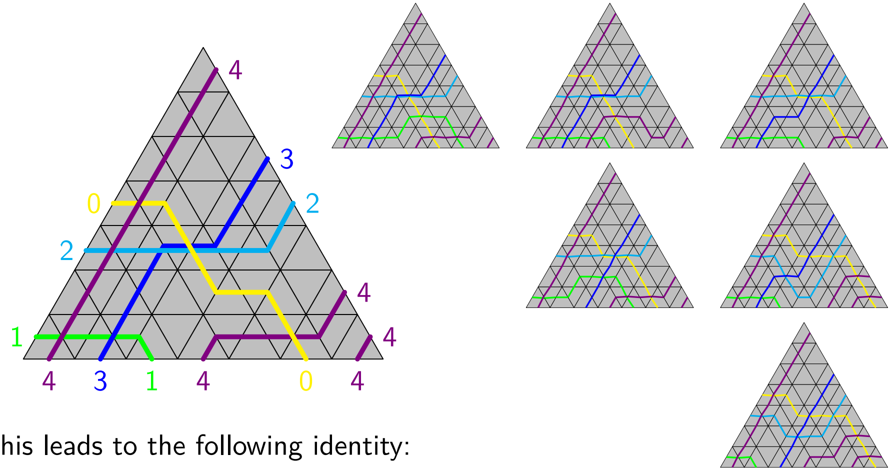




This leads to the following identity:

$$\mathfrak{S}^{4132567}\mathfrak{S}^{2543167} = \mathfrak{S}^{6352147} + \mathfrak{S}^{5632147} + \mathfrak{S}^{5462137} + \mathfrak{S}^{6432157} + \mathfrak{S}^{6523147} + \mathfrak{S}^{7342156} + \mathfrak{S}^{7253146}$$

Almost separated descent example



This leads to the following identity:

$$\mathfrak{S}^{4|13|2567}\mathfrak{S}^{25|4|3|167} = \mathfrak{S}^{6|35|2|147} + \mathfrak{S}^{56|3|2|147}$$
$$+ \mathfrak{S}^{5|46|2|137} + \mathfrak{S}^{6|4|3|2|157} + \mathfrak{S}^{6|5|23|147} + \mathfrak{S}^{7|34|2|156} + \mathfrak{S}^{7|25|3|146}$$

The representation theory

- In general, one builds solutions of YBE (and from there, an exactly solvable model) out of the representation theory of Yangians (or quantized loop algebras).
- The Schubert model (pipe dreams) is based on $\mathcal{Y}(\mathfrak{a}_d)$ where $d = |D(\sigma)|$.
- One could reformulate the search for Schubert puzzles as: finding a Yangian containing $\mathcal{Y}(\mathfrak{a}_d)$ as a subalgebra, with various combinatorial constraints coming from the geometry of root systems.
- For example, the separated descent model is also based on $\mathcal{Y}(\mathfrak{a}_n)$ with $n = |D(\pi)| + |D(\rho)|$.
- The almost separated descent model is based on $\mathcal{Y}(\mathfrak{d}_n)$.
- For technical reasons, we've (so far) restricted the search to simply laced Lie algebras.

The representation theory

- In general, one builds solutions of YBE (and from there, an exactly solvable model) out of the representation theory of Yangians (or quantized loop algebras).
- The Schubert model (pipe dreams) is based on $\mathcal{Y}(\mathfrak{a}_d)$ where $d = |D(\sigma)|$.
- One could reformulate the search for Schubert puzzles as: finding a Yangian containing $\mathcal{Y}(\mathfrak{a}_d)$ as a subalgebra, with various combinatorial constraints coming from the geometry of root systems.
- For example, the separated descent model is also based on $\mathcal{Y}(\mathfrak{a}_n)$ with $n = |D(\pi)| + |D(\rho)|$.
- The almost separated descent model is based on $\mathcal{Y}(\mathfrak{d}_n)$.
- For technical reasons, we've (so far) restricted the search to simply laced Lie algebras.

The representation theory

- In general, one builds solutions of YBE (and from there, an exactly solvable model) out of the representation theory of Yangians (or quantized loop algebras).
- The Schubert model (pipe dreams) is based on $\mathcal{Y}(\mathfrak{a}_d)$ where $d = |D(\sigma)|$.
- One could reformulate the search for Schubert puzzles as: finding a Yangian containing $\mathcal{Y}(\mathfrak{a}_d)$ as a subalgebra, with various combinatorial constraints coming from the geometry of root systems.
- For example, the separated descent model is also based on $\mathcal{Y}(\mathfrak{a}_n)$ with $n = |D(\pi)| + |D(\rho)|$.
- The almost separated descent model is based on $\mathcal{Y}(\mathfrak{d}_n)$.
- For technical reasons, we've (so far) restricted the search to simply laced Lie algebras.

The representation theory

- In general, one builds solutions of YBE (and from there, an exactly solvable model) out of the representation theory of Yangians (or quantized loop algebras).
- The Schubert model (pipe dreams) is based on $\mathcal{Y}(\mathfrak{a}_d)$ where $d = |D(\sigma)|$.
- One could reformulate the search for Schubert puzzles as: finding a Yangian containing $\mathcal{Y}(\mathfrak{a}_d)$ as a subalgebra, with various combinatorial constraints coming from the geometry of root systems.
- For example, the separated descent model is also based on $\mathcal{Y}(\mathfrak{a}_n)$ with $n = |D(\pi)| + |D(\rho)|$.
- The almost separated descent model is based on $\mathcal{Y}(\mathfrak{d}_n)$.
- For technical reasons, we've (so far) restricted the search to simply laced Lie algebras.

The representation theory

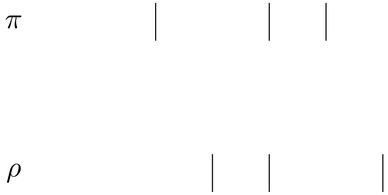
- In general, one builds solutions of YBE (and from there, an exactly solvable model) out of the representation theory of Yangians (or quantized loop algebras).
- The Schubert model (pipe dreams) is based on $\mathcal{Y}(\mathfrak{a}_d)$ where $d = |D(\sigma)|$.
- One could reformulate the search for Schubert puzzles as: finding a Yangian containing $\mathcal{Y}(\mathfrak{a}_d)$ as a subalgebra, with various combinatorial constraints coming from the geometry of root systems.
- For example, the separated descent model is also based on $\mathcal{Y}(\mathfrak{a}_n)$ with $n = |D(\pi)| + |D(\rho)|$.
- The almost separated descent model is based on $\mathcal{Y}(\mathfrak{d}_n)$.
- For technical reasons, we've (so far) restricted the search to simply laced Lie algebras.

The representation theory

- In general, one builds solutions of YBE (and from there, an exactly solvable model) out of the representation theory of Yangians (or quantized loop algebras).
- The Schubert model (pipe dreams) is based on $\mathcal{Y}(\mathfrak{a}_d)$ where $d = |D(\sigma)|$.
- One could reformulate the search for Schubert puzzles as: finding a Yangian containing $\mathcal{Y}(\mathfrak{a}_d)$ as a subalgebra, with various combinatorial constraints coming from the geometry of root systems.
- For example, the separated descent model is also based on $\mathcal{Y}(\mathfrak{a}_n)$ with $n = |D(\pi)| + |D(\rho)|$.
- The almost separated descent model is based on $\mathcal{Y}(\mathfrak{d}_n)$.
- For technical reasons, we've (so far) restricted the search to simply laced Lie algebras.

3-step to 5-step

Let $\pi, \rho \in \mathcal{S}_\infty$ with $\#D(\pi) = \#D(\rho) = 3$,
common middle descent:



3-step to 5-step

Let $\pi, \rho \in \mathcal{S}_\infty$ with $\#D(\pi) = \#D(\rho) = 3$,
common middle descent:

$$\pi \qquad 2 \mid 1 \quad 5 \mid 4 \mid 3 \quad 6 \quad 7$$

$$\rho \qquad 4 \quad 5 \mid 3 \mid 2 \quad 6 \mid 1 \quad 7$$

3-step to 5-step

Let $\pi, \rho \in \mathcal{S}_\infty$ with $\#D(\pi) = \#D(\rho) = 3$,
common middle descent:

$$\begin{array}{lc|cc|c|ccc} \pi & 2 & 1 & 5 & 4 & 3 & 6 & 7 \\ \omega_1 & 0 & 1 & 1 & 2 & 3 & 3 & 3 \end{array}$$

$$\begin{array}{lcc|c|cc|cc} \omega_2 & 0 & 0 & 1 & 2 & 2 & 3 & 3 \\ \rho & 4 & 5 & 3 & 2 & 6 & 1 & 7 \end{array}$$

3-step to 5-step

Let $\pi, \rho \in \mathcal{S}_\infty$ with $\#D(\pi) = \#D(\rho) = 3$,
common middle descent:

$$
\begin{array}{c|c|cc|c|ccc}
\pi & 2 & 1 & 5 & 4 & 3 & 6 & 7 \\
\omega_1 & 0 & 1 & 1 & 2 & 3 & 3 & 3
\end{array}
$$

$$
\begin{array}{cc|c|c|c|c|cc}
\omega_3 & 0 & 1 & 2 & 3 & 4 & 5 & 5
\end{array}
$$

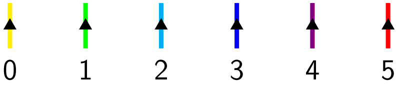
$$
\begin{array}{ccc|c|cc|cc}
\omega_2 & 0 & 0 & 1 & 2 & 2 & 3 & 3 \\
\rho & 4 & 5 & 3 & 2 & 6 & 1 & 7
\end{array}
$$

3-step to 5-step

Let $\pi, \rho \in \mathcal{S}_\infty$ with $\#D(\pi) = \#D(\rho) = 3$,
common middle descent:

π	2	1	5	4	3	6	7
ω_1	0	1	1	2	3	3	3
ω_3	0	1	2	3	4	5	5
ω_2	0	0	1	2	2	3	3
ρ	4	5	3	2	6	1	7

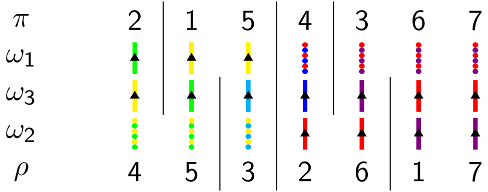
We represent the ω_3 digits as oriented coloured paths:



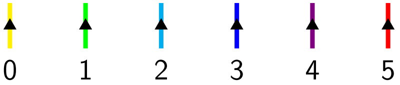
We also have unoriented paths made of two colours, e.g., .

3-step to 5-step

Let $\pi, \rho \in \mathcal{S}_\infty$ with $\#D(\pi) = \#D(\rho) = 3$,
common middle descent:



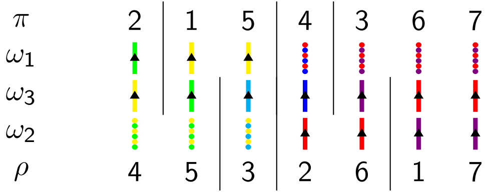
We represent the ω_3 digits as oriented coloured paths:



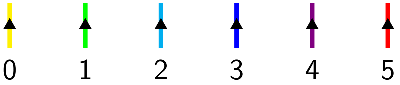
We also have unoriented paths made of two colours, e.g., .

3-step to 5-step

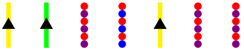
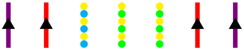
Let $\pi, \rho \in \mathcal{S}_\infty$ with $\#D(\pi) = \#D(\rho) = 3$, common middle descent:



We represent the ω_3 digits as oriented coloured paths:



We also have unoriented paths made of two colours, e.g., .
We use ω_1 to encode π, ω_2 to encode ρ, and ω_3 to encode σ:

$\lambda =$  $\mu =$ 

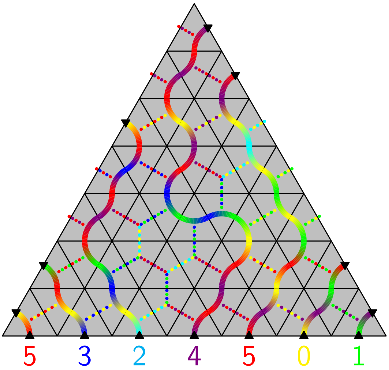
3-step to 5-step rule

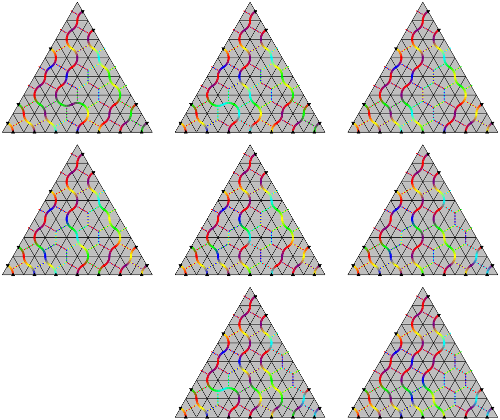
Theorem

Let $\pi, \rho \in \mathcal{S}_\infty$ as above, and $\sigma \in \mathcal{S}_\infty$ such that $\ell(\sigma) = \ell(\pi) + \ell(\rho)$. The coefficient of $\mathfrak{S}_\sigma$ in the expansion of $\mathfrak{S}_\pi \mathfrak{S}_\rho$ is the number of puzzles made of oriented colored paths and unoriented bicolored paths with the following two types of triangles:

and their 180 degree rotations,
where in the first, the three paths can be freely permuted,
and in the second, all colors must be present.

3-step to 5-step example

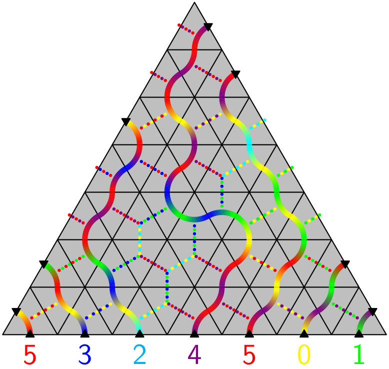


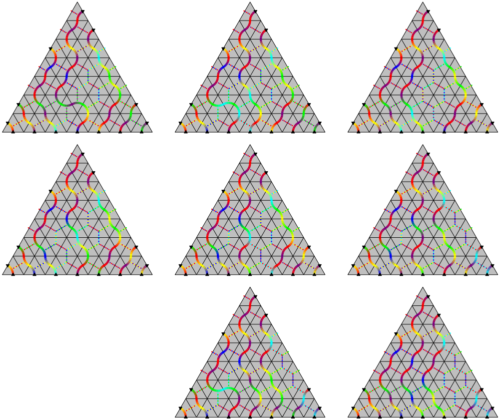


This leads to the following identity:

$$\mathfrak{S}^{2154367}\mathfrak{S}^{4532617} = \mathfrak{S}^{6732415} + \mathfrak{S}^{5734216} + \mathfrak{S}^{5742316}$$
$$+\mathfrak{S}^{7435216} + \mathfrak{S}^{7532416} + \mathfrak{S}^{7452316} + \mathfrak{S}^{6472315} + \mathfrak{S}^{5672314} + \mathfrak{S}^{5473216}$$

3-step to 5-step example





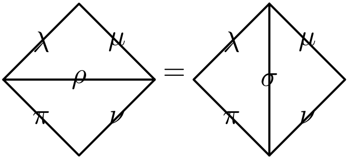
This leads to the following identity:

$$\mathfrak{S}^{2|15|4|367}\mathfrak{S}^{45|3|26|17} = \mathfrak{S}^{67|3|24|15} + \mathfrak{S}^{57|34|2|16} + \mathfrak{S}^{57|4|23|16}$$
$$+\mathfrak{S}^{7|4|35|2|16}+\mathfrak{S}^{7|5|3|24|16}+\mathfrak{S}^{7|45|23|16}+\mathfrak{S}^{6|47|23|15}+\mathfrak{S}^{567|23|14}+\mathfrak{S}^{5|47|3|2|16}$$

Further result. Associativity

Imposing associativity $(\mathfrak{S}^\lambda \mathfrak{S}^\mu)\mathfrak{S}^\nu = \mathfrak{S}^\lambda(\mathfrak{S}^\mu \mathfrak{S}^\nu)$ leads to quadratic constraints for the structure constants $c_\nu^{\lambda\mu}$:

$$\sum_\rho c_\rho^{\lambda\mu} c_\pi^{\rho\nu} = \sum_\sigma c_\pi^{\lambda\sigma} c_\sigma^{\mu\nu}$$



Is there a natural bijection?

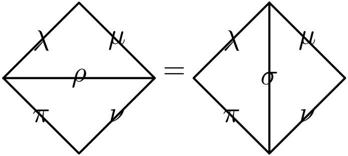
Integrability provides a linear algebraic answer:

Assoc

Further result. Associativity

Imposing associativity $(\mathfrak{S}^\lambda \mathfrak{S}^\mu)\mathfrak{S}^\nu = \mathfrak{S}^\lambda(\mathfrak{S}^\mu \mathfrak{S}^\nu)$ leads to quadratic constraints for the structure constants $c_\nu^{\lambda\mu}$:

$$\sum_\rho c_\rho^{\lambda\mu} c_\pi^{\rho\nu} = \sum_\sigma c_\pi^{\lambda\sigma} c_\sigma^{\mu\nu}$$



Is there a natural bijection?

Integrability provides a linear algebraic answer:

Assoc