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Loop Models and

Alternating Sign Matrices

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• The $O(1)$ loop model: ◇ Definition

◇ Boundary Conditions: periodic/closed, odd/even, distinct/identified

• Alternating Sign Matrices: ◇ Definition

◇ Symmetry classes: HTSASM, VSASM

◇ Relation to 6-vertex model

◇ Results on the enumeration problem

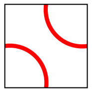
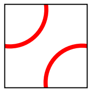
• $O(1)$ loop model $\leftrightarrow$ ASM: ◇ Formulation of the sum rules

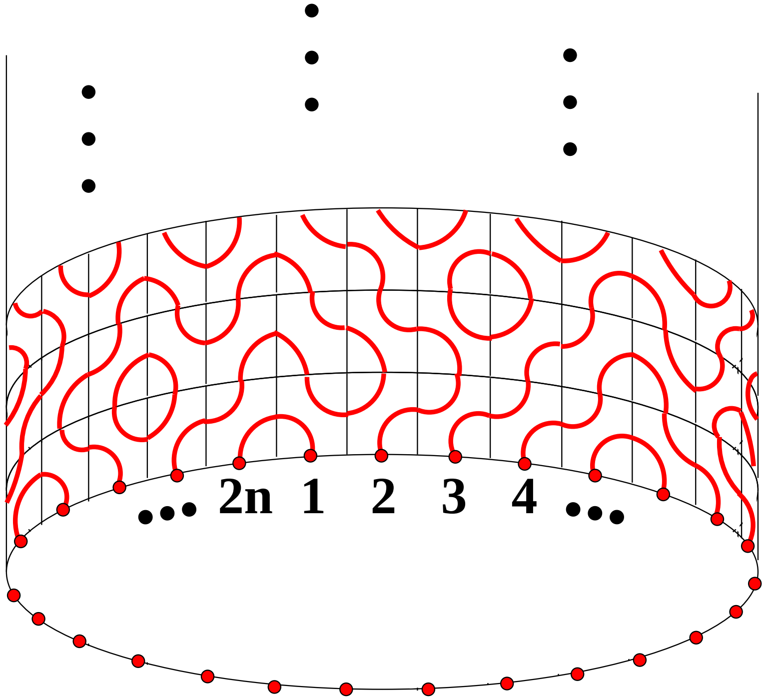
◇ Proof of the sum rules

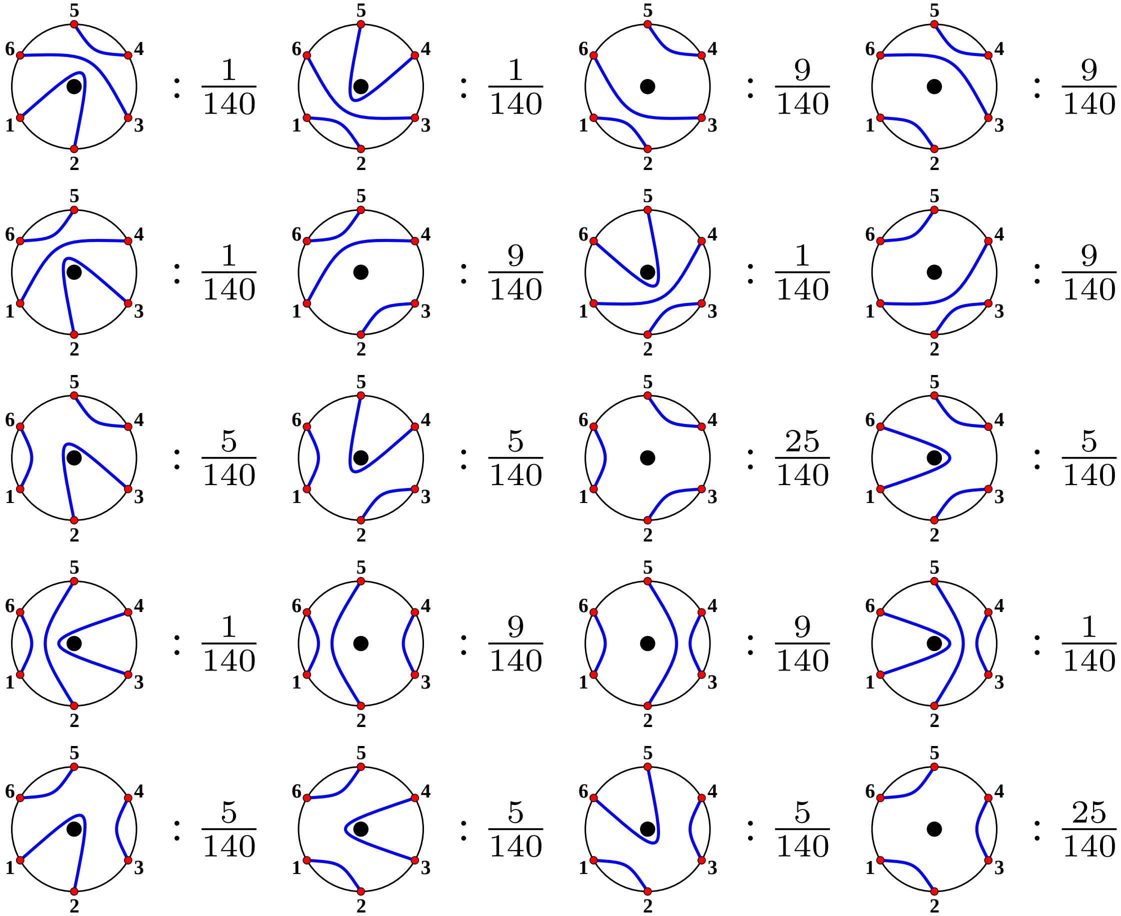
◇ An open problem: the Razumov–Stroganov conjecture

`math-ph/0603009`

Definition of the $O(1)$ loop model

Fill some two-dimensional surface with boundary with plaquettes:  with probability p,  with probability $1-p$. Simplest case: semi-infinite cylinder with perimeter $L = 2n$.

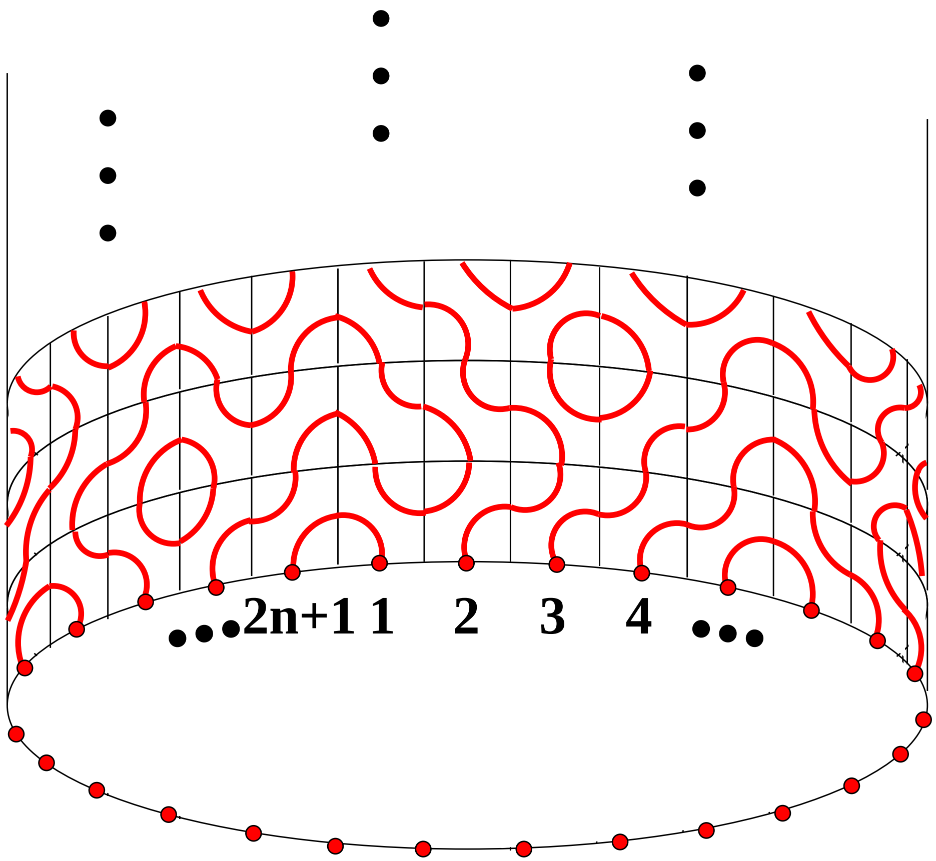


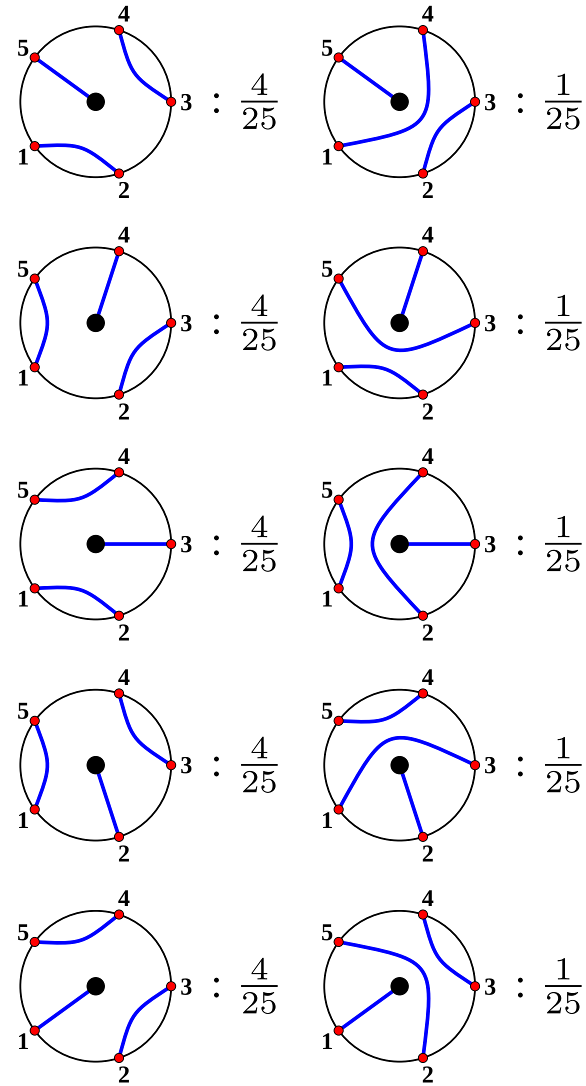


What is the probability law of the connectivity of the external points? $\rightarrow \left|\Psi^{2n,*}\right\rangle$

The $O(1)$ loop model: odd size

If the perimeter is odd: $L = 2n+1$, one external point will be unconnected (connected to infinity):

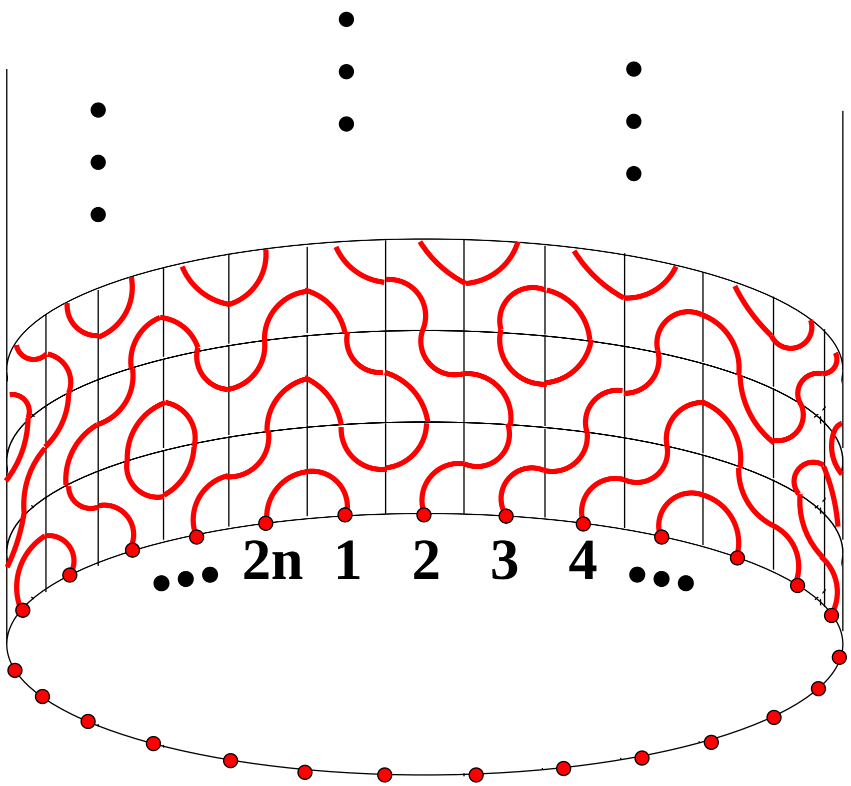


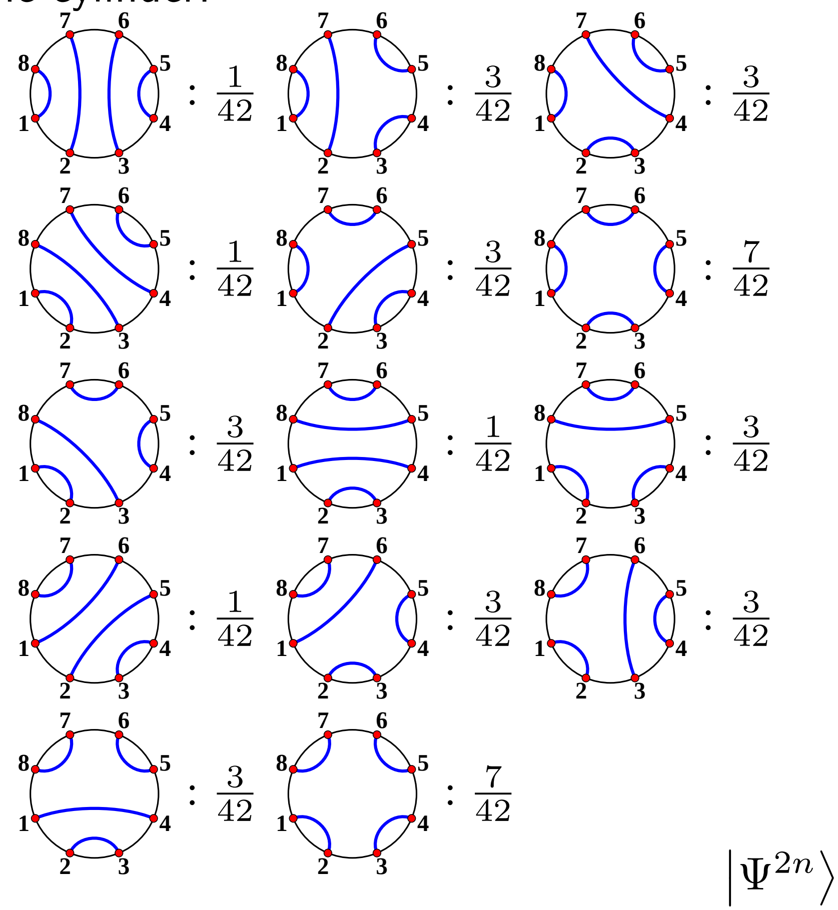


$\left|\Psi^{2n+1}\right\rangle$

The $O(1)$ loop model: identified connectivities

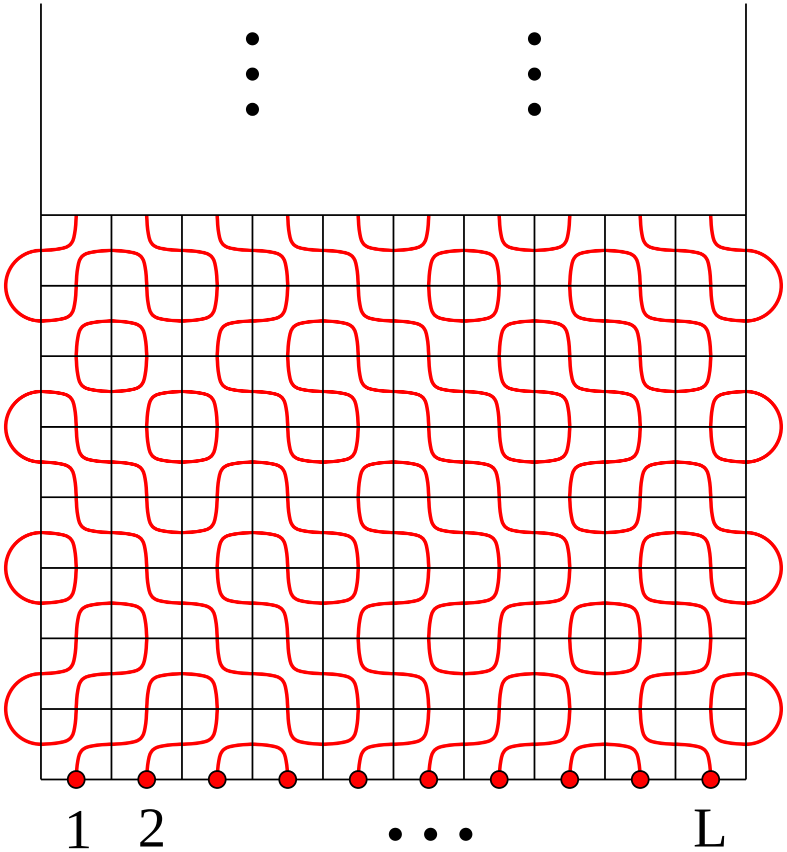
For even L, we can ignore how the paths wrap around the cylinder:

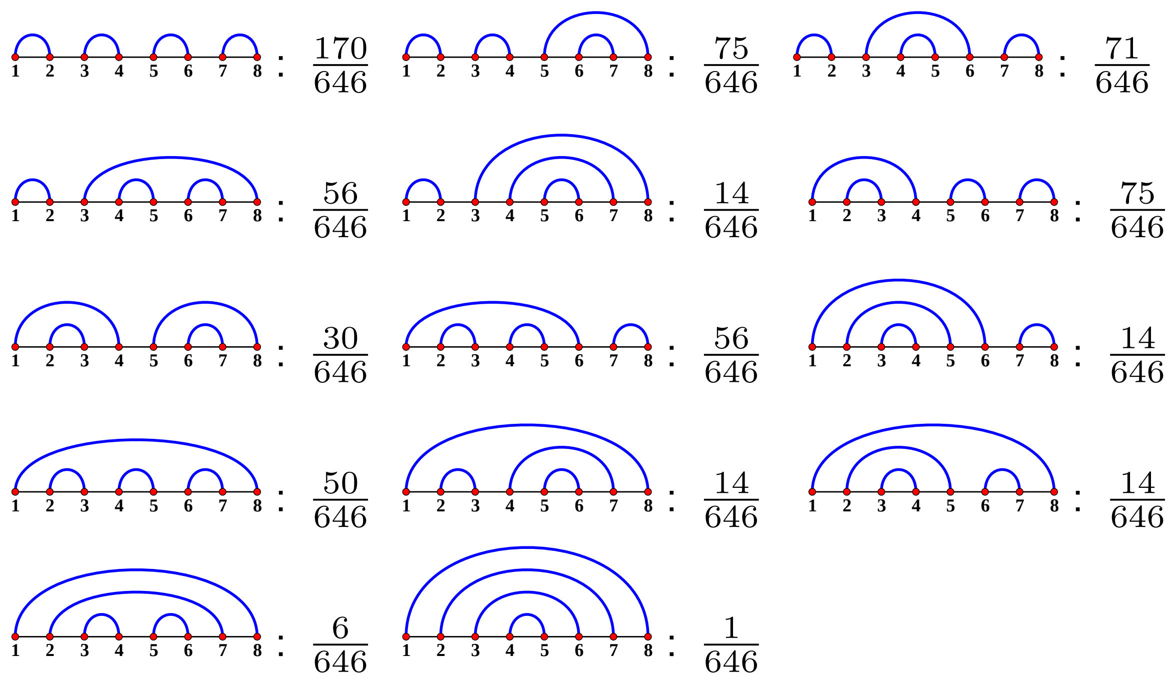




The $O(1)$ loop model: closed boundary conditions

Finally, we can consider the model on a strip of width L:





$\left|\Psi^{2n,closed}\right\rangle$

Some empirical observations

These probabilities satisfy many remarkable combinatorial properties [Batchelor, de Gier, Nienhuis; Razumov, Stroganov]:

$\Psi^{2n,*}_{max} = (4/3)^{n-1} \prod_{j=0}^{n-1} \frac{(2j+1)^2}{(3j+1)(3j+2)} = 1/2, 3/10, 5/28, 7/66, 9/143, \ldots$

$\Psi^{2n+1}_{max} = (4/3)^{n} \left(\frac{1\times 3\times\cdots\times(2n-1)}{2\times 5\times\cdots\times(3n-1)}\right)^2 = 1/3, 4/25, 1/12, \ldots$

$\Psi^{2n}_{max} = \frac{(2n-1)!(2n-2)!}{(3n-2)!(n-1)!} = 1, 1/2, 2/7, 1/6, 14/143, \ldots$

$\Psi^{2n,closed}_{max} = (2/3)^{n} \prod_{j=0}^{n-1} \frac{(4j+3)(4j+1)}{(3j+2)(6j+1)} = 1, 2/3, 11/26, 5/19, \ldots$

Various correlators are also conjectured, e.g.

[de Gier] The average number of “nests” of the connectivity on a strip of width L is

$$\sum_\pi \Psi^{L,closed}_\pi n_\pi = L \prod_{j=1}^{L-1} \frac{3j+1}{3j+2} \sim \frac{\Gamma(5/6)}{\sqrt{\pi}} (2L)^{2/3} \quad L \to \infty$$

The smallest component Ψ_{min} is always the inverse of an integer. The latter is related to. . .

Alternating Sign Matrices

$n \times n$ matrices with entries $0, +1, -1$ such that

$\star$ signs $+1$ and -1 alternate along each row and each column

$\star$ the sum is $+1$ along each row and each column

Example: There are seven 3×3 ASM :

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}, \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}, \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}, \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}, \begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}$$

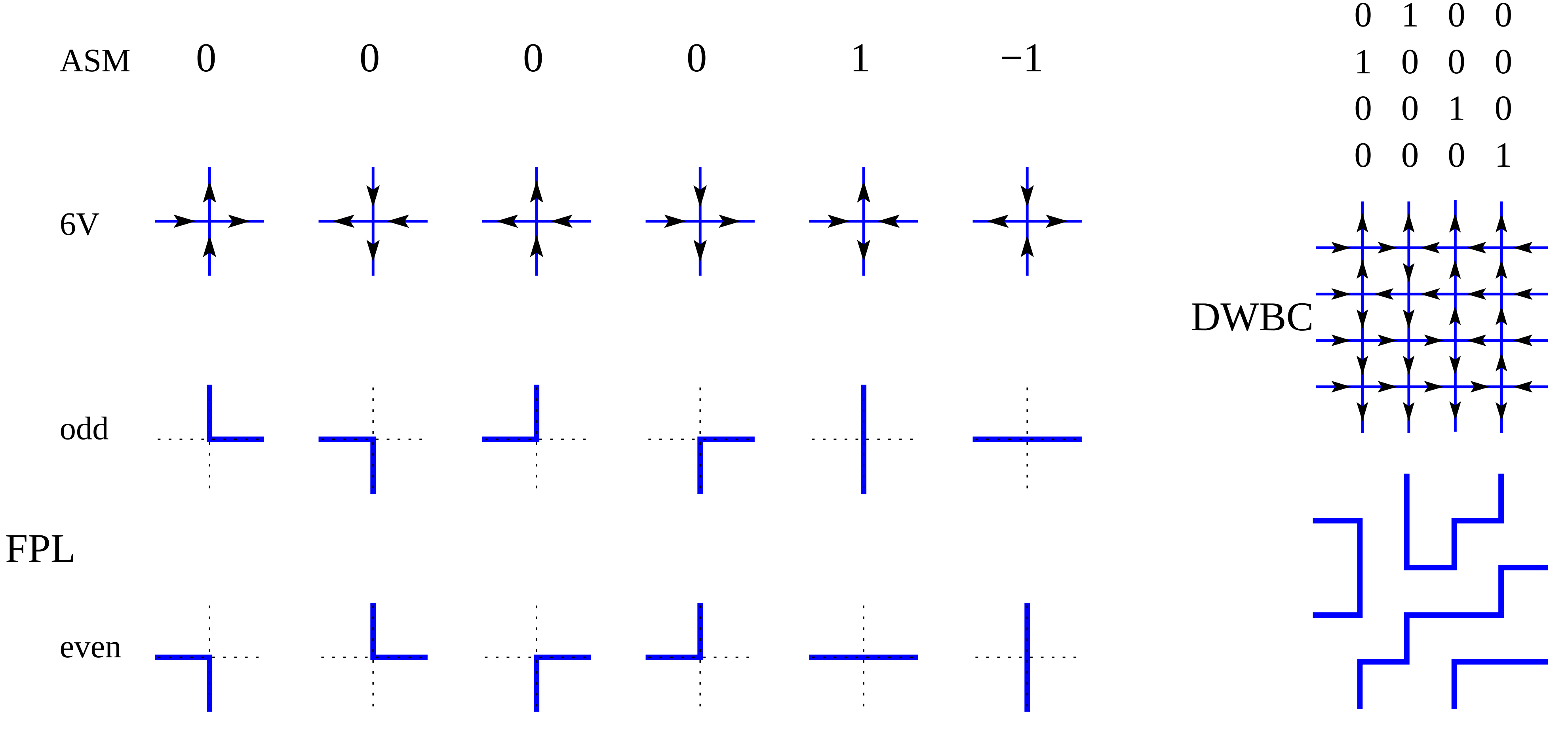
Number of ASM of size n:

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} \;\; = 1, 2, 7, 42, 429, \cdots$$

conjectured by Mills, Robbins and Rumsey (1983), proved by Zeilberger (1996) and Kuperberg (1996).

$\rightarrow$ we describe now Kuperberg's method. . .

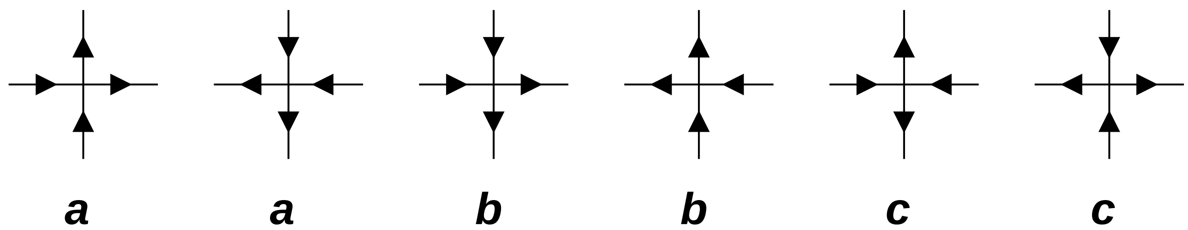
ASM ↔ 6 Vertex ↔ FPL



6 Vertex Model with DWBC: Izergin–Korepin formula

Associate to each horizontal line of the grid a parameter x_i and to each vertical line a parameter y_i.

The weight $w(x, y)$ at a vertex depends on the parameters x, y of the lines and is equal to:



$$a(x, y) = q^{-1/2}x - q^{1/2}y \qquad b(x, y) = q^{-1/2}y - q^{1/2}x \qquad c(x, y) = (q^{-1} - q)(x\,y)^{1/2}$$

$$A_n(x_1, \ldots, x_n; y_1, \ldots, y_n) \equiv \sum_{\text{6v DWBC configs}} \prod_{i,j=1}^{n} w(x_i, y_j)$$

Izergin–Korepin determinant formula ('87):

$$A_n(x_1, \ldots, x_n; y_1, \ldots, y_n) = \frac{\prod_{i,j=1}^{n} a(x_i, y_j) b(x_i, y_j)}{\prod_{i<j} (x_i - x_j)(y_i - y_j)} \det_{i,j=1\ldots n} \left(\frac{c(x_i, y_j)}{a(x_i, y_j) b(x_i, y_j)} \right)$$

Kuperberg ('98): set $q = \mathrm{e}^{2i\pi/3}$ and $x_i = y_i = 1 \Rightarrow$ recover Zeilberger's formula for A_n.

NB: $A_n(x_1, \ldots, x_n; y_1, \ldots, y_n)$ is a symmetric function of the x_i, and of the y_i for generic q. However it is a fully symmetric function of **all** arguments at $q = \mathrm{e}^{2i\pi/3}$.

Symmetry classes of ASM

One can consider subsets of ASM which possess certain symmetries. In most cases, Kuperberg and others have found expressions generalizing the IK formula, leading to an exact enumeration.

Half-Turn symmetric Alternating Sign Matrices (HTSASM):

Ex: $\begin{pmatrix} 0 & 0 & + & 0 & 0 & 0 \\ + & 0 & - & 0 & + & 0 \\ 0 & 0 & + & 0 & 0 & 0 \\ 0 & 0 & 0 & + & 0 & 0 \\ 0 & + & 0 & - & 0 & + \\ 0 & 0 & 0 & + & 0 & 0 \end{pmatrix}$

$$A_{\mathrm{HT}}(L) = \begin{cases} \prod_{j=0}^{n-1} \frac{3j+2}{3j+1} \left(\frac{(3j+1))!}{(n+j)!} \right)^2 = 2, 10, 140 \dots & L = 2n \\ \prod_{j=1}^{n} \frac{4}{3} \left(\frac{(3j)!j!}{(2j)!^2} \right)^2 = 1, 3, 25 \dots & L = 2n+1 \end{cases}$$

Vertically Symmetric Alternating Sign Matrices (VSASM):

Ex: $\begin{pmatrix} 0 & 0 & 0 & + & 0 & 0 & 0 \\ 0 & 0 & + & - & + & 0 & 0 \\ + & 0 & - & + & - & 0 & + \\ 0 & 0 & + & - & + & 0 & 0 \\ 0 & + & - & + & - & + & 0 \\ 0 & 0 & + & - & + & 0 & 0 \\ 0 & 0 & 0 & + & 0 & 0 & 0 \end{pmatrix}$

$$A^V_{2n+1} = \prod_{j=0}^{n-1} (3j+2) \frac{(2j+1)!(6j+3)!}{(4j+2)!(4j+3)!} = 1, 1, 3, 26, 646 \dots$$

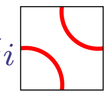
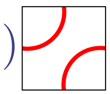
Okada has found alternate formulae (with spectral parameters) in terms of Schur functions.

Sum rule

Theorem [DF-ZJ '04]:

$$\Psi^{2n}_{min} = \frac{1}{A_n}$$

Proof. Define the **inhomogeneous transfer matrix**:

$T(z_1, \ldots, z_{2n}) = \prod_{i=1}^{2n} \Big(t_i$  $+ (1 - t_i)$  $\Big)$

with $t_i = \frac{q\, z_i - t}{q\, t - z_i}$ and $q = \mathrm{e}^{2i\pi/3}$. The equilibrium eigenvector is given by

$$T(z_1 \ldots, z_{2n}) \left|\Psi^{2n}(z_1, \ldots, z_{2n})\right\rangle = \left|\Psi^{2n}(z_1, \ldots, z_{2n})\right\rangle$$

Remark: when all z_i are equal, the model is homogeneous and $\left|\Psi^{2n}\right\rangle$ is (up to normalization) the desired vector of probabilities.

Normalize $\left|\Psi^{2n}\right\rangle$ so that its entries are **coprime** polynomials of $z_1, \ldots, z_{2n}$.

Proof of sum rule cont'd

⋆ *Polynomiality.*

The components of $\left|\Psi^{2n}(z_1,\ldots,z_{2n})\right\rangle$ are homogenous polynomials of total degree $n(n-1)$ and of partial degree at most $n-1$ in each z_i.

⋆ *Factorization and symmetry.*

$$\Psi_\pi^{2n}(z_1,\ldots,z_{2n}) = \left(\prod_{s\in E_\pi} \prod_{\substack{i,j\in s\\ i<j}} (q\,z_i - z_j) \right) \Phi_\pi(z_1,\ldots,z_{2n})$$

where Φ_π is a polynomial symmetric in the set of variables $\{z_i, i\in s\}$ for each subset s.

The sum of components is a symmetric polynomial of all z_i.

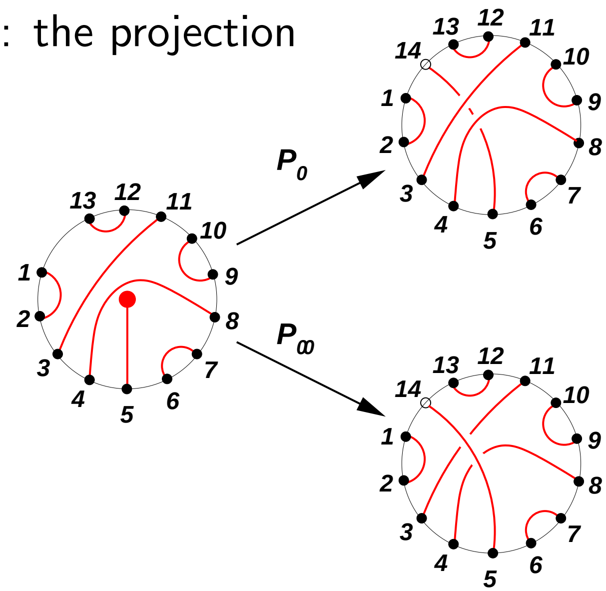
⋆ *Recursion relations.*

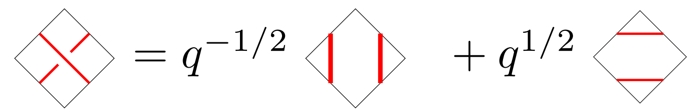
Components $\Psi_\pi^{2n}(z_1,\ldots,z_{2n})$ satisfy linear recursion relations; in particular, the sum is entirely determined by recursion relations also satisfied by the Izergin–Korepin determinant; therefore

$$\sum_\pi \Psi_\pi^{2n}(z_1,\ldots,z_{2n}) = A_n(z_1,\ldots,z_{2n})$$

Sum rule: other boundary conditions

⋆ From even size to odd size: the projection





Undercrossings and overcrossings should be understood in the skein module of the Jones polynomial; they correspond to zero / infinite spectral parameters. $\Rightarrow A_{n+1}(z_1,\ldots,z_{2n+1},0) | \sum_\pi \Psi_\pi^{2n+1}$ and $A_{n+1}(z_1,\ldots,z_{2n+1},\infty) | \sum_\pi \Psi_\pi^{2n+1}$.

Quasi-theorem [DF, ZJ, Z '06]:

$$\sum_\pi \Psi_\pi^{2n+1} = A_{n+1}(z_1,\ldots,z_{2n+1},\infty) A_{n+1}(z_1,\ldots,z_{2n+1},0) = A_{2n+1}^{HT}(z_1,\ldots,z_{2n+1})$$

As a corollary, when all z_i are equal, $\Psi_{min}^{2n+1} = \frac{1}{A_{2n+1}^{HT}}$.

Sum rule: other boundary conditions cont'd

⋆ From odd size back to even size: the embedding.

Similary, there is an (injective) mapping from even-sized link patterns with **unidentified connectivities** to odd-sized link patterns.

Quasi-theorem [DF, ZJ, Z '06]:

$$\sum_{\pi} \Psi_{\pi}^{2n,*} = A_{2n+1}^{HT}(z_1, \dots, z_{2n}, 0) = A_{2n}^{HT}(z_1, \dots, z_{2n})$$

As a corollary, $\Psi_{min}^{2n,*} = \frac{1}{A_{2n}^{HT}}$.

⋆ On a strip of width $2n$: PDF has found an expression for the sum of entries with spectral parameters.

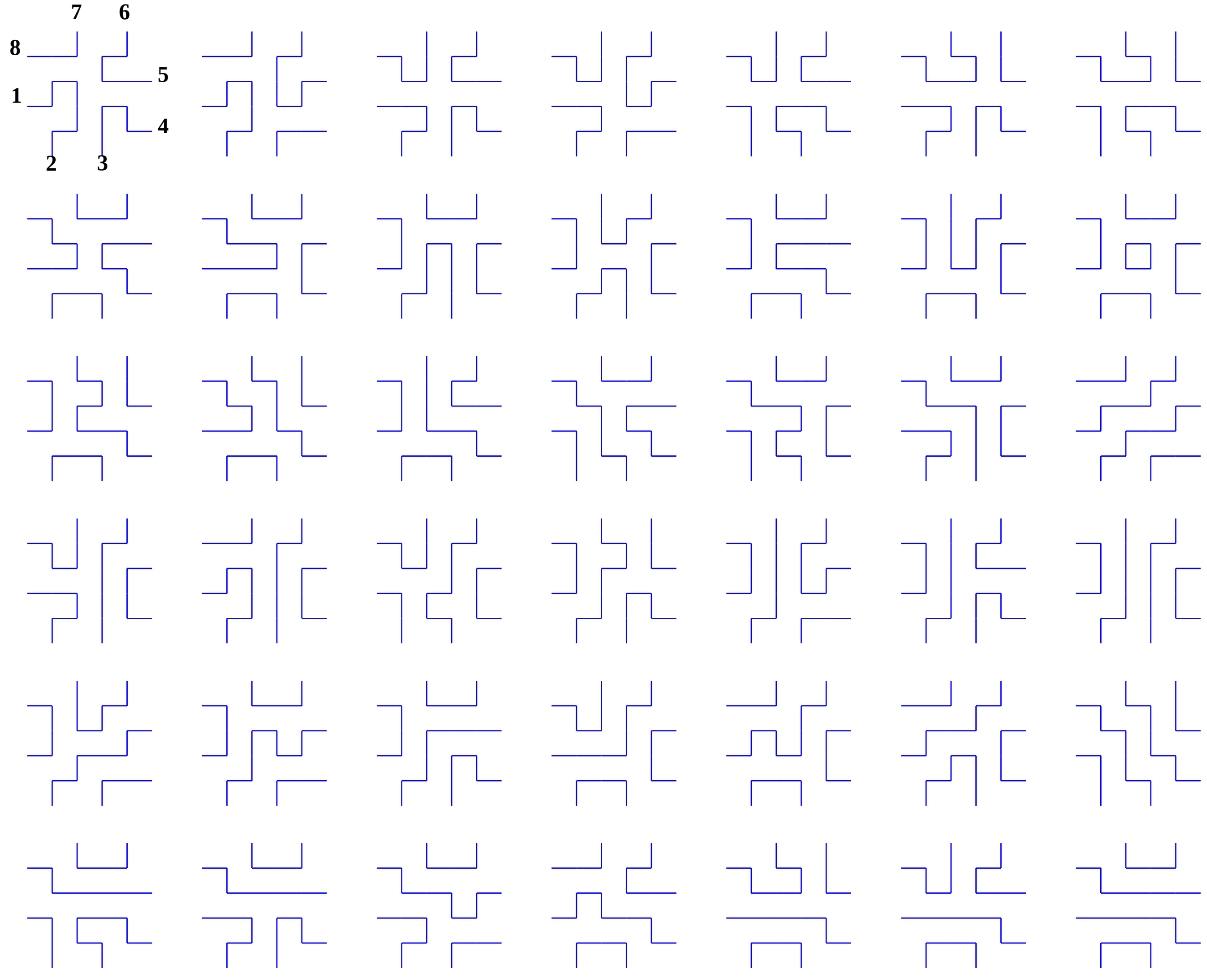
It corresponds to the partition function of UASMs, which generalize VSASMs.

$$\sum_{\pi} \Psi_{\pi}^{2n,closed} = A_{2n}^{U}(z_1, \dots, z_{2n})$$

As a corollary, $\Psi_{min}^{2n,closed} = \frac{1}{A_{2n+1}^{V}}$.

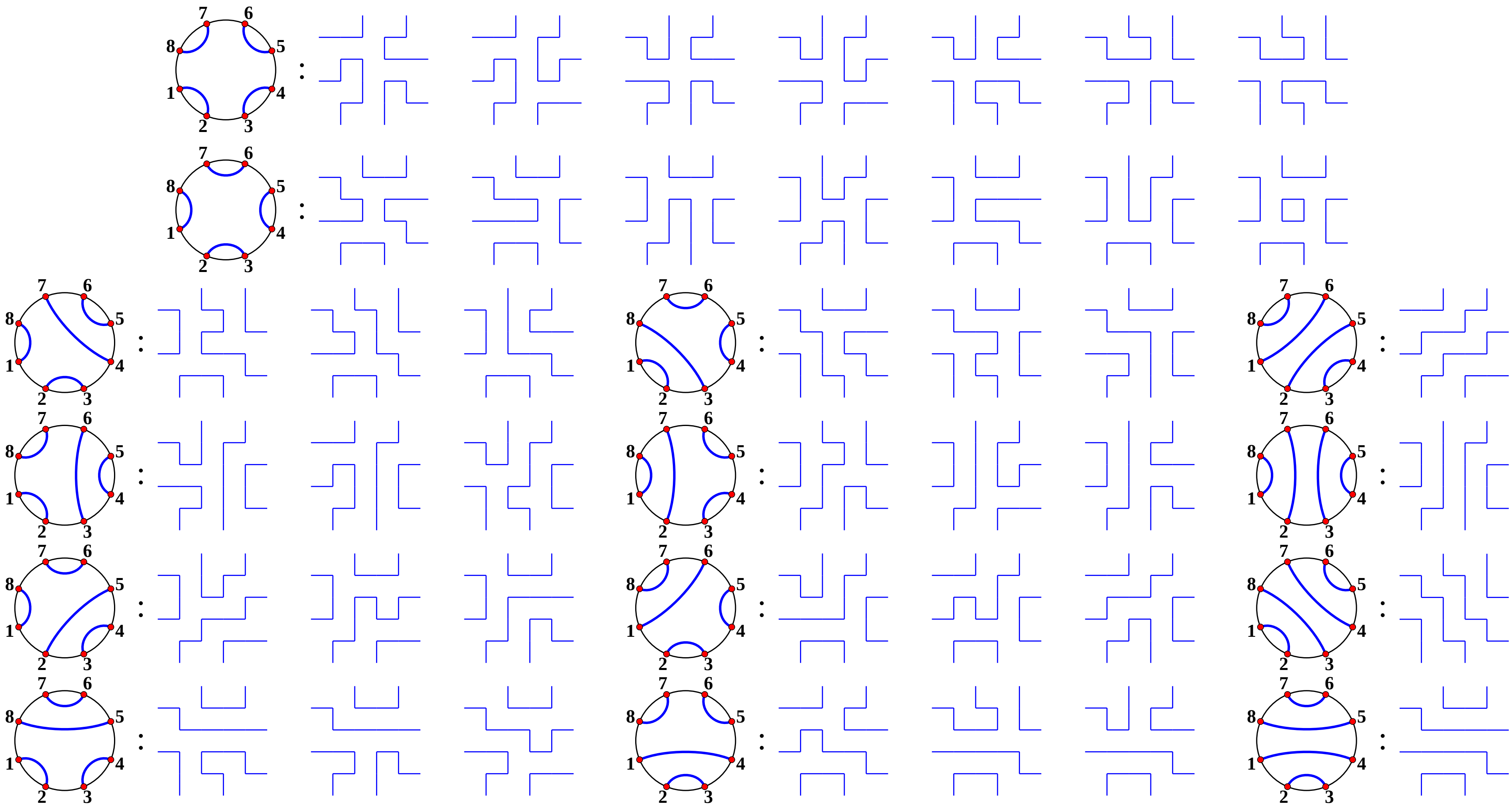
The RS conjecture: Fully Packed Loops

Example: The 42 FPL on a 4×4 grid



Fully Packed Loops cont'd

FPL configurations fall into Connectivity Classes with a given link pattern π of their external links:



Call $A_n(\pi)$ the FPL configurations with pattern π.

Razumov–Stroganov conjecture

The equilibrium state of the loop model with identified connectivities is given up to normalization by

$$\left|\Psi^{2n}\right\rangle = \sum_{\pi} A_n(\pi)\left|\pi\right\rangle$$

i.e. with proper normalization, each component is given by $\Psi^{2n}(\pi) = A_n(\pi)/A_n$ (Razumov & Stroganov '01).

⋆ There are variants for odd size, unidentified connectivities, closed boundary conditions, ...

$$\left|\Psi^{2n,*}\right\rangle = \sum_{\pi} A_{2n}^{HT}(\pi)\left|\pi\right\rangle$$

$$\left|\Psi^{2n+1}\right\rangle = \sum_{\pi} A_{2n+1}^{HT}(\pi)\left|\pi\right\rangle$$

$$\left|\Psi^{2n,closed}\right\rangle = \sum_{\pi} A_{2n+1}^{V}(\pi)\left|\pi\right\rangle$$