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The six-vertex model with domain wall boundary conditions

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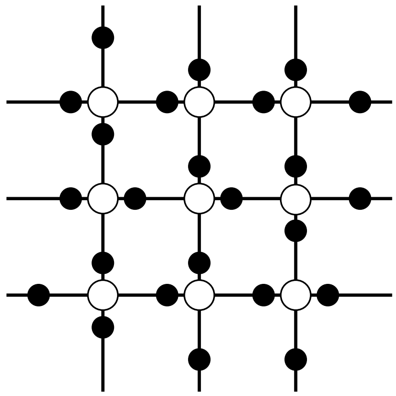
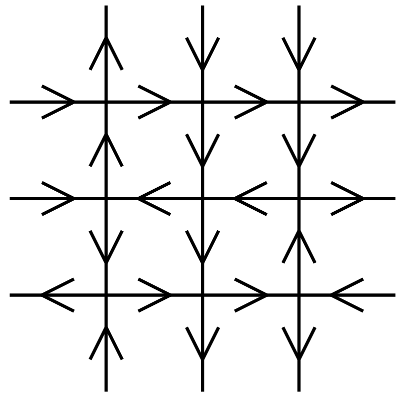
- Review of the six-vertex model.
- Determinant formula for the partition function with domain wall boundary conditions.
- Thermodynamic limit: Toda equations. Matrix model solution.
- Connection with domino tilings (dimers on square lattice).

The six-vertex model

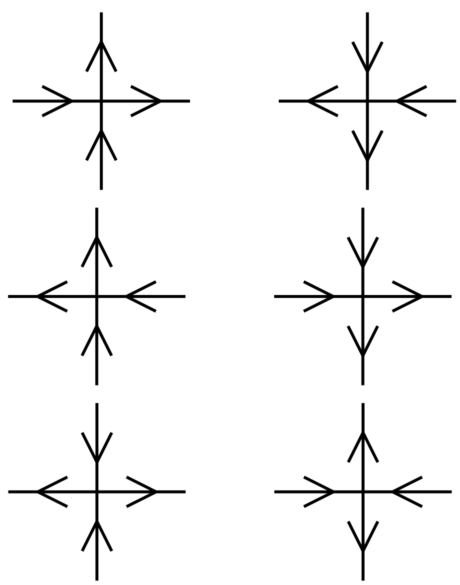




$$Z = \sum_{\text{arrow configurations}} \prod_{\text{all vertices}} \mathrm{e}^{-\varepsilon/T}$$

The Boltzmann weights are:

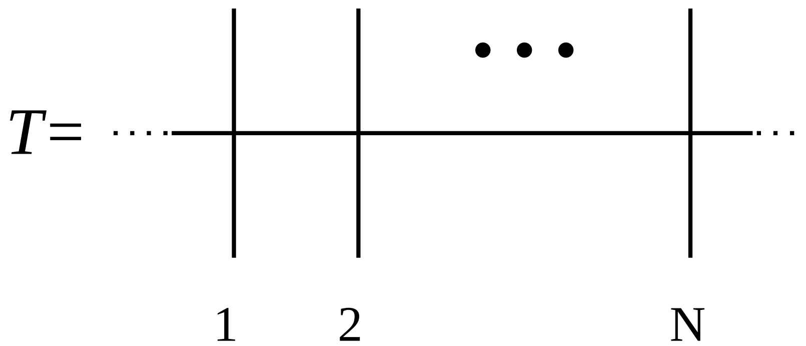
$$\mathrm{e}^{-\varepsilon/T} = \begin{cases} a \\ b \\ c \end{cases}$$



- Periodic Boundary Conditions: $(M \times N)$

Solution in the transfer matrix formalism (Lieb, Sutherland)

$$Z = \operatorname{tr} T^M$$

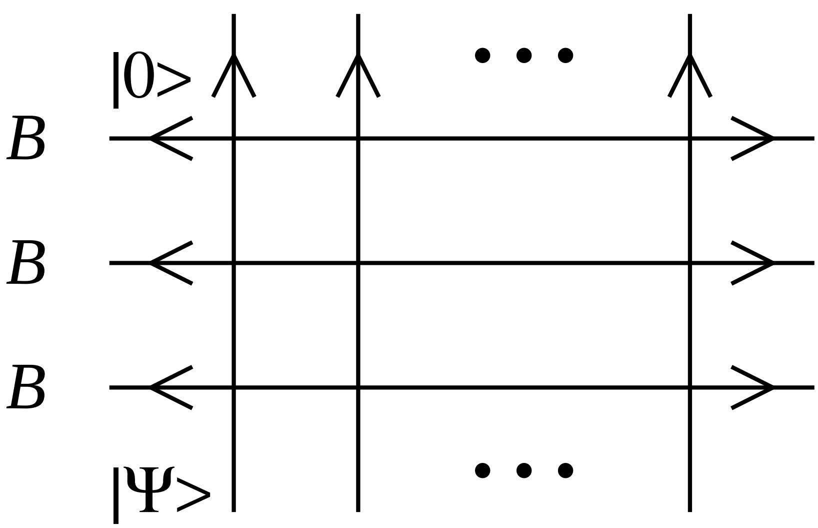


T is an operator acting on $\mathbb{C}^{2^N}$.

Thermodynamic limit $(M, N \to \infty)$: $Z \sim \lambda^M$ where λ is the largest eigenvalue of T.

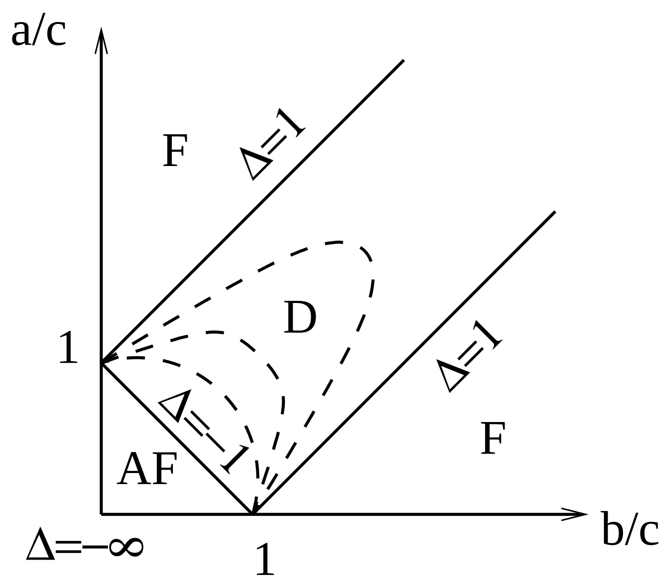
Exact diagonalization of T via **Bethe Ansatz**:

$$|\Psi\rangle = B(\lambda_1) \ldots B(\lambda_n) \, |0\rangle$$



Three phases distinguished by the value of the parameter

$$\Delta = \frac{a^2 + b^2 - c^2}{2ab}$$



- $\Delta > 1$: ferroelectric phase.

Frozen phase. $Z \sim \max(a, b)^{MN}$

- $-1 < \Delta < 1$: disordered phase.

Critical phase. Introduce the parametrization

$$a = \sin(\gamma - t) \qquad b = \sin(\gamma + t) \qquad c = \sin(2\gamma)$$

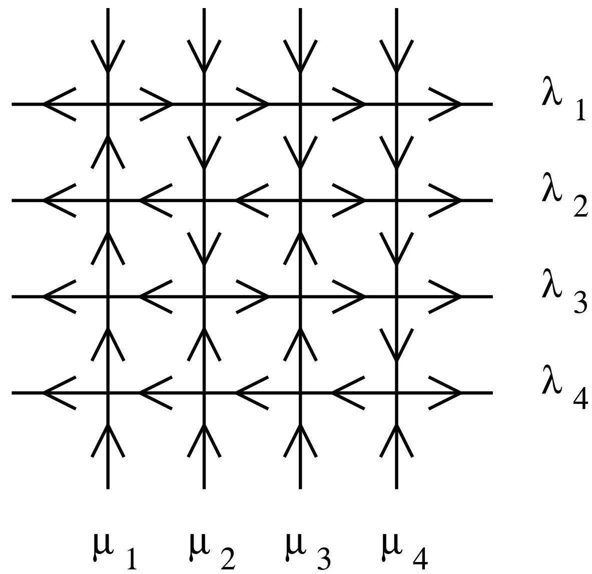
$Z \sim \mathrm{e}^{-MN\ F}$ with

$$F = -\int_{-\infty}^{+\infty} \mathrm{d}x\, \frac{\sinh 2(\gamma + t)x \sinh(\pi - 2\gamma)x}{2x \sinh \pi x \cosh 2\gamma x}$$

- $\Delta < -1$: anti-ferroelectric phase.

Non-critical phase.

• Domain Wall Boundary Conditions: ($N \times N$ square lattice)



General inhomogeneous model:

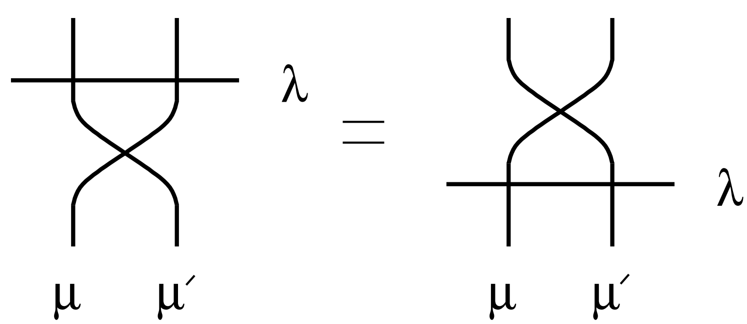
the weight of the vertex at row i, column k is

$$a = \sinh(\lambda_i - \mu_k - \gamma)$$

$$b = \sinh(\lambda_i - \mu_k + \gamma)$$

$$c = \sinh(2\gamma)$$

Yang–Baxter equation:



The partition function $Z_N(\{\lambda_i\},\{\mu_k\})$ is entirely determined by the following four properties: (Izergin, Korepin)

a) $Z_1 = \sinh(2\gamma)$.

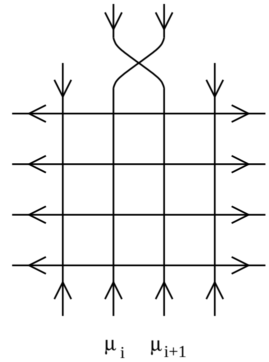
b) $Z_N(\{\lambda_i\},\{\mu_k\})$ is a symmetric function of the $\{\lambda_i\}$ and of the $\{\mu_k\}$.

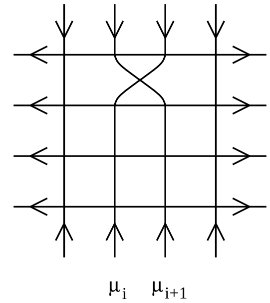
c) $Z_N(\{\lambda_i\},\{\mu_k\}) = \mathrm{e}^{-(N-1)\lambda_i} P_{N-1}(\mathrm{e}^{2\lambda_i})$ where P_{N-1} is a polynomial of degree $N-1$, and similarly for the μ_k.

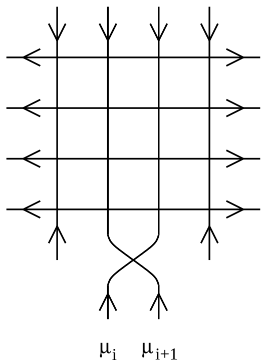
d) $Z_N(\{\lambda_i\},\{\mu_k\})$ obeys the following recursion relation:

$$Z_N(\{\lambda_i\},\{\mu_k\})_{|\lambda_j-\mu_l=\gamma} = \sinh(2\gamma) \prod_{\substack{1\le k\le N \\ k\neq l}} \sinh(\lambda_j-\mu_k+\gamma) \prod_{\substack{1\le i\le N \\ i\neq j}} \sinh(\lambda_i-\mu_l+\gamma)\; Z_{N-1}(\{\lambda_i\}_{i\neq j},\{\mu_k\}_{k\neq l})$$

Proof of b):

$$R_{\downarrow\downarrow}(\mu_i - \mu_{i+1}) Z_N(\{\ldots \mu_i, \mu_{i+1} \ldots\}) =$$ 

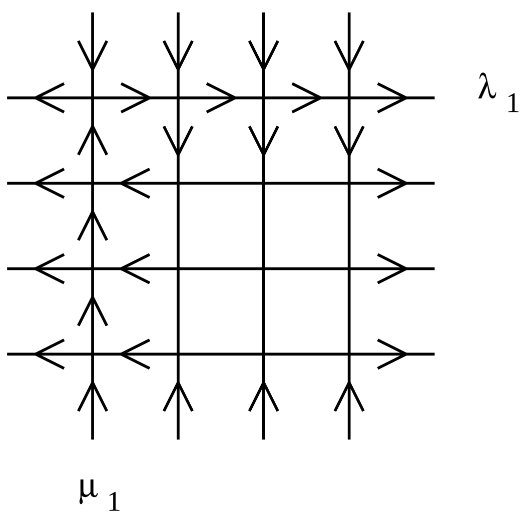
$$=$$ 

$$= \cdots =$$ 

$$= R_{\uparrow\uparrow}(\mu_i - \mu_{i+1}) Z_N(\{\ldots \mu_{i+1}, \mu_i \ldots\})$$

Proof of d):

Because of property b), we can assume that $j = l = 1$. Since $\lambda_k - \mu_l = \gamma$ implies $a(\lambda_j - \mu_l) = 0$, all configurations are of the form



$\Rightarrow$ Determinant formula for $Z_N(\{\lambda_i\},\{\mu_k\})$:

$$Z_N(\{\lambda_i\},\{\mu_k\}) = \frac{\prod_{1\le i,k\le N}\sinh(\lambda_i-\mu_k+\gamma)\sinh(\lambda_i-\mu_k-\gamma)}{\prod_{1\le i<j\le N}\sinh(\lambda_i-\lambda_j)\prod_{1\le k<l\le N}\sinh(\mu_k-\mu_l)} \det_{1\le i,k\le N}\left[\frac{\sinh(2\gamma)}{\sinh(\lambda_i-\mu_k+\gamma)\sinh(\lambda_i-\mu_k-\gamma)}\right]$$

Homogeneous limit: $\lambda_i-\mu_k\equiv t$.

$$Z_N(t) = \frac{(\sinh(t+\gamma)\sinh(t-\gamma))^{N^2}}{\left(\prod_{n=0}^{N-1} n!\right)^2}\det_{1\le i,k\le N}\left[\frac{\mathrm{d}^{i+k-2}}{\mathrm{d}t^{i+k-2}}\phi(t)\right]$$

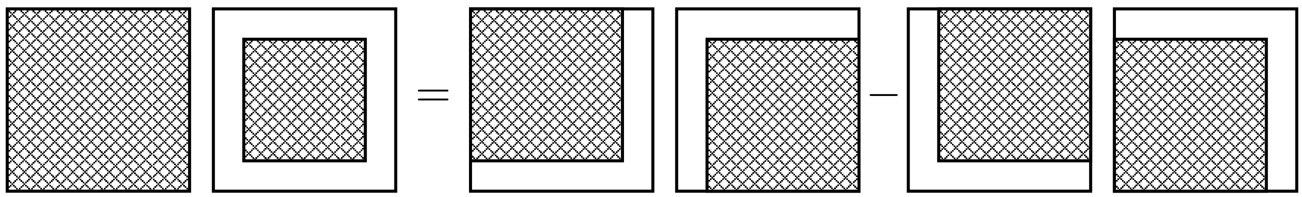
with

$$\phi(t)\equiv\frac{\sinh(2\gamma)}{\sinh(t+\gamma)\sinh(t-\gamma)}$$

- Toda equations

$$\tau_N = \det_{1\le i,k\le N}\left[\frac{\mathrm{d}^{i+k-2}}{\mathrm{d}t^{i+k-2}}\phi(t)\right]$$

Apply Jacobi's determinant identity



to τ_{N+1}:

$$\tau_N \tau_N'' - {\tau_N'}^2 = \tau_{N+1}\tau_{N-1} \qquad \forall N \ge 1$$

or

$$(\log \tau_N)'' = \frac{\tau_{N+1}\tau_{N-1}}{\tau_N^2} \qquad \forall N \ge 1$$

which is manifestly equivalent to the usual Toda equations: $\mathrm{e}^{\varphi_N} = \tau_N/\tau_{N-1}$

$$\varphi_N'' = \mathrm{e}^{\varphi_{N+1}-\varphi_N} - \mathrm{e}^{\varphi_N-\varphi_{N-1}} \qquad \forall N \ge 2$$

This is Toda semi-infinite chain.

• Thermodynamic limit = large N limit

Expected asymptotic behavior:

$$\tau_N = \left(\prod_{n=0}^{N-1} n! \right)^2 \mathrm{e}^{N^2 f(t) + \cdots}$$

Assumption: The Z_N have a smooth large N limit. Plugging this into

$$(\log \tau_N)'' = \frac{\tau_{N+1}\tau_{N-1}}{\tau_N^2} \qquad \forall N \geq 1$$

yields an ordinary differential equations for f:

$$f'' = \mathrm{e}^{2f}$$

General solution:

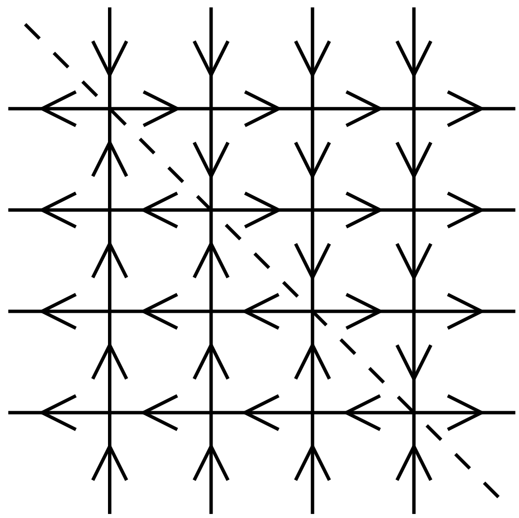
$$\mathrm{e}^{f(t)} = \frac{\alpha}{\sinh(\alpha(t - t_0))}$$

with α real or purely imaginary.

F. Ferroelectric phase:

$$a = \sinh(t - \gamma) \qquad b = \sinh(t + \gamma) \qquad c = \sinh(2\gamma)$$

with $|\gamma| < t$. Dominant configuration:



The solution of the ODE is

$$e^{f(t)} = \frac{1}{\sinh(t - |\gamma|)}$$

so that finally

$$\lim_{N \to \infty} Z_N^{1/N^2} = \sinh(t + |\gamma|) = \max(a, b)$$

which is identical to the PBC result.

D. Disordered phase:

$$a = \sin(\gamma - t) \qquad b = \sin(\gamma + t) \qquad c = \sin(2\gamma)$$

with $|t| < \gamma$, $0 < \gamma < \pi/2$. Must be an even function of t:

$$\mathrm{e}^{f(t)} = \frac{\alpha}{\cos(\alpha t)}$$

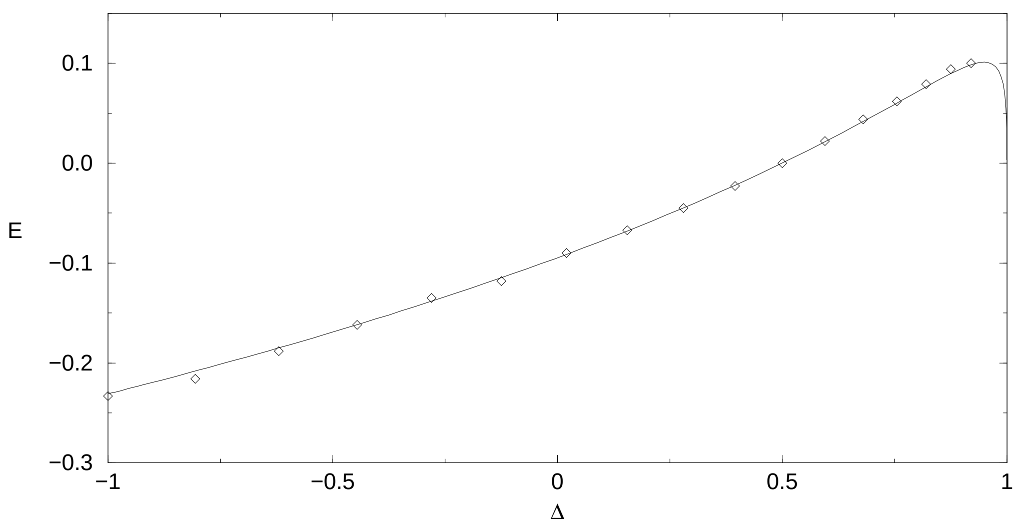
Use boundary conditions at $t = \pm\gamma$: cancellation of zeroes implies $\alpha = \frac{\pi}{2\gamma}$.

$$\lim_{N\to\infty} Z_N^{1/N^2} = \sin(\gamma - t)\sin(\gamma + t)\frac{\pi/2\gamma}{\cos(\pi t/2\gamma)}$$

Example: pure entropy ($T = \infty$). $t = 0$, $\gamma = \pi/3$.

$Z_N^{1/N^2} \to \frac{3\sqrt{3}}{4} \approx 1.30$ (Kuperberg). As opposed to PBC: $Z_N^{1/N^2} \to \frac{8}{3\sqrt{3}} \approx 1.54$ (Lieb).

Montecarlo simulations.



Pictures reminiscent of random tilings. Connection?

• Hänkel matrices, Toda hierarchy, one-matrix model

Let us write

$$\phi(t) = \int \mathrm{d}\lambda f(\lambda)\, \mathrm{e}^{t\lambda}$$

so that

$$\phi^{(k)}(t) = \int \mathrm{d}\lambda f(\lambda)\, \lambda^k \mathrm{e}^{t\lambda}$$

Then we can rewrite the determinant

$$\begin{aligned}
\tau_N &= \det_{1\le i,k\le N} \left[\frac{\mathrm{d}^{i+k-2}}{\mathrm{d}t^{i+k-2}} \phi(t) \right] \\
&= \int \mathrm{d}\lambda_1 \ldots \mathrm{d}\lambda_N \sum_{\sigma\in\mathcal{S}_N} (-1)^\sigma \prod_{i=1}^N \left[f(\lambda_i) \mathrm{e}^{t\lambda_i} \lambda_i^{\sigma(i)+i-2} \right] \\
&= \frac{1}{N!} \int \mathrm{d}\lambda_1 \ldots \mathrm{d}\lambda_N \Delta(\lambda_i)^2 \prod_{i=1}^N \left[f(\lambda_i) \mathrm{e}^{t\lambda_i} \right]
\end{aligned}$$

where $\Delta(\lambda_i) = \prod_{i<j}(\lambda_i - \lambda_j)$. This is the expression in terms of eigenvalues λ_i of the one-matrix integral

$$\tau_N = \int \mathrm{d}M\, \mathrm{e}^{\mathrm{tr}\, V(M)}$$

with $V(x) = tx + \log f(x)$. More generally, if

$$\phi(t_1, \ldots, t_q \ldots) = \int \mathrm{d}\lambda f(\lambda)\, \mathrm{e}^{\sum_{q\ge 1} t_q \lambda^q}$$

then the τ_N are tau-functions of the whole Toda chain hierarchy with respect to the times t_q. (cf Adler, van Moerbeke)

• Solution via Matrix Models:

D. Disordered phase: $(|t| < \gamma < \pi/2)$

$$\phi(t) = \frac{\sin(2\gamma)}{\sin(\gamma+t)\sin(\gamma-t)} = \int_{-\infty}^{+\infty} \mathrm{d}\lambda\, \mathrm{e}^{t\lambda} \frac{\sinh \frac{\lambda}{2}(\pi - 2\gamma)}{\sinh \frac{\lambda}{2}\pi}$$

Key insight from matrix models: $\lambda \sim N$.

Rescaling: $\mu = \gamma\lambda/N$. Then

$$\tau_N \sim c_N\, \gamma^{-N^2} \int_{-\infty}^{+\infty} \mathrm{d}\mu_1 \ldots \mathrm{d}\mu_N \Delta(\mu_i)^2 \prod_{i=1}^{N} \left[\frac{\sinh N\mu_i(\frac{\pi}{2\gamma} - 1)}{\sinh N\mu_i \frac{\pi}{2\gamma}} \mathrm{e}^{N\frac{t}{\gamma}\mu_i} \right]$$

One can simplify: $\frac{\sinh N\mu(\frac{\pi}{2\gamma}-1)}{\sinh N\mu\frac{\pi}{2\gamma}} \sim \mathrm{e}^{-N|\mu|}$.

$$\tau_N \sim c_N\, \gamma^{-N^2} \int_{-\infty}^{+\infty} \mathrm{d}\mu_1 \ldots \mathrm{d}\mu_N \Delta(\mu_i)^2 \mathrm{e}^{N\sum_i(\frac{t}{\gamma}\mu_i - |\mu_i|)}$$

$\Rightarrow \mathrm{e}^{f(t)} = \frac{1}{\gamma}\Phi\left(\frac{t}{\gamma}\right)$. To compute Φ one must actually solve the matrix model.

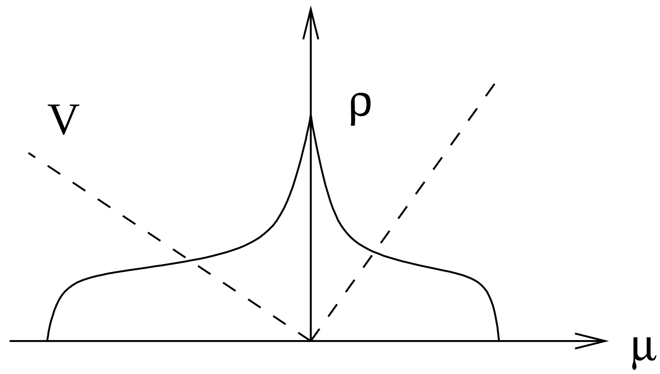
AF. Anti-ferroelectric phase: $(|t| < \gamma)$

$$\phi(t) = \frac{\sinh(2\gamma)}{\sinh(\gamma+t)\sinh(\gamma-t)} = \sum_{l=-\infty}^{\infty} \mathrm{e}^{2tl}\mathrm{e}^{-2\gamma|l|}$$

Naïvely, after rescaling: $\mu = 2\gamma l/N$, same as **D** $\Rightarrow$ no phase transition between **D** and **AF**!!

• Resolution of the paradox:

Density of eigenvalues $\rho(\mu)$ in the large N limit in phase **D**:



In phase **AF**, $|l_i - l_j| \geq 1 \Rightarrow$ the density must satisfy

$$\rho(\mu) \leq \frac{1}{2\gamma}$$

$\Rightarrow$ Saturation of eigenvalues in the valley.

Solution given in terms of elliptic integrals...

• Phase transition **D** / **AF**:

Estimate of free energy singularity near $\gamma = 0$:

$$\rho(\mu) \sim \frac{1}{\pi^2} \log|\mu|$$

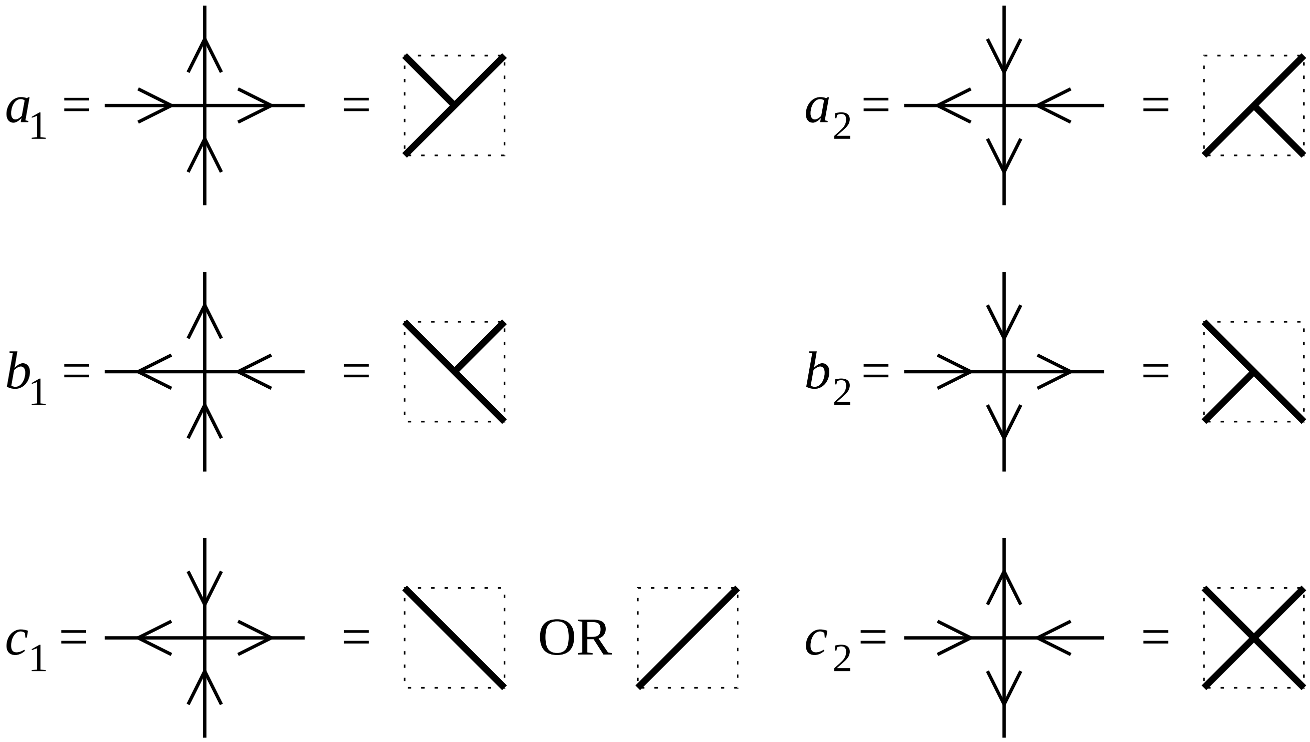
$\Rightarrow$ the saturated region has a width $\Delta\mu \sim \exp(-\pi^2/2\gamma)$.

More explicitly ($\gamma \propto \sqrt{T_c - T}$)

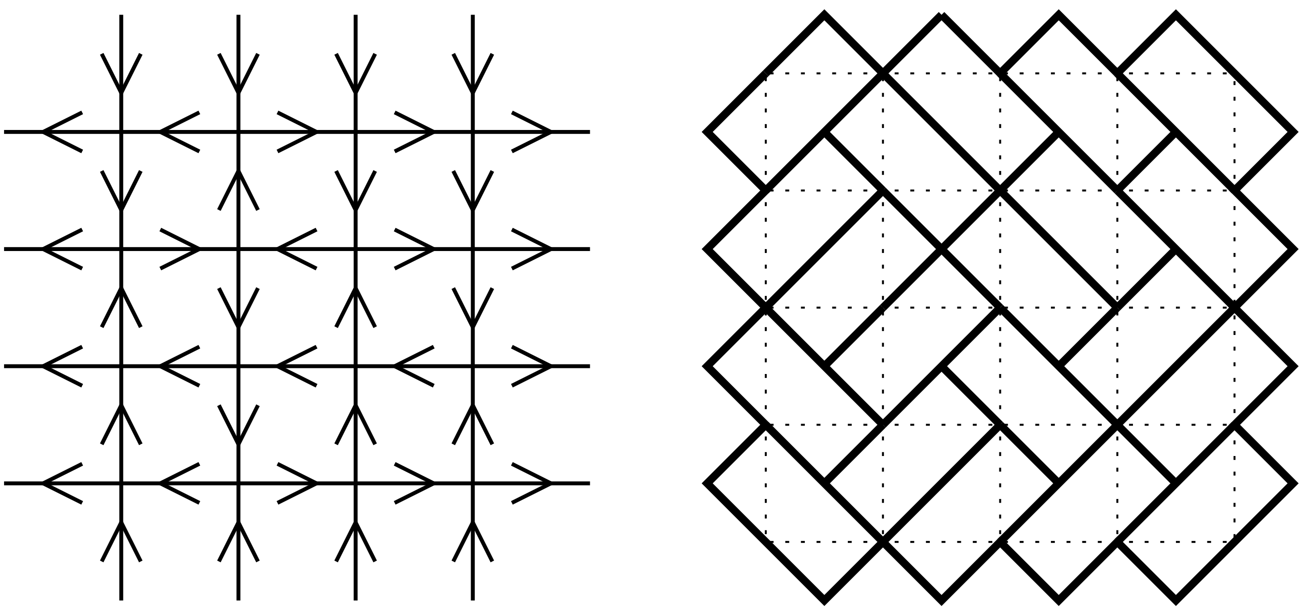
$$F_{\text{sing}} \propto \mathrm{e}^{-C/\sqrt{T_c - T}}$$

Same leading singularity as in PBC.

• Connection with domino tilings



Example with DWBC:



Domain Wall Boundary Conditions ⟷ Aztec Diamond shape.

Each vertex of type c_1 gives rise to **two** local domino configurations.

With Domain Wall Boundary Conditions, $\#c_1 = \#c_2 + N$ and therefore $\#c_1 = \frac{1}{2}\#c + \frac{N}{2}$.

$$
\begin{aligned}
\#\text{Domino tilings of order } N &= \sum_{\text{6v configurations}} 2^{\#c_1} \\
&= 2^{N/2} Z_N(a = b = 1, c = \sqrt{2})
\end{aligned}
$$

The counting of domino tilings (of the Aztec Diamond) is equivalent to the six-vertex model at $a = b = 1$, $c = \sqrt{2}$ (with DWBC).

Remark: using the even sub-lattice instead of the odd sub-lattice, one finds

$$\#\text{Domino tilings of order } N = 2^{-(N+1)/2} Z_{N+1}(a = b = 1, c = \sqrt{2})$$

and therefore (Elkies, Kuperberg, Larsen, Propp)

$$Z_N(a = b = 1, c = \sqrt{2}) = 2^{N^2/2}$$

- (Jockush, Propp, Shor) Fix $\epsilon > 0$. Then for all sufficiently large N, all but an ϵ fraction of the domino tilings of the Aztec diamond of order N will have a temperate zone whose boundary stays uniformly within distance ϵN of the inscribed circle.

- (Cohn, Elkies, Propp) Computation of a certain one point function $\mathcal{P}(x, y)$ in the thermodynamic limit:

$$\mathcal{P}(x, y) = \begin{cases} 0 & x^2 + y^2 \geq 1/2 \text{ and } y < 1/2 \\ 1 & x^2 + y^2 \geq 1/2 \text{ and } y > 1/2 \\ \frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{2y-1}{\sqrt{1-2(x^2+y^2)}}\right) & x^2 + y^2 < 1/2 \end{cases}$$

Prospects:

- Local free energy (1-point function)? Determinant formula? Comparison with translationally invariant case?
- Subdominant corrections? Either via connection with matrix models, or using more traditional methods from classical integrable differential equations.
- Applications to combinatorics (Alternating Sign Matrices, Self-Complementary Totally Symmetric Plane Partitions ...)