Quiver varieties and quantum Knizhnik–Zamolodchikov equation

P. Zinn-Justin

Laboratoire de Physique Théorique des Hautes Energies

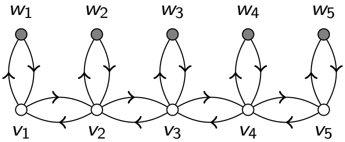
UPMC Université Paris 6 and CNRS

Introduction

- Ten years ago, P. Di Francesco, A. Knutson and myself investigated a mysterious new connection: some quantum integrable systems effectively performed computations in algebraic geometry (equivariant cohomology).
- My interest has been revived by the recent work of Maulik and Okounkov on quantum cohomology and quantum groups, which formalizes the appearance of quantum integrable systems in the context of geometric representation theory (in a fairly general setting), and also connects to a number of hot topics, including $\mathcal{N} = 1$ SUSY gauge theories and the AGT conjecture.
- The goal of this talk (and upcoming paper) is to reanalyze and generalize this connection in view of recent developments.

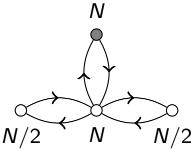
Quiver varieties

Start with an A_{k-1} Nakajima quiver, i.e., a comb-like quiver with arrows doubled:



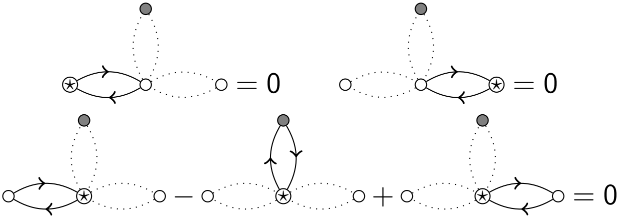
The numbers are the dimensions of the vector spaces at each vertex: $\dim V_i = v_i$, $\dim W_i = w_i$. The arrows are linear maps.
Also set $N = \sum_{i=1}^{k-1} w_i$, $M = \sum_{i=1}^{k-1} i\, w_i$.

A simpler example:



Moment map conditions

We impose that at each white ("unframed") vertex, the (signed) sum of length two paths is zero:



where paths start and end at the marked white vertex.

Quotients

We want to take the quotient by $G_v := \prod_{i=1}^{k-1} GL(V_i)$.
The naive quotient is

$$\mathfrak{M}_0 = \{\text{arrows subject to } m.m. = 0\}//G_v = \{\text{closed } G_v - \text{orbits}\}$$

We shall describe it more explicitly in what follows.

There is a "better" quotient (GIT quotient):

$$\mathfrak{M} = \{\text{arrows subject to } m.m. = 0\}//_\chi G_v = \{\text{stable } G_v - \text{orbits}\}$$

with a map $p : \mathfrak{M} \to \mathfrak{M}_0$ which is a (symplectic, semi-small) resolution of singularities (of its image).

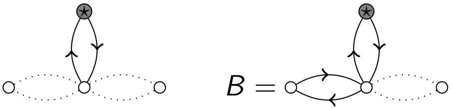
Description of $\mathfrak{M}_0$

$\mathfrak{M}_0$ is a (singular) affine variety, which can be described in terms of the natural G_v-invariants [Lusztig]:

- Paths from one gray ("framed") vertex to another (possibly itself).
- Traces of closed cycles.

Here, only paths occur:

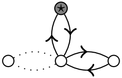
$A =$ $B =$

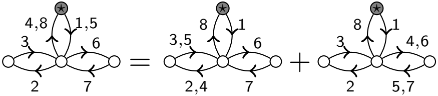


Description of $\mathfrak{M}_0$ cont'd

One has $\mathfrak{M}_0(v) \subset \mathfrak{M}_0(v')$ when $v \leq v'$ (and it stabilizes eventually), so in what follows we mean $\mathfrak{M}_0 := \mathfrak{M}_0(v_\infty)$.

Then the only relations satisfied by the invariants come from the moment map conditions.

For example here $A^2 - B =$  and then

$B(A^2 - B) =$ 

$= 0$

So we have an explicit description

$$\mathfrak{M}_0 = \{A, B \ N \times N \text{ matrices} : B(A^2 - B) = (A^2 - B)B = A^3 - (AB + BA) = 0\}$$

Combinatorial data

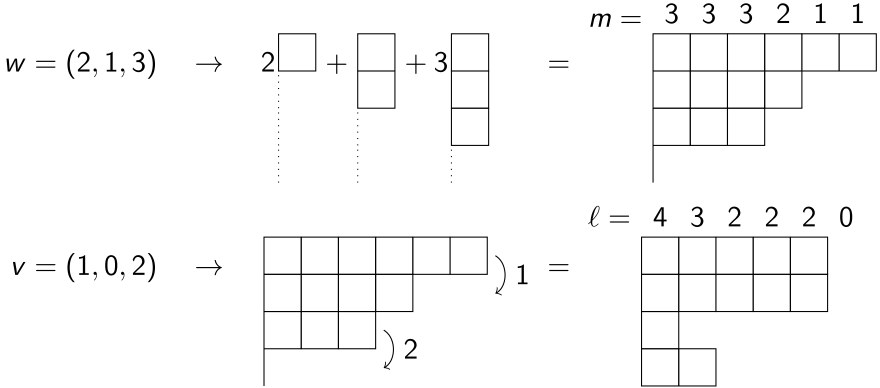
There is a second description of $\mathfrak{M}_0$ (or for general v, $p(\mathfrak{M})$) in terms of a transverse slice of a nilpotent orbit closure. [Mirković, Vybornov '09]

Define two partitions (with $N = \sum_{i=1}^{k-1} w_i$ parts) out of w and v:

- Define the $GL(k)$ weights $\mu = \sum_{i=1}^{k-1} w_i \omega_i$ and $\lambda = \sum_{i=1}^{k-1} w_i \omega_i - \sum_{i=1}^{k-1} v_i \alpha_i$. (lifted from $SL(k)$)
- Draw them as box diagrams; they both have $M = \sum_{i=1}^{k-1} i\, w_i$ boxes.
- $m = (m_1, \ldots, m_N)$ and $\ell = (\ell_1, \ldots, \ell_N)$ are the numbers of boxes in each column of μ and λ.
  (which implies $\sum_{i=1}^{N} m_i = \sum_{i=1}^{N} \ell_i = M$)

NB: if one relaxes the condition that m be ordered, making it a composition rather than a partition, then m is characterized by $m \in \{1, \ldots, k-1\}^N$ such that $\#\{a : m_a = i\} = w_i$.

Combinatorial data: example



Mirković–Vybornov slice

Then $\mathfrak{M}_0 \cong \overline{\mathcal{O}_\lambda} \cap \mathcal{T}_\mu$, where

- $\mathcal{O}_\lambda$ is the orbit of nilpotent operators with Jordan type λ, (i.e., sizes of Jordan blocks $= \ell_i$)
- $\mathcal{T}_\mu = x + \mathcal{T}'_\mu$,
  - x is a nilpotent operator with Jordan type μ, which we shall always choose to be in Jordan form.
  - $\mathcal{T}'_\mu$ is a certain linear subspace (each block of $\mathcal{T}_\mu$ has "companion form", i.e., $\mathcal{T}'_\mu$ consists of the last row of each block, possibly truncated to its leftmost square sub-block if its width is larger than its height).

Mirković–Vybornov slice: example

$m = (\underbrace{2, \ldots, 2}_{N})$, so the slice in $\mathfrak{gl}(M = 2N)$ is

$$\mathcal{T}_\mu = \left\{ \begin{array}{|cc|cc|c|} \hline 0 & 1 & 0 & 0 & \cdots \\ \star & \star & \star & \star & \cdots \\ \hline 0 & 0 & 0 & 1 & \cdots \\ \star & \star & \star & \star & \cdots \\ \hline \vdots & \vdots & \vdots & \vdots & \ddots \\ \hline \end{array} \right\} \cong \left\{ \begin{array}{c} \begin{matrix} N & & N \end{matrix} \\ \begin{pmatrix} 0 & 1 \\ -B & -A \end{pmatrix} \begin{matrix} N \\ N \end{matrix} \end{array} \right\}$$

Assuming $v = v_\infty$ and N even, $\ell = (\underbrace{4, \ldots, 4}_{N/2})$ so $\overline{\mathcal{O}_\lambda} = \{X^4 = 0\}$.

Plugging $X^4 = 0$ into the slice above leads to the same equations as before.

Borel/Cartan subgroups

Suppose the direct sum $W := \bigoplus_{i=1}^{k-1} W_i$ of spaces at framed vertices has a basis $(e_1, \ldots, e_N)$, such that $e_a \in W_{m_a}$ for some $m \in \{1, \ldots, k-1\}^N$ with $\#\{a : m_a = i\} = w_i$.

To each such (ordered) basis is associated a Borel subgroup, with its unipotent subgroup, Cartan torus and their Lie algebras:

$$\mathfrak{n} = \mathfrak{gl}(W) \cap \{\text{strict upper triangular matrices}\}$$
$$T = GL(W) \cap \{\text{diagonal matrices}\}$$

Tensor product variety: definition

We define

$$\mathfrak{Z}_0 = \mathfrak{M}_0 \cap \{\text{paths} \in \mathfrak{n}\}$$

as well as $\mathfrak{Z} = p^{-1}(\mathfrak{Z}_0)$. The irreducible components of $\mathfrak{Z}$ are Lagrangian.

Equivalently, one can embed $\mathfrak{n}$ in $\mathfrak{gl}(M)$ by considering matrices that are proportional to the identity in each Jordan block, and $\mathfrak{M}_0$ in $\mathfrak{gl}(M)$ using the Mirković–Vybornov slice. Then

$$\mathfrak{Z}_0 = \mathfrak{M}_0 \cap \mathfrak{n}$$

In our example,

$$\mathfrak{Z}_0 = \{A, B\ N \times N \text{ strict upper triangular matrices} : \\ B(A^2 - B) = (A^2 - B)B = A^3 - (AB + BA) = 0\}$$

Tensor product variety: RT interpretation

Also associated to $(m_1, \ldots, m_N)$ is a decomposition of the $SL(k)$ weight $\sum_{i=1}^{k-1} w_i \omega_i = \omega_{m_1} + \cdots + \omega_{m_N}$, and the invariant subspace of the tensor product of corresponding irreps:

$$\left(L_{\omega_{m_1}} \otimes \cdots \otimes L_{\omega_{m_N}}\right)^{SL(k)}$$

Assume now $k|M$, so that this space is nonzero. Then it is isomorphic to $H_{top}(\mathfrak{Z}_0)$, the top-dimensional Borel–Moore homology of $\mathfrak{Z}_0$ (which has a natural basis of irreducible components of $\mathfrak{Z}_0$).

Remarks:

- This space is also encoded in the intersection homology of $\mathfrak{M}_0$.
- More general tensor product varieties can be considered (Borel $\to$ parabolic), which correspond to tensor products of non-fundamental irreps.

Torus action

Each path is an element of $\mathfrak{n}$. T acts (linearly) on $\mathfrak{n}$ by conjugation; an extra $\mathbb{C}^\times$ acts by scaling paths according to the rule: $X \mapsto t^{|X|}X$, where $|X|$ is the length of the path X.

Equivalently, $T \times \mathbb{C}^\times$ acts (linearly) on $\mathcal{T}'_\mu \cap \mathfrak{n}$ by conjugation by diagonal matrices (with repeats for each block) and scaling the entry on last row of each block with $t^{\text{perimeter}-2(\text{column}-1)}$.

$T \times \mathbb{C}^\times$ leaves $\mathfrak{Z}_0$, and therefore its irreducible components $\mathfrak{Z}_{0,\alpha}$ stable.

Considering $\mathfrak{Z}_0 = \mathfrak{M}_0 \cap \mathfrak{n} \overset{j}{\hookrightarrow} \mathcal{T}'_\mu \cap \mathfrak{n}$, we can define their $(T \times \mathbb{C}^\times)$-equivariant (co)homology class:

$$\Psi_\alpha := j_*[\mathfrak{Z}_{0,\alpha}] \in H^*_{T\times\mathbb{C}^\times}(\mathcal{T}'_\mu \cap \mathfrak{n}) \cong \mathbb{Z}[\hbar/2, z_1, \ldots, z_N]$$

where $\hbar/2 \sim$ scaling, $z_i \sim e_i$ direction.

Remark: such polynomials are called multidegrees, because they generalize the notion of degree of projective varieties. (up to normalization, they are equivariant volumes)

Example: $N = 4$

With the same example as before in size 4:

$$\mathfrak{Z}_0 = \{A, B \ 4 \times 4 \text{ strict upper triangular matrices} : \\ B(A^2 - B) = (A^2 - B)B = A^3 - (AB + BA) = 0\}$$

$\mathfrak{Z}_0$ has 3 irreducible components, and their multidegrees are:

$$\Psi_{\begin{smallmatrix}1&2\\1&2\\3&4\\3&4\end{smallmatrix}} = (\hbar + z_3 - z_4)(\hbar + z_1 - z_2)(2\hbar + z_3 - z_4)(2\hbar + z_1 - z_2)$$

$$\Psi_{\begin{smallmatrix}1&2\\1&3\\2&4\\3&4\end{smallmatrix}} = (2\hbar + z_3 - z_4)(2\hbar + z_2 - z_3)(2\hbar + z_1 - z_2)(3\hbar + z_1 - z_4)$$

$$\Psi_{\begin{smallmatrix}1&3\\1&3\\2&4\\2&4\end{smallmatrix}} = (\hbar + z_2 - z_3)(2\hbar + z_2 - z_3)(3\hbar + z_1 - z_4)(4\hbar + z_1 - z_4)$$

Howe duality

We shall think of $(L_{\omega_{m_1}} \otimes \cdots \otimes L_{\omega_{m_N}})^{SL(k)}$ as a weight space of a $GL(N)$ irrep. Start with

$$(\bigwedge(\mathbb{C}^k))^{\otimes N} = \bigwedge(\mathbb{C}^k \otimes \mathbb{C}^N) = \bigoplus_\lambda L_\lambda^{GL(k)} \otimes L_{\lambda'}^{GL(N)}$$

where the sum is over partitions λ with at most k rows and N columns.

Inside $GL(N)$ sits the maximal torus $(\mathbb{C}^\times)^N$ of diagonal matrices, which allows to distinguish $\bigwedge(\mathbb{C}^k) = \bigoplus_{m\geq 0} L_{\omega_m}^{GL(k)}$. This way, we find

$$L_{\omega_{m_1}}^{GL(k)} \otimes \cdots \otimes L_{\omega_{m_N}}^{GL(k)} = \bigoplus_\lambda L_\lambda^{GL(k)} \otimes L_{\lambda'}^{GL(N)}|_{\text{weight space }(m_1,\ldots,m_N)}$$

In particular, the multiplicity of $L_\lambda^{GL(k)}$ is the number of SSYT of shape λ' with m_i times the letter i.

Geometric Satake correspondence

The geometric Satake correspondence relates the representation theory of G to geometric data of $G^\vee$.

One builds a certain finite-dimensional subvariety of the affine Grassmannian $\hat{Gr}(G^\vee)$ based of $G^\vee$ whose intersection homology forms an irrep of G.

Here $G \cong G^\vee = GL(N)$. In type A, a slice of this subvariety is isomorphic to $\mathfrak{M}_0$ [Mirković, Vybornov '09] (and components of $\mathfrak{Z}_0$ are slices of so-called Mirković–Vilonen cycles). By taking appropriate representatives in $\hat{Gr}(G^\vee) = G^\vee(\mathbb{C}((t)))/G^\vee(\mathbb{C}[[t]])$ we shall show this isomorphism explicitly.

Construction

We assume here, as in the example, that $m_i = m$, so that the weight space $(m, \ldots, m)$ is central.

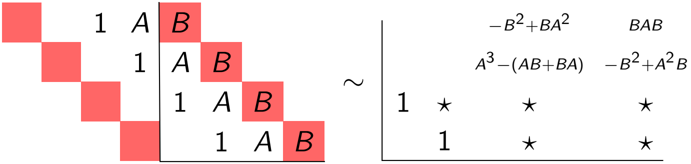
- Consider infinite (N, N)-periodic matrices of the form

$$U = \left\{\begin{matrix} 1 & A & B & \cdots & & & \\ & 1 & A & B & \cdots & & \\ & & 1 & A & B & \cdots & \end{matrix}\right\}$$

  where 1's are at $column - row = m$.
- Intersect U with rank conditions on $N \times N$ blocks obtained as follows: put the number of ℓ's (parts of λ') equal to i $(= \lambda_i - \lambda_{i+1}$, with $\lambda_0 \equiv N)$ on every block at $column - row = i$. Then every north-west block-submatrix of $M \in U$ must have rank at most the sum of numbers in the corresponding region.

Construction: example

Set $k = 4$, $m = 2$, $\lambda = (N/2, N/2, N/2, N/2)$,
$\ell = \lambda' = (\underbrace{4, \ldots, 4}_{N/2}, \underbrace{0, \ldots, 0}_{N/2})$. Then the numbers are



where each red box means the number $N/2$.

In particular the north-west submatrix shown has rank at most $2N$.

After row manipulations, all remaining entries must be zero.

The meta-theorem

The main point of this talk is to make sense of the following

Theorem

The (Ψ_α) form a solution of the (level 1) rational quantum Knizhnik–Zamolodchikov equation.

In the case of $w_i = N\delta_{i,1}$ ("orbital varieties" and "extended Joseph polynomials"), this was observed in [Di Francesco, ZJ, '05] and proven in [Rimányi, Tarasov, Varchenko, ZJ, '12].

Quantum Knizhnik–Zamolodchikov equation

The quantum Knizhnik–Zamolodchikov equation is a system of holonomic first order (q-)difference equations equations that appears:

- in the study of form factors of integrable models [Smirnov, '86]
- in the representation theory of quantum affine algebras [Frenkel, Reshetikhin '92] and Yangians
- in the study of correlation functions of integrable models [Jimbo, Miwa et al, '93]
- in relation to the representation theory of affine Hecke algebra and DAHA [Cherednik, Pasquier, '90s]
- in the exact finite size computation of the ground state of integrable models at roots of unity [Di Francesco, ZJ, '05] $\rightarrow$ applications to combinatorics

Definition of Ψ

Define $\Psi = j_*$ viewed as an element of $\mathcal{H} \otimes \mathbb{Z}[\hbar/2, z_1, \ldots, z_N]$ where

$$\mathcal{H} := \left(L^*_{\omega_{m_1}} \otimes \cdots \otimes L^*_{\omega_{m_N}}\right)^{SL(k)}$$

More explicitly, if u^α is the basis dual to the $[\mathfrak{Z}_{0,\alpha}]$, then

$$\Psi = \sum_\alpha \Psi_\alpha \, u^\alpha$$

In order to emphasize that $\mathfrak{Z}_0$ depends on a choice of basis $(e_1, \ldots, e_N)$, and Ψ depends on a choice of $(m_1, \ldots, m_N)$, we shall also write

$$\mathfrak{Z}_0 = \mathfrak{Z}_0^{e_1,\ldots,e_N}, \quad \Psi = \Psi^{m_1,\ldots,m_N}, \quad \mathcal{H} = \mathcal{H}^{m_1,\ldots,m_N}$$

Exchange relation

The exchange relation is the "nonaffine" part of the qKZ equation.

Theorem

There exist $\check{R}_i(z) \in \mathcal{L}(\mathcal{H}^{m_1,\ldots,m_i,m_{i+1},\ldots,m_N}, \mathcal{H}^{m_1,\ldots,m_{i+1},m_i,\ldots,m_N}) \otimes \mathbb{C}(z,\hbar)$*,* $i = 1,\ldots,N-1$*, which satisfy*

$$\begin{aligned} \check{R}_i(u)\check{R}_{i+1}(u+v)\check{R}_i(v) &= \check{R}_{i+1}(v)\check{R}_i(u+v)\check{R}_{i+1}(u) & i &= 1,\ldots,N-2 \\ \check{R}_i(u)\check{R}_i(-u) &= 1 & i &= 1,\ldots,N-1 \\ \check{R}_i(u)\check{R}_j(v) &= \check{R}_i(v)\check{R}_j(u) & |i-j| &> 1 \end{aligned}$$

such that

$$\Psi^{m_1,\ldots,m_{i+1},m_i,\ldots,m_N}(z_1,\ldots,z_{i+1},z_i,\ldots,z_N) = \check{R}_i(z_i - z_{i+1})\Psi^{m_1,\ldots,m_i,m_{i+1},\ldots,m_N}(z_1,\ldots,z_i,z_{i+1},\ldots,z_N)$$

The R-matrix of [Maulik, Okounkov]

In [MO], the R-matrix is defined as follows:

1. Start with the irreducible components $\mathfrak{Z}_\alpha$ for two different orderings of the same basis (e_i): $\mathfrak{Z}_\alpha^{\cdots e_i, e_{i+1} \cdots}$ and $\mathfrak{Z}_\alpha^{\cdots e_{i+1}, e_i \cdots}$. (note: same torus T)
2. Consider their T-equivariant cohomology classes $[\mathfrak{Z}_\alpha]$ in $\mathfrak{M}$.
3. Make a (triangular) change of basis to the stable sets:

$$\widetilde{[\mathfrak{Z}_\alpha]} = \sum_\beta c_\alpha^\beta [\mathfrak{Z}_\beta]$$

4. Then $\tilde{R}_i$ is the "matrix of change of basis" (as $H^*_{T\times\mathbb{C}^\times}(\cdot)$-module)

$$[\mathfrak{Z}_\alpha^{\widetilde{\cdots e_{i+1}, e_i \cdots}}] = \sum_\beta (\tilde{R}_i)_\alpha^\beta [\mathfrak{Z}_\beta^{\widetilde{\cdots e_i, e_{i+1} \cdots}}]$$

Step 3 ensures "locality", but we don't need it (except for the consequence that $\tilde{R}_i$ depends on $\hbar$ and $z := z_i - z_{i+1}$ only); so we redefine

$$[\mathfrak{Z}_\alpha^{\cdots e_{i+1}, e_i \cdots}] = \sum_\beta (R_i)_\alpha^\beta [\mathfrak{Z}_\beta^{\cdots e_i, e_{i+1} \cdots}]$$

Connection to [Maulik, Okounkov]

Now apply $\mathfrak{M} \xrightarrow{p} \mathfrak{M}_0 \xhookrightarrow{i} \mathcal{T}'_\mu$: the pushforward $p_*[\mathfrak{Z}_\alpha]$ is either zero or $[\mathfrak{Z}_{0,\alpha}]$ (geometric analogue of going from weight space to multiplicity space). So we still have

$$i_*[\mathfrak{Z}_{0,\alpha}^{\dots e_{i+1},e_i\dots}] = \sum_\beta (R_i)_\alpha^\beta \, i_*[\mathfrak{Z}_{0,\beta}^{\dots e_i,e_{i+1}\dots}]$$

$i_*[\mathfrak{Z}_{0,\alpha}]$ is not quite yet Ψ_α for two reasons:

- (a trivial one) The embedding space is $\mathcal{T}'_\mu$ and not $\mathcal{T}'_\mu \cap \mathfrak{n}$ (paying attention that $\mathfrak{n}$ depends on the ordering of the basis).
- (an important one) To compute Ψ_α, we parameterize the torus T differently depending on whether we are considering $\mathfrak{Z}_{0,\alpha}^{\dots,e_i,e_{i+1},\dots}$ or $\mathfrak{Z}_{0,\alpha}^{\dots,e_{i+1},e_i,\dots}$.

Therefore, introducing the operator τ_i that switches z_i and z_{i+1}:

$$i_*[\mathfrak{Z}_{0,\alpha}^{\dots e_i,e_{i+1}\dots}] = f\,\Psi_\alpha^{\dots m_i,m_{i+1}\dots}$$
$$i_*[\mathfrak{Z}_{0,\alpha}^{\dots e_{i+1},e_i\dots}] = \tau_i(f\,\Psi_\alpha^{\dots m_{i+1},m_i\dots})$$

Connection to [Maulik, Okounkov], end

Finally, changing the normalization of the R-matrix to

$$\check{R}_i := \frac{f}{\tau_i f} R_i$$

we find

$$\tau_i \Psi_{\alpha}^{\dots m_{i+1}, m_i \dots} = \sum_{\beta} (\check{R}_i)_{\alpha}^{\beta} \Psi_{\beta}^{\dots m_i, m_{i+1} \dots}$$

Remarks:

- Applying i_* "loses information", so that naively the exchange relation cannot be taken as a definition of the R-matrix; but actually, it does define it uniquely.
- $\tilde{R}_i$ is local (wrt tensor product), whereas R_i isn't.
- By definition, the R_i satisfy the relations of the corresponding Weyl group (here, symmetric group $\mathcal{S}_N$); taking into account once again the reparameterization of the torus, we obtain the relations of the theorem.

Definition of Vertex Operators

Consider (type I, dual) level 1 vertex operators (VOs) $\Phi(z)$, i.e., intertwiners

$$\Phi_\nu(z) : \mathcal{V} \otimes L_\nu(z) \to \mathcal{V}$$

where $\mathcal{V}$ is an appropriate level 1 representation of the Yangian double $\widehat{DY(\mathfrak{sl}(k))}$, and $L_\nu(z)$ is the evaluation representation of $\widehat{DY(\mathfrak{sl}(k))}$ (i.e., level 0 and isomorphic to L_ν as a $\mathfrak{sl}(k)$-module). In level 1, one must have $\nu = \omega_i$.

Let $|0\rangle \in \mathcal{V}$ be the highest weight vector of the basic level 1 representation, and $\langle 0| \in \mathcal{V}^*$ be the lowest weight vector of its dual.

Theorem

Ψ *is proportional to the VEV of a product of VOs:*

$$\Psi^{m_1,\dots,m_N}(z_1,\dots,z_N) = \kappa(z_1,\dots,z_N)\,\langle 0|\,\Phi_{\omega_{m_1}}(z_1)\dots\Phi_{\omega_{m_N}}(z_N)\,|0\rangle$$

Remarks

- We can be slightly more explicit in the "stable" basis, where locality is apparent: the indexing set for components of $\mathfrak{Z}$ / stable sets is of the form $\alpha = (\alpha_1, \ldots, \alpha_N)$ (where α_i is a subset of cardinality m_i of $\{1, \ldots, k\}$) with a weight constraint, and then we have

$$\tilde{\Psi}_\alpha = \sum_\beta c_\alpha^\beta \Psi_\beta = \kappa(z_1, \ldots, z_N) \langle 0| \Phi^{(\alpha_1)}(z_1) \ldots \Phi^{(\alpha_N)}(z_N) |0\rangle$$

  where $\Phi^{(\alpha_i)}(z) : \mathcal{V} \to \mathcal{V}$ is the expansion of $\Phi_{\omega_{m_i}}(z)$ in the standard basis of L_{ω_i}.
- This result, just like the previous one, can actually be generalized to arbitrary v (not necessarily equal to v_∞).
- I do not know any geometric proof of this result.

Byproduct 1: fusion and quiver varieties

In the process, we find a beautiful correspondence between two concepts:

- On the integrable side, the fusion procedure allows to write

$$\Phi_{\omega_\ell}(z) = \Phi(z - \frac{\ell-1}{2}\hbar)\Phi(z - \frac{\ell-3}{2}\hbar)\dots\Phi(z + \frac{\ell-1}{2}\hbar)$$

  where $\Phi := \Phi_{\omega_1}$, and embedding $L_{\omega_\ell} \hookrightarrow L_{\omega_1}^{\otimes \ell}$ is implied.
- On the geometric side, the Mirković–Vybornov transverse slice of nilpotent orbits allows to write an equality of multidegrees w.r.t. the appropriate torus. The torus $(\mathbb{C}^\times)^{M+1}$ acts on the whole of $\mathfrak{gl}(M)$ with weights of the form $\hbar + z_{\text{row}} - z_{\text{column}}$; but the form of x precisely enforces inside each block $\hbar + z_{i;k} - z_{i;k+1} = 0$, i.e., $z_{i;k} = z_i + (k - \frac{\ell+1}{2})\hbar$, which corresponds to restricting to the subtorus $(\mathbb{C}^\times)^{N+1}$.

Byproduct 2: qKZ equation

VEVs of VOs (of a Yangian) are known to satisfy the (rational quantum, or difference) Knizhnik–Zamolodchikov equation.

Explicitly, the Ψ_α satisfy the following system of equations:

- The exchange relation:

$$\Psi^{m_1,\ldots,m_{i+1},m_i,\ldots,m_N}(z_1,\ldots,z_{i+1},z_i,\ldots,z_N) = \check{R}_i(z_i - z_{i+1})\Psi^{m_1,\ldots,m_i,m_{i+1},\ldots,m_N}(z_1,\ldots,z_i,z_{i+1},\ldots,z_N)$$

- The cyclicity condition:

$$\Psi^{m_2,\ldots,m_N,m_1}(z_2,\ldots,z_N,z_1+(k+1)\hbar) = \varepsilon^{m_1}\rho\,\Psi^{m_1,\ldots,m_N}(z_1,\ldots,z_N)$$

where $\varepsilon = (-1)^{\frac{M}{k}-1}$, and the rotation operator ρ has an explicit combinatorial definition (promotion), i.e., $\rho_\alpha^\beta = \delta_\alpha^{\rho(\beta)}$.

Conclusion and prospects

- We have found that multidegrees of irreducible components of tensor product quiver varieties are polynomial solutions of the qKZ equation, though a full geometric proof is lacking. (only the "nonaffine" part of qKZ is geometric)
- In particular, the occurrence of the corresponding ADE Yangian (at both zero and nonzero level!) is surprising.

Various possible generalizations:

- What to do with other untwisted/twisted affine algebras/Yangians? (I have worked out $A_2^{(2)}$)
- Consider quivers not based on GL. Should be related to qKZ in other types (see [Ponsaing, ZJ '14]), i.e., integrable models with boundaries.
- Is there a similar connection between higher level polynomial solutions of the qKZ equation (see for example [Fonseca, ZJ '13]) and geometry?
- Generalizations to K-theory? elliptic cohomology??