Exactly solvable tiling models, Schur functions and the Littlewood–Richardson rule

P. Zinn-Justin

LPTHE, Université Paris 6

U.C. Berkeley – Dec 2012

Part I

Schur functions

The general linear group

[H. Barcelo, A. Ram, Combinatorial representation theory]; [A. Molev, Gelfand–Tsetlin bases for classical Lie algebras]

Let $G = GL(n,\mathbb{C})$, T be the subgroup of diagonal invertible matrices. Let $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$ be the Lie algebra of G. $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-$, with $\mathfrak{t}$ (resp. $\mathfrak{n}_+$, $\mathfrak{n}_-$) generated by $h_i = E_{i,i}$, $i = 1,\ldots,n$ (resp., $e_i = E_{i,i+1}$ and $f_i = E_{i+1,i}$, $i = 1,\ldots,n-1$).

A partition is a nonincreasing sequence of nonnegative integers $(\lambda_1,\ldots,\lambda_n)$, up to addition/removal of an arbitrary number of zeroes at the end. We represent partitions using Young diagrams:

$(4,2,1) =$

Nonzero entries are called parts. $|\lambda| = \sum_i \lambda_i$.

The general linear group

[H. Barcelo, A. Ram, Combinatorial representation theory]; [A. Molev, Gelfand–Tsetlin bases for classical Lie algebras]

Let $G = GL(n, \mathbb{C})$, T be the subgroup of diagonal invertible matrices. Let $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ be the Lie algebra of G. $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-$, with $\mathfrak{t}$ (resp. $\mathfrak{n}_+$, $\mathfrak{n}_-$) generated by $h_i = E_{i,i}$, $i = 1, \ldots, n$ (resp., $e_i = E_{i,i+1}$ and $f_i = E_{i+1,i}$, $i = 1, \ldots, n-1$).

A partition is a nonincreasing sequence of nonnegative integers $(\lambda_1, \ldots, \lambda_n)$, up to addition/removal of an arbitrary number of zeroes at the end. We represent partitions using Young diagrams:

$(4, 2, 1) =$

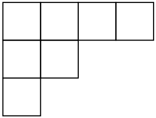
Nonzero entries are called parts. $|\lambda| = \sum_i \lambda_i$.

The general linear group

[H. Barcelo, A. Ram, Combinatorial representation theory]; [A. Molev, Gelfand–Tsetlin bases for classical Lie algebras]

Let $G = GL(n, \mathbb{C})$, T be the subgroup of diagonal invertible matrices. Let $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ be the Lie algebra of G. $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-$, with $\mathfrak{t}$ (resp. $\mathfrak{n}_+$, $\mathfrak{n}_-$) generated by $h_i = E_{i,i}$, $i = 1, \ldots, n$ (resp., $e_i = E_{i,i+1}$ and $f_i = E_{i+1,i}$, $i = 1, \ldots, n-1$).

A partition is a nonincreasing sequence of nonnegative integers $(\lambda_1, \ldots, \lambda_n)$, up to addition/removal of an arbitrary number of zeroes at the end. We represent partitions using Young diagrams:

$(4, 2, 1) =$ 

Nonzero entries are called parts. $|\lambda| = \sum_i \lambda_i$.

Irreducible representations

We are interested in irreducible polynomial representations of G. We shall need the following classical

Theorem

Irreducible polynomial representations ρ_λ of G are indexed by partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ with at most n parts. They are highest weight representation, i.e., have a highest weight vector v_λ (unique up to multiplication by a scalar):

$$\rho_\lambda(h_i)v_\lambda = \lambda_i v_\lambda \qquad \rho_\lambda(e_i)v_\lambda = 0$$

The proof involves explicit construction of $\rho_\lambda : G \to GL(V_\lambda)$ out of the fundamental representation. . .

Irreducible representations

We are interested in irreducible polynomial representations of G. We shall need the following classical

Theorem

Irreducible polynomial representations ρ_λ of G are indexed by partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ with at most n parts. They are highest weight representation, i.e., have a highest weight vector v_λ (unique up to multiplication by a scalar):

$$\rho_\lambda(h_i)v_\lambda = \lambda_i v_\lambda \qquad \rho_\lambda(e_i)v_\lambda = 0$$

The proof involves explicit construction of $\rho_\lambda : G \to GL(V_\lambda)$ out of the fundamental representation…

Examples

- The fundamental representation. G acts on $V_{\square} = \mathbb{C}^n$ in the natural way. $\lambda = (1, 0, \ldots, 0) = \square$.

$$v_{\square} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad h_i v = \delta_{i1} v$$

- $V_{\square} \otimes V_{\square}$ decomposes into symmetric/skew-symmetric subspaces:

$$V_{\square} \otimes V_{\square} = V_{\square\square} \oplus V_{\substack{\square \\ \square}}$$

with $v_{\square\square} = v_{\square} \otimes v_{\square}$, $v_{\substack{\square \\ \square}} = v_{\square} \otimes (f_1 v_{\square}) - (f_1 v_{\square}) \otimes v_{\square}$.

Examples

- The fundamental representation. G acts on $V_{\square} = \mathbb{C}^n$ in the natural way. $\lambda = (1, 0, \ldots, 0) = \square$.

$$v_{\square} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad h_i v = \delta_{i1} v$$

- $V_{\square} \otimes V_{\square}$ decomposes into symmetric/skew-symmetric subspaces:

$$V_{\square} \otimes V_{\square} = V_{\square\square} \oplus V_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$$

with $v_{\square\square} = v_{\square} \otimes v_{\square}$, $v_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = v_{\square} \otimes (f_1 v_{\square}) - (f_1 v_{\square}) \otimes v_{\square}$.

Schur–Weyl duality

In general, the natural actions on $V_\square^{\otimes r}$ of $GL(n)$ and $\mathcal{S}_r$ are commutants of each other, so that

$$V_\square^{\otimes r} = \bigoplus_{\lambda=(\lambda_1,\ldots,\lambda_n):|\lambda|=r} W_\lambda \otimes V_\lambda$$

where W_λ is an irreducible representation of $\mathcal{S}_r$, and V_λ an irreducible representation of $GL(n)$.

Characters

Define
$$\hat{s}_\lambda : G \to \mathbb{C}$$
$$g \mapsto \mathrm{tr}_{V_\lambda}(\rho_\lambda(g))$$

$\lambda \mapsto \hat{s}_\lambda$ is a map from $(Rep(G), \oplus, \otimes)$ to the class functions on G.

A generic matrix is diagonalizable, therefore $\hat{s}_\lambda(g)$ only depends on the eigenvalues $\{x_1, \ldots, x_n\}$ of g. Applying $\hat{s}_\lambda$ to an element of T implies

$$\hat{s}_\lambda(g) = s_\lambda(x_1, \ldots, x_n)$$

where $s_\lambda(x_1, \ldots, x_n)$ is a symmetric polynomial in the $\{x_i\}$ of degree $|\lambda|$, called Schur polynomial.

$\lambda \mapsto s_\lambda$ extends linearly into an isomorphism of graded rings from $(Rep(G), \oplus, \otimes)$ to $\mathbb{Z}[x_1, \ldots, x_n]^{\mathcal{S}_n}$ (symmetric polynomials in n variables).

Characters

Define
$$\hat{s}_\lambda : G \to \mathbb{C}$$
$$g \mapsto \mathrm{tr}_{V_\lambda}(\rho_\lambda(g))$$

$\lambda \mapsto \hat{s}_\lambda$ is a map from $(Rep(G), \oplus, \otimes)$ to the class functions on G.

A generic matrix is diagonalizable, therefore $\hat{s}_\lambda(g)$ only depends on the eigenvalues $\{x_1, \ldots, x_n\}$ of g. Applying $\hat{s}_\lambda$ to an element of T implies

$$\hat{s}_\lambda(g) = s_\lambda(x_1, \ldots, x_n)$$

where $s_\lambda(x_1, \ldots, x_n)$ is a symmetric polynomial in the $\{x_i\}$ of degree $|\lambda|$, called Schur polynomial.

$\lambda \mapsto s_\lambda$ extends linearly into an isomorphism of graded rings from $(Rep(G), \oplus, \otimes)$ to $\mathbb{Z}[x_1, \ldots, x_n]^{\mathcal{S}_n}$ (symmetric polynomials in n variables).

Characters

Define
$$\hat{s}_\lambda : G \to \mathbb{C}$$
$$g \mapsto \mathrm{tr}_{V_\lambda}(\rho_\lambda(g))$$

$\lambda \mapsto \hat{s}_\lambda$ is a map from $(Rep(G), \oplus, \otimes)$ to the class functions on G.

A generic matrix is diagonalizable, therefore $\hat{s}_\lambda(g)$ only depends on the eigenvalues $\{x_1, \ldots, x_n\}$ of g. Applying $\hat{s}_\lambda$ to an element of T implies

$$\hat{s}_\lambda(g) = s_\lambda(x_1, \ldots, x_n)$$

where $s_\lambda(x_1, \ldots, x_n)$ is a symmetric polynomial in the $\{x_i\}$ of degree $|\lambda|$, called Schur polynomial.

$\lambda \mapsto s_\lambda$ extends linearly into an isomorphism of graded rings from $(Rep(G), \oplus, \otimes)$ to $\mathbb{Z}[x_1, \ldots, x_n]^{\mathcal{S}_n}$ (symmetric polynomials in n variables).

Examples

- The fundamental representation:

$$s_{\square}(x_1,\ldots,x_n) = \operatorname{tr}\operatorname{diag}(x_1,\ldots,x_n) = \sum_{i=1}^{n} x_i$$

- $V_{\square\square}$ and $V_{\substack{\square\\\square}}$:

$$s_{\square\square}(x_1,\ldots,x_n) = \sum_{1\le i\le j\le n} x_i x_j$$

$$s_{\substack{\square\\\square}}(x_1,\ldots,x_n) = \sum_{1\le i< j\le n} x_i x_j$$

Note that $s_{\square\square}(x_1,\ldots,x_n) + s_{\substack{\square\\\square}}(x_1,\ldots,x_n) = (\sum_{i=1}^{n} x_i)^2 = s_{\square}(x_1,\ldots,x_n)^2$, as expected from the decomposition of the tensor product.

Examples

- The fundamental representation:

$$s_{\square}(x_1,\dots,x_n) = \operatorname{tr}\operatorname{diag}(x_1,\dots,x_n) = \sum_{i=1}^{n} x_i$$

- $V_{\square\square}$ and $V_{\substack{\square\\\square}}$:

$$s_{\square\square}(x_1,\dots,x_n) = \sum_{1\le i\le j\le n} x_i x_j$$

$$s_{\substack{\square\\\square}}(x_1,\dots,x_n) = \sum_{1\le i<j\le n} x_i x_j$$

Note that $s_{\square\square}(x_1,\dots,x_n) + s_{\substack{\square\\\square}}(x_1,\dots,x_n) = (\sum_{i=1}^{n} x_i)^2 = s_{\square}(x_1,\dots,x_n)^2$, as expected from the decomposition of the tensor product.

Examples

- The fundamental representation:

$$s_{\square}(x_1,\ldots,x_n) = \operatorname{tr}\operatorname{diag}(x_1,\ldots,x_n) = \sum_{i=1}^{n} x_i = p_1$$

- $V_{\square\square}$ and $V_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$:

$$s_{\square\square}(x_1,\ldots,x_n) = \sum_{1\le i\le j\le n} x_i x_j = \frac{1}{2}p_1^2 + \frac{1}{2}p_2$$

$$s_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}(x_1,\ldots,x_n) = \sum_{1\le i< j\le n} x_i x_j = \frac{1}{2}p_1^2 - \frac{1}{2}p_2$$

Note that $s_{\square\square}(x_1,\ldots,x_n) + s_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}(x_1,\ldots,x_n) = (\sum_{i=1}^n x_i)^2 = s_{\square}(x_1,\ldots,x_n)^2$, as expected from the decomposition of the tensor product.

Schur–Weyl II

Consider an element $g = \mathrm{diag}(x_1, \ldots, x_n) \in T$ and $\sigma \in \mathcal{S}_r$. Then

$$\mathrm{tr}_{V_{\square}^{\otimes r}}(g\sigma) = \sum_{\lambda=(\lambda_1,\ldots,\lambda_n),\, |\lambda|=r} \hat{\chi}_\lambda(\sigma) s_\lambda(x_1, \ldots, x_n)$$

where $\hat{\chi}_\lambda(\sigma)$ is the character of the irrep λ of $\mathcal{S}_r$. If σ has α_k cycles of length k, then $\hat{\chi}_\lambda(\sigma) = \chi_\lambda(\alpha)$.

The left hand side is easily computed to be $\prod_{k\geq 1} p_k^{\alpha_k}$, where the p_k are the power sums: $p_k(x_1, \ldots, x_n) = \sum_{i=1}^n x_i^k$.

This formula can be inverted (orthogonality of characters):

$$s_\lambda(x_1, \ldots, x_n) = \sum_{\alpha_k \geq 0, \sum_k \alpha_k = |\lambda|} \prod_{k\geq 1} \frac{1}{k^{\alpha_k} \alpha_k!} \chi_\lambda(\alpha) \prod_{k \geq 1} p_k(x_1, \ldots, x_n)^{\alpha_k}$$

More combinatorial formula?

Schur–Weyl II

Consider an element $g = \mathrm{diag}(x_1, \ldots, x_n) \in T$ and $\sigma \in \mathcal{S}_r$. Then

$$\mathrm{tr}_{V_{\square}^{\otimes r}}(g\sigma) = \sum_{\lambda=(\lambda_1,\ldots,\lambda_n),\, |\lambda|=r} \hat{\chi}_\lambda(\sigma) s_\lambda(x_1, \ldots, x_n)$$

where $\hat{\chi}_\lambda(\sigma)$ is the character of the irrep λ of $\mathcal{S}_r$. If σ has α_k cycles of length k, then $\hat{\chi}_\lambda(\sigma) = \chi_\lambda(\alpha)$.

The left hand side is easily computed to be $\prod_{k\geq 1} p_k^{\alpha_k}$, where the p_k are the power sums: $p_k(x_1, \ldots, x_n) = \sum_{i=1}^n x_i^k$.

This formula can be inverted (orthogonality of characters):

$$s_\lambda(x_1, \ldots, x_n) = \sum_{\alpha_k \geq 0, \sum_k \alpha_k = |\lambda|} \prod_{k\geq 1} \frac{1}{k^{\alpha_k} \alpha_k!} \chi_\lambda(\alpha) \prod_{k\geq 1} p_k(x_1, \ldots, x_n)^{\alpha_k}$$

More combinatorial formula?

Schur–Weyl II

Consider an element $g = \mathrm{diag}(x_1, \ldots, x_n) \in T$ and $\sigma \in \mathcal{S}_r$. Then

$$\mathrm{tr}_{V_{\square}^{\otimes r}}(g\sigma) = \sum_{\lambda=(\lambda_1,\ldots,\lambda_n),\, |\lambda|=r} \hat{\chi}_\lambda(\sigma) s_\lambda(x_1, \ldots, x_n)$$

where $\hat{\chi}_\lambda(\sigma)$ is the character of the irrep λ of $\mathcal{S}_r$. If σ has α_k cycles of length k, then $\hat{\chi}_\lambda(\sigma) = \chi_\lambda(\alpha)$.

The left hand side is easily computed to be $\prod_{k\geq 1} p_k^{\alpha_k}$, where the p_k are the power sums: $p_k(x_1, \ldots, x_n) = \sum_{i=1}^n x_i^k$.

This formula can be inverted (orthogonality of characters):

$$s_\lambda(x_1, \ldots, x_n) = \sum_{\alpha_k \geq 0, \sum_k \alpha_k = |\lambda|} \prod_{k\geq 1} \frac{1}{k^{\alpha_k} \alpha_k!} \chi_\lambda(\alpha) \prod_{k\geq 1} p_k(x_1, \ldots, x_n)^{\alpha_k}$$

More combinatorial formula?

Schur–Weyl II

Consider an element $g = \mathrm{diag}(x_1, \ldots, x_n) \in T$ and $\sigma \in \mathcal{S}_r$. Then

$$\mathrm{tr}_{V_{\square}^{\otimes r}}(g\sigma) = \sum_{\lambda=(\lambda_1,\ldots,\lambda_n),\, |\lambda|=r} \hat{\chi}_\lambda(\sigma) s_\lambda(x_1, \ldots, x_n)$$

where $\hat{\chi}_\lambda(\sigma)$ is the character of the irrep λ of $\mathcal{S}_r$. If σ has α_k cycles of length k, then $\hat{\chi}_\lambda(\sigma) = \chi_\lambda(\alpha)$.

The left hand side is easily computed to be $\prod_{k\geq 1} p_k^{\alpha_k}$, where the p_k are the power sums: $p_k(x_1, \ldots, x_n) = \sum_{i=1}^n x_i^k$.

This formula can be inverted (orthogonality of characters):

$$s_\lambda(x_1, \ldots, x_n) = \sum_{\alpha_k \geq 0, \sum_k \alpha_k = |\lambda|} \prod_{k\geq 1} \frac{1}{k^{\alpha_k} \alpha_k!} \chi_\lambda(\alpha) \prod_{k\geq 1} p_k(x_1, \ldots, x_n)^{\alpha_k}$$

More combinatorial formula?

Weight space decomposition

Elements of T are simultaneously diagonalizable in $V_\square$, and therefore in any V_λ. This means every V_λ has a weight space decomposition:

$$V_\lambda = \bigoplus_{\mu \in \mathbb{Z}^n_{\geq 0}} V_{\lambda;\mu} \qquad V_{\lambda;\mu} = \{v \in V_\lambda : \rho_\lambda(h_i)v = \mu_i v\}$$

which implies in turn

$$s_\lambda(x_1, \ldots, x_n) = \sum_{\mu \in \mathbb{Z}^n_{\geq 0}} K_{\lambda;\mu} \prod_{i=1}^{n} x_i^{\mu_i}$$

where $K_{\lambda;\mu} = \dim V_{\lambda;\mu}$. (when μ is also a partition, which we can always reduce to, $K_{\lambda;\mu}$ is called a Kostka number)

Gelfand–Tsetlin induction

Theorem

The decomposition of V_λ, $\lambda = (\lambda_1, \ldots, \lambda_n)$ as a representation of $GL(n-1) \subset GL(n)$ is of the form

$$V_\lambda = \bigoplus_{\substack{\mu=(\mu_1,\ldots,\mu_{n-1})\\ \lambda_i \geq \mu_i \geq \lambda_{i+1}}} V_\mu$$

(in particular it's multiplicity-free)

$\lambda =$

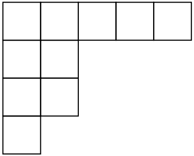
Gelfand–Tsetlin induction

Theorem

The decomposition of V_λ, $\lambda = (\lambda_1, \ldots, \lambda_n)$ as a representation of $GL(n-1) \subset GL(n)$ is of the form

$$V_\lambda = \bigoplus_{\substack{\mu=(\mu_1,\ldots,\mu_{n-1})\\ \lambda_i \geq \mu_i \geq \lambda_{i+1}}} V_\mu$$

(in particular it's multiplicity-free)

$\lambda =$ 

Gelfand–Tsetlin induction

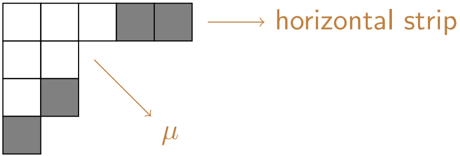
Theorem

The decomposition of V_λ, $\lambda = (\lambda_1, \ldots, \lambda_n)$ as a representation of $GL(n-1) \subset GL(n)$ is of the form

$$V_\lambda = \bigoplus_{\substack{\mu=(\mu_1,\ldots,\mu_{n-1}) \\ \lambda_i \geq \mu_i \geq \lambda_{i+1}}} V_\mu$$

(in particular it's multiplicity-free)

$\lambda =$



Gelfand–Tsetlin induction cont'd

One can iterate the process: $GL(n) \supset GL(n-1) \supset \cdots \supset GL(1)$, until we reach irreps of $GL(1)$ which are one-dimensional. This way, we decompose V_λ into a direct sum of one-dimensional subspaces indexed by Gelfand–Tsetlin patterns:

$$
\begin{array}{ccccccccccccc}
\lambda_{n,1} & & \geq & & \lambda_{n,2} & & \geq & \cdots & \geq & \lambda_{n,n-1} & \geq & & \lambda_{n,n} \\
& \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\leq$} & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\leq$} & \rotatebox{45}{$\leq$} & & \rotatebox{-45}{$\geq$} & \rotatebox{45}{$\leq$} & \\
& \lambda_{n-1,1} & & \geq & \lambda_{n-1,2} & & \geq & \cdots & \geq & \lambda_{n-1,n-1} & & & \\
& & \rotatebox{-45}{$\geq$} & \rotatebox{45}{$\leq$} & & \rotatebox{-45}{$\geq$} & & & \rotatebox{45}{$\leq$} & & & & \\
& & \ddots & & & & & & \iddots & & & & \\
& & & \rotatebox{-45}{$\geq$} & & & & \rotatebox{45}{$\leq$} & & & & & \\
& & & & \lambda_{2,1} & & \geq & \lambda_{2,2} & & & & & \\
& & & & & \rotatebox{-45}{$\geq$} & \rotatebox{45}{$\leq$} & & & & & & \\
& & & & & & \lambda_{1,1} & & & & & &
\end{array}
$$

A combinatorial formula

Assume $GL(n-1)$ is matrices of the form $\begin{pmatrix} 1 & 0 \\ 0 & \star \end{pmatrix}$. Then each such one-dimensional subspace is a weight space, with weight given by $\mu_i = |\lambda_i| - |\lambda_{i-1}|$, $i = 1, \ldots, n$. (conventionally $|\lambda_0| = 0$)

Therefore $K_{\lambda,\mu} = \#\{\text{GT patterns} : \mu_i = |\lambda_i| - |\lambda_{i-1}|\}$, which leads to the following explicit formula for Schur functions:

$$s_\lambda(x_1, \ldots, x_n) = \sum_{\text{GT patterns } (\lambda_{i,j})} \prod_{i=1}^{n} x_i^{|\lambda_i| - |\lambda_{i-1}|}$$

Remark: bijection with SSYT...

A combinatorial formula

Assume $GL(n-1)$ is matrices of the form $\begin{pmatrix} 1 & 0 \\ 0 & \star \end{pmatrix}$. Then each such one-dimensional subspace is a weight space, with weight given by $\mu_i = |\lambda_i| - |\lambda_{i-1}|$, $i = 1, \ldots, n$. (conventionally $|\lambda_0| = 0$)

Therefore $K_{\lambda,\mu} = \#\{\text{GT patterns} : \mu_i = |\lambda_i| - |\lambda_{i-1}|\}$, which leads to the following explicit formula for Schur functions:

$$s_\lambda(x_1, \ldots, x_n) = \sum_{\text{GT patterns } (\lambda_{i,j})} \prod_{i=1}^{n} x_i^{|\lambda_i| - |\lambda_{i-1}|}$$

Remark: bijection with SSYT...

A combinatorial formula

Assume $GL(n-1)$ is matrices of the form $\begin{pmatrix} 1 & 0 \\ 0 & \star \end{pmatrix}$. Then each such one-dimensional subspace is a weight space, with weight given by $\mu_i = |\lambda_i| - |\lambda_{i-1}|$, $i = 1, \ldots, n$. (conventionally $|\lambda_0| = 0$)

Therefore $K_{\lambda,\mu} = \#\{\text{GT patterns} : \mu_i = |\lambda_i| - |\lambda_{i-1}|\}$, which leads to the following explicit formula for Schur functions:

$$s_\lambda(x_1, \ldots, x_n) = \sum_{\text{GT patterns } (\lambda_{i,j})} \prod_{i=1}^{n} x_i^{|\lambda_i| - |\lambda_{i-1}|}$$

Remark: bijection with SSYT...

Stability

From the construction above we conclude that

$$s_\lambda(x_1,\ldots,x_n)=\sum_{\substack{\mu=(\mu_1,\ldots,\mu_{n-1})\\ \lambda_i\ge\mu_i\ge\lambda_{i+1}}} x_n^{|\lambda|-|\mu|}\, s_\mu(x_1,\ldots,x_{n-1})$$

In particular,

$$s_\lambda(x_1,\ldots,x_{n-1},0)=\begin{cases}0 & \text{if } \lambda_n\neq 0\\ s_\lambda(x_1,\ldots,x_{n-1}) & \text{if } \lambda_n=0\end{cases}$$

where recall that $(\lambda_1,\ldots,\lambda_{n-1},0)\equiv(\lambda_1,\ldots,\lambda_{n-1})$.

Stability

From the construction above we conclude that

$$s_\lambda(x_1,\ldots,x_n) = \sum_{\substack{\mu=(\mu_1,\ldots,\mu_{n-1})\\ \lambda_i\ge\mu_i\ge\lambda_{i+1}}} x_n^{|\lambda|-|\mu|} s_\mu(x_1,\ldots,x_{n-1})$$

In particular,

$$s_\lambda(x_1,\ldots,x_{n-1},0) = \begin{cases} 0 & \text{if } \lambda_n \neq 0 \\ s_\lambda(x_1,\ldots,x_{n-1}) & \text{if } \lambda_n = 0 \end{cases}$$

where recall that $(\lambda_1,\ldots,\lambda_{n-1},0) \equiv (\lambda_1,\ldots,\lambda_{n-1})$.

Schur functions

Consider the ring of symmetric functions R, obtained as the inverse limit of $R_n = \mathbb{Z}[x_1, \ldots, x_n]^{\mathcal{S}_n}$ where $R_n \to R_{n-1}$ by setting $x_n = 0$. (see also wikipedia)

Due to the stability property above, Schur polynomials $(s_\lambda(x_1, \ldots, x_n))_{n\in\mathbb{N}}$ define an element of R called Schur function and also denoted s_λ.

Remark: power sums $p_k = \sum_i x_i^k$ are also symmetric functions.

Schur functions

Consider the ring of symmetric functions R, obtained as the inverse limit of $R_n = \mathbb{Z}[x_1, \ldots, x_n]^{\mathcal{S}_n}$ where $R_n \to R_{n-1}$ by setting $x_n = 0$. (see also wikipedia)

Due to the stability property above, Schur polynomials $(s_\lambda(x_1, \ldots, x_n))_{n\in\mathbb{N}}$ define an element of R called Schur function and also denoted s_λ.

Remark: power sums $p_k = \sum_i x_i^k$ are also symmetric functions.

Schur functions

Consider the ring of symmetric functions R, obtained as the inverse limit of $R_n = \mathbb{Z}[x_1, \ldots, x_n]^{\mathcal{S}_n}$ where $R_n \to R_{n-1}$ by setting $x_n = 0$. (see also wikipedia)

Due to the stability property above, Schur polynomials $(s_\lambda(x_1, \ldots, x_n))_{n\in\mathbb{N}}$ define an element of R called Schur function and also denoted s_λ.

Remark: power sums $p_k = \sum_i x_i^k$ are also symmetric functions.

Skew Schur functions

More generally, given a Young diagram λ with n boxes, decompose V_λ w.r.t. the action of $GL(m)$, $m < n$:

$$s_\lambda(x_1, \ldots, x_n) = \sum_\mu s_{\lambda/\mu}(x_{m+1}, \ldots, x_n) s_\mu(x_1, \ldots, x_m)$$

where the summation is over $\mu \subset \lambda$ with m rows. $s_{\lambda/\mu}$ is called a skew Schur polynomial; it is sum over Gelfand–Tsetlin trapezoids.

By stability as before, one can define a skew Schur function which satisfies the same relation with no restriction on μ.

Skew Schur functions

More generally, given a Young diagram λ with n boxes, decompose V_λ w.r.t. the action of $GL(m)$, $m < n$:

$$s_\lambda(x_1, \ldots, x_n) = \sum_\mu s_{\lambda/\mu}(x_{m+1}, \ldots, x_n) s_\mu(x_1, \ldots, x_m)$$

where the summation is over $\mu \subset \lambda$ with m rows. $s_{\lambda/\mu}$ is called a skew Schur polynomial; it is sum over Gelfand–Tsetlin trapezoids.

By stability as before, one can define a skew Schur function which satisfies the same relation with no restriction on μ.

Grassmannians

[W. Fulton, Young tableaux with applications to representation theory and geometry]
Let $Gr(n, d)$ be the Grassmannian:

$$Gr(n, d) = \{V \subset \mathbb{C}^d : \dim V = n\}$$

It's a projective algebraic variety of dimension $n(d-n)$. Its cohomology ring has rank $\binom{d}{n}$ (by localization). There are many explicit descriptions of it.

Cohomology of Grassmannians I

Shortcut: define $X = \{u \in Mat(d,n) : \operatorname{rank}(u) = n\}$.
$G = GL(n) \supset T = (\mathbb{C}^\times)^n$ acts on it by right multiplication, and
$X \xrightarrow{p} X/G \cong Gr(n,d)$.

Then

$$H^*(Gr(n,d)) \cong H^*_G(X) \xleftarrow{\iota^*} H^*_G(Mat(d,n))$$

and

$$H^*_G(Mat(d,n)) \cong H^*_G(\cdot) \cong H^*_T(\cdot)^{\mathcal{S}_n} \cong \mathrm{Sym}(T^*)^{\mathcal{S}_n} \cong \mathbb{Z}[x_1,\dots,x_n]^{\mathcal{S}_n}$$

where $\deg x_i = 2$.

Cohomology of Grassmannians I

Shortcut: define $X = \{u \in Mat(d,n) : \text{rank}(u) = n\}$.
$G = GL(n) \supset T = (\mathbb{C}^\times)^n$ acts on it by right multiplication, and
$X \xrightarrow{p} X/G \cong Gr(n,d)$.

Then

$$H^*(Gr(n,d)) \cong H^*_G(X) \xleftarrow{\iota^*} H^*_G(Mat(d,n))$$

and

$$H^*_G(Mat(d,n)) \cong H^*_G(\cdot) \cong H^*_T(\cdot)^{\mathcal{S}_n} \cong \mathrm{Sym}(T^*)^{\mathcal{S}_n} \cong \mathbb{Z}[x_1,\ldots,x_n]^{\mathcal{S}_n}$$

where $\deg x_i = 2$.

Cohomology of Grassmannians II

Start from the exact sequence of tautological vector bundles:

$$0 \to V \to \mathbb{C}^d \to \mathbb{C}^d/V \to 0$$

This implies

$$c(z;V)c(z;\mathbb{C}^d/V) = 1$$

where $c(z,\cdot)$ is the generating series of Chern classes.

Correspondence: $\prod_{i=1}^n (1 + x_i z) = c(z;V)$, so that

$$\prod_{i=1}^n \frac{1}{1+x_i z} = \text{polynomial of deg } d-n \text{ in } z \qquad (\circ)$$

This defines the kernel of ι^*.

Cohomology of Grassmannians II

Start from the exact sequence of tautological vector bundles:

$$0 \to V \to \mathbb{C}^d \to \mathbb{C}^d/V \to 0$$

This implies

$$c(z;V)c(z;\mathbb{C}^d/V) = 1$$

where $c(z,\cdot)$ is the generating series of Chern classes.

Correspondence: $\prod_{i=1}^n (1 + x_i z) = c(z;V)$, so that

$$\prod_{i=1}^{n} \frac{1}{1+x_i z} = \text{polynomial of deg } d-n \text{ in } z \qquad (\circ)$$

This defines the kernel of ι^*.

Cohomology of Grassmannians II

Theorem

- $H^*(Gr(n,d)) = \mathbb{Z}[x_1,\ldots,x_n]^{\mathcal{S}_n}/(\circ)$.
- *The* $\mathring{s}_\lambda(x_1,\ldots,x_n)$, $\lambda \subset n \times (d-n)$, *form a basis of* $H^*(Gr(n,d))$.
- $\mathring{s}_\lambda(x_1,\ldots,x_n) = 0$ *if* $\lambda \not\subset n \times (d-n)$.

Note that there are indeed $\binom{d}{n}$ such Young diagrams, as they are in bijection with binary strings of n $\bullet$ and $d-n$ $\circ$ (more on that later).

Geometric interpretation of Schur polynomials?

Cohomology of Grassmannians II

Theorem

- *$H^*(Gr(n,d)) = \mathbb{Z}[x_1,\ldots,x_n]^{\mathcal{S}_n}/(\circ)$.*
- *The $\mathring{s}_\lambda(x_1,\ldots,x_n)$, $\lambda \subset n\times(d-n)$, form a basis of $H^*(Gr(n,d))$.*
- *$\mathring{s}_\lambda(x_1,\ldots,x_n) = 0$ if $\lambda \not\subset n\times(d-n)$.*

Note that there are indeed $\binom{d}{n}$ such Young diagrams, as they are in bijection with binary strings of n $\bullet$ and $d-n$ $\circ$ (more on that later).

Geometric interpretation of Schur polynomials?

Cohomology of Grassmannians II

Theorem

- *$H^*(Gr(n,d)) = \mathbb{Z}[x_1,\ldots,x_n]^{\mathcal{S}_n}/(\circ)$.*
- *The $\mathring{s}_\lambda(x_1,\ldots,x_n)$, $\lambda \subset n \times (d-n)$, form a basis of $H^*(Gr(n,d))$.*
- *$\mathring{s}_\lambda(x_1,\ldots,x_n) = 0$ if $\lambda \not\subset n \times (d-n)$.*

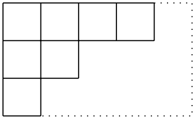
Note that there are indeed $\binom{d}{n}$ such Young diagrams, as they are in bijection with binary strings of n $\bullet$ and $d-n$ $\circ$ (more on that later).

Geometric interpretation of Schur polynomials?

Definition of Schubert varieties

Define the matrix Schubert variety [Knutson, Miller] $S_\lambda = \mathfrak{b}'_- u_\lambda\, \mathfrak{g}$, $\lambda \subset n \times (d-n)$, where

$$u_\lambda = \sum_{i=1}^{n} E_{\lambda_{n+1-i}+i,i}$$

$\lambda =$ 

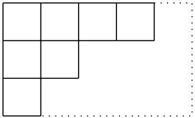
$$u_\lambda = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ & 0 & 0 \\ & 1 & 0 \\ & & 0 \\ & & 0 \\ & & 1 \\ & & \end{pmatrix}$$

and the Schubert variety $\mathring{S}_\lambda = p(S_\lambda \cap X) = p(\overline{B'_- u_\lambda\, G})$.

Definition of Schubert varieties

Define the matrix Schubert variety [Knutson, Miller] $S_\lambda = \overline{\mathfrak{b}'_- u_\lambda \, \mathfrak{g}}$, $\lambda \subset n \times (d-n)$, where

$$u_\lambda = \sum_{i=1}^{n} E_{\lambda_{n+1-i}+i,i}$$

$\lambda =$ 

$$u_\lambda = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ & 0 & 0 \\ & 1 & 0 \\ & & 0 \\ & & 0 \\ & & 1 \end{pmatrix}$$

and the Schubert variety $\mathring{S}_\lambda = p(S_\lambda \cap X) = p(\overline{B'_- u_\lambda \, G})$.

Cohomology classes of Schubert varieties

S_λ (resp. $\mathring{S}_\lambda$) is an irreducible (closed) subvariety of $Mat(d,n)$ (resp. $Gr(n,d)$) of codimension $|\lambda|$. S_λ is G-invariant.

By pushforward,

$$s_\lambda(x_1,\ldots,x_n) = [S_\lambda]_G$$

and

$$\mathring{s}_\lambda(x_1,\ldots,x_n) = [\mathring{S}_\lambda]$$

Cohomology classes of Schubert varieties

S_λ (resp. $\mathring{S}_\lambda$) is an irreducible (closed) subvariety of $Mat(d,n)$ (resp. $Gr(n,d)$) of codimension $|\lambda|$. S_λ is G-invariant.

By pushforward,

$$s_\lambda(x_1,\ldots,x_n) = [S_\lambda]_G$$

and

$$\mathring{s}_\lambda(x_1,\ldots,x_n) = [\mathring{S}_\lambda]$$

Examples

- $\lambda = \varnothing$. $S_\lambda = Mat(d,n)$, $\mathring{S}_\lambda = Gr(n,d)$, $s_\lambda = 1$.
- $\lambda = n \times (d-n)$. $S_\lambda = \left\{ \begin{pmatrix} 0 \\ \star \end{pmatrix} \begin{matrix} d-n \\ n \end{matrix} \right\}$. $\mathring{S}_\lambda = \{\mathbb{C}^d\}$. $s_\lambda = \prod_{i=1}^n x_i^{d-n}$.
- $\lambda = \square$. $S_\lambda = \left\{ \begin{pmatrix} a \\ \star \end{pmatrix} \begin{matrix} n \\ d-n \end{matrix}, \det a = 0 \right\}$. $s_\lambda = \sum_{i=1}^n x_i$.

Examples

- $\lambda = \varnothing$. $S_\lambda = Mat(d,n)$, $\mathring{S}_\lambda = Gr(n,d)$, $s_\lambda = 1$.
- $\lambda = n \times (d-n)$. $S_\lambda = \left\{ \begin{pmatrix} 0 \\ \star \end{pmatrix} \begin{smallmatrix} d-n \\ n \end{smallmatrix} \right\}$. $\mathring{S}_\lambda = \{\mathbb{C}^d\}$. $s_\lambda = \prod_{i=1}^n x_i^{d-n}$.
- $\lambda = \square$. $S_\lambda = \left\{ \begin{pmatrix} a \\ \star \end{pmatrix} \begin{smallmatrix} n \\ d-n \end{smallmatrix}, \det a = 0 \right\}$. $s_\lambda = \sum_{i=1}^n x_i$.

Examples

- $\lambda = \varnothing$. $S_\lambda = Mat(d,n)$, $\mathring{S}_\lambda = Gr(n,d)$, $s_\lambda = 1$.
- $\lambda = n \times (d-n)$. $S_\lambda = \left\{ \begin{pmatrix} 0 \\ \star \end{pmatrix} \begin{matrix} d-n \\ n \end{matrix} \right\}$. $\mathring{S}_\lambda = \{\mathbb{C}^d\}$. $s_\lambda = \prod_{i=1}^n x_i^{d-n}$.
- $\lambda = \square$. $S_\lambda = \left\{ \begin{pmatrix} a \\ \star \end{pmatrix} \begin{matrix} n \\ d-n \end{matrix}, \det a = 0 \right\}$. $s_\lambda = \sum_{i=1}^n x_i$.

Equivariant cohomology

There is another group, namely $G' = GL(d)$, which naturally acts on $Gr(n,d)$. It acts transitively, so let's restrict to $T' = (\mathbb{C}^\times)^d$.
The cohomology ring $H^*_{T'}(Gr(n,d))$ is a module over $H^*_{T'}(\cdot) = \mathbb{Z}[y_1,\ldots,y_d]$. Using the same argument as before, we find that it is a quotient of $\mathbb{Z}[x_1,\ldots,x_n,y_1,\ldots,y_d]^{\mathcal{S}_n}$ by

$$c(z;V)c(z;\mathbb{C}^d/V) = \prod_{i=1}^{d}(1+y_\ell z) \qquad (\circ)$$

Define the factorial Schur polynomial

$$s_\lambda(x_1,\ldots,x_n;y_1,\ldots,y_d) = [S_\lambda]_{G\times T'}$$

so that $\mathring{s}_\lambda(x_1,\ldots,x_n;y_1,\ldots,y_d) = [\mathring{S}_\lambda]_{T'}$.

Equivariant cohomology

There is another group, namely $G' = GL(d)$, which naturally acts on $Gr(n,d)$. It acts transitively, so let's restrict to $T' = (\mathbb{C}^\times)^d$.
The cohomology ring $H^*_{T'}(Gr(n,d))$ is a module over $H^*_{T'}(\cdot) = \mathbb{Z}[y_1,\ldots,y_d]$. Using the same argument as before, we find that it is a quotient of $\mathbb{Z}[x_1,\ldots,x_n,y_1,\ldots,y_d]^{\mathcal{S}_n}$ by

$$c(z;V)c(z;\mathbb{C}^d/V) = \prod_{i=1}^{d}(1+y_\ell z) \qquad (\circ)$$

Define the factorial Schur polynomial

$$s_\lambda(x_1,\ldots,x_n;y_1,\ldots,y_d) = [S_\lambda]_{G\times T'}$$

so that $\mathring{s}_\lambda(x_1,\ldots,x_n;y_1,\ldots,y_d) = [\mathring{S}_\lambda]_{T'}$.

Examples

- $\lambda = \varnothing$. $S_\lambda = Mat(d, n)$, $\mathring{S}_\lambda = Gr(n, d)$, $s_\lambda = 1$.

- $\lambda = n \times (d-n)$. $S_\lambda = \left\{ \begin{pmatrix} 0 \\ \star \end{pmatrix} \begin{matrix} d-n \\ n \end{matrix} \right\}$. $\mathring{S}_\lambda = \{\mathbb{C}^d\}$.
  $s_\lambda = \prod_{i=1}^{n} \prod_{j=1}^{d-n} (x_i - y_j)$.

- $\lambda = \square$. $S_\lambda = \left\{ \begin{pmatrix} a \\ \star \end{pmatrix} \begin{matrix} n \\ d-n \end{matrix}, \det a = 0 \right\}$. $s_\lambda = \sum_{i=1}^{n} x_i - \sum_{j=1}^{n} y_j$.

- $\lambda = \ydiagram{2,1}$, $n = 2$. $S_\lambda = \left\{ \begin{pmatrix} 0 & 0 \\ a_{11} & a_{12} \\ a_{21} & a_{22} \\ \star & \star \\ \vdots & \vdots \end{pmatrix}, \det a = 0 \right\}$,
  $s_\lambda = (x_1 - y_1)(x_2 - y_1)(x_1 + x_2 - y_2 - y_3)$.

Examples

- $\lambda = \varnothing$. $S_\lambda = Mat(d,n)$, $\mathring{S}_\lambda = Gr(n,d)$, $s_\lambda = 1$.

- $\lambda = n \times (d-n)$. $S_\lambda = \left\{ \begin{pmatrix} 0 \\ \star \end{pmatrix} \begin{matrix} d-n \\ n \end{matrix} \right\}$. $\mathring{S}_\lambda = \{\mathbb{C}^d\}$.
  $s_\lambda = \prod_{i=1}^{n} \prod_{j=1}^{d-n} (x_i - y_j)$.

- $\lambda = \square$. $S_\lambda = \left\{ \begin{pmatrix} a \\ \star \end{pmatrix} \begin{matrix} n \\ d-n \end{matrix}, \det a = 0 \right\}$. $s_\lambda = \sum_{i=1}^{n} x_i - \sum_{j=1}^{n} y_j$.

- $\lambda = \ydiagram{2,1}$, $n = 2$. $S_\lambda = \left\{ \begin{pmatrix} 0 & 0 \\ a_{11} & a_{12} \\ a_{21} & a_{22} \\ \star & \star \\ \vdots & \vdots \end{pmatrix}, \det a = 0 \right\}$,
  $s_\lambda = (x_1 - y_1)(x_2 - y_1)(x_1 + x_2 - y_2 - y_3)$.

Examples

- $\lambda = \varnothing$. $S_\lambda = Mat(d,n)$, $\mathring{S}_\lambda = Gr(n,d)$, $s_\lambda = 1$.

- $\lambda = n \times (d-n)$. $S_\lambda = \left\{ \begin{pmatrix} 0 \\ \star \end{pmatrix} \begin{array}{c} d-n \\ n \end{array} \right\}$. $\mathring{S}_\lambda = \{\mathbb{C}^d\}$.
  $s_\lambda = \prod_{i=1}^{n} \prod_{j=1}^{d-n} (x_i - y_j)$.

- $\lambda = \square$. $S_\lambda = \left\{ \begin{pmatrix} a \\ \star \end{pmatrix} \begin{array}{c} n \\ d-n \end{array}, \det a = 0 \right\}$. $s_\lambda = \sum_{i=1}^{n} x_i - \sum_{j=1}^{n} y_j$.

- $\lambda = \begin{smallmatrix} \square & \square \\ \square & \end{smallmatrix}$, $n = 2$. $S_\lambda = \left\{ \begin{pmatrix} 0 & 0 \\ a_{11} & a_{12} \\ a_{21} & a_{22} \\ \star & \star \\ \vdots & \vdots \end{pmatrix}, \det a = 0 \right\}$,
  $s_\lambda = (x_1 - y_1)(x_2 - y_1)(x_1 + x_2 - y_2 - y_3)$.

Examples

- $\lambda = \varnothing$. $S_\lambda = Mat(d,n)$, $\mathring{S}_\lambda = Gr(n,d)$, $s_\lambda = 1$.

- $\lambda = n \times (d-n)$. $S_\lambda = \left\{ \begin{pmatrix} 0 \\ \star \end{pmatrix} \begin{smallmatrix} d-n \\ n \end{smallmatrix} \right\}$. $\mathring{S}_\lambda = \{\mathbb{C}^d\}$.
  $s_\lambda = \prod_{i=1}^n \prod_{j=1}^{d-n} (x_i - y_j)$.

- $\lambda = \ydiagram{1}$. $S_\lambda = \left\{ \begin{pmatrix} a \\ \star \end{pmatrix} \begin{smallmatrix} n \\ d-n \end{smallmatrix}, \det a = 0 \right\}$. $s_\lambda = \sum_{i=1}^n x_i - \sum_{j=1}^n y_j$.

- $\lambda = \ydiagram{2,1}$, $n = 2$. $S_\lambda = \left\{ \begin{pmatrix} 0 & 0 \\ a_{11} & a_{12} \\ a_{21} & a_{22} \\ \star & \star \\ \vdots & \vdots \end{pmatrix}, \det a = 0 \right\}$,
  $s_\lambda = (x_1 - y_1)(x_2 - y_1)(x_1 + x_2 - y_2 - y_3)$.

Transition

Plücker embedding

$\mathbb{P}(\cdot)$

Grassmannian $Gr(n,d)$	first quantized (free) fermions, $\Lambda^n(\mathcal{H})$, $\dim \mathcal{H} = d$
$d \to \infty$: $BU(n)$, rep theory of $GL(n)$	first quantized (free) fermions, $\Lambda^n(\mathcal{H})$, $\dim \mathcal{H} = \infty$
$d, n \to \infty$: $BU(\infty)$, rep theory of $GL(\infty)$	second quantized (free) fermions, Fock space $\mathcal{F} \sim \Lambda(\mathcal{H})$.

Fermionic operators

References in [HDR].

Fermionic operators ψ_k, $\psi^\star_k$, $k \in \mathbb{Z}$, with anti-commutation relations (Clifford algebra):

$$[\psi^\star_k, \psi_\ell]_+ = \delta_{k\ell} \qquad [\psi_k, \psi_\ell]_+ = [\psi^\star_k, \psi^\star_\ell]_+ = 0$$

Generating series:

$$\psi(z) = \sum_{k\in\mathbb{Z}} \psi_k\, z^{-k}, \qquad \psi^\star(z) = \sum_{k\in\mathbb{Z}} \psi^\star_k\, z^k$$

Fermionic operators

References in [HDR].

Fermionic operators ψ_k, $\psi^\star_k$, $k \in \mathbb{Z}$, with anti-commutation relations (Clifford algebra):

$$[\psi^\star_k, \psi_\ell]_+ = \delta_{k\ell} \qquad [\psi_k, \psi_\ell]_+ = [\psi^\star_k, \psi^\star_\ell]_+ = 0$$

Generating series:

$$\psi(z) = \sum_{k\in\mathbb{Z}} \psi_k\, z^{-k}, \qquad \psi^\star(z) = \sum_{k\in\mathbb{Z}} \psi^\star_k\, z^k$$

Fermionic operators

References in [HDR].

Fermionic operators ψ_k, $\psi^\star_k$, $k \in \mathbb{Z}$, with anti-commutation relations (Clifford algebra):

$$[\psi^\star_k, \psi_\ell]_+ = \delta_{k\ell} \qquad [\psi_k, \psi_\ell]_+ = [\psi^\star_k, \psi^\star_\ell]_+ = 0$$

Generating series:

$$\psi(z) = \sum_{k\in\mathbb{Z}} \psi_k \, z^{-k}, \qquad \psi^\star(z) = \sum_{k\in\mathbb{Z}} \psi^\star_k \, z^k$$

Fock space

Start from $\mathbb{C}^2 = \langle \bullet, \circ \rangle$. Then the Fock space $\mathcal{F}$ is the subspace of $(\mathbb{C}^2)^{\otimes \mathbb{Z}}$ generated by the $(a_k) \in \{\bullet, \circ\}^{\mathbb{Z}}$ such that $a_k = \bullet$ (resp. $\circ$) sufficient far to the left (resp. right).

ψ_k and $\psi_k^\star$ act as annihilation/creation operators:

$$\psi_k \,|\cdots \underset{k}{\circ} \cdots\rangle = 0 \qquad \psi_k \,|\cdots \underset{k}{\bullet} \cdots\rangle = (-1)^{\#\text{ particles to the right}} \,|\cdots \underset{k}{\circ} \cdots\rangle$$

$$\psi_k^\star \,|\cdots \underset{k}{\bullet} \cdots\rangle = 0 \qquad \psi_k^\star \,|\cdots \underset{k}{\circ} \cdots\rangle = (-1)^{\#\text{ particles to the right}} \,|\cdots \underset{k}{\bullet} \cdots\rangle$$

Example:

$$\psi_1^\star \cdots \underset{-3}{\bullet} \underset{-2}{\bullet} \underset{-1}{\bullet} \underset{0}{\bullet} \,\Big|\, \underset{1}{\circ} \underset{2}{\circ} \underset{3}{\bullet} \underset{4}{\circ} \cdots =$$

$$- \cdots \underset{-3}{\bullet} \underset{-2}{\bullet} \underset{-1}{\bullet} \underset{0}{\bullet} \,\Big|\, \underset{1}{\bullet} \underset{2}{\circ} \underset{3}{\bullet} \underset{4}{\circ} \cdots$$

Fock space

Start from $\mathbb{C}^2 = \langle \bullet, \circ \rangle$. Then the Fock space $\mathcal{F}$ is the subspace of $(\mathbb{C}^2)^{\otimes \mathbb{Z}}$ generated by the $(a_k) \in \{\bullet, \circ\}^{\mathbb{Z}}$ such that $a_k = \bullet$ (resp. $\circ$) sufficient far to the left (resp. right).

ψ_k and $\psi_k^\star$ act as annihilation/creation operators:

$$\psi_k \left| \cdots \underset{k}{\circ} \cdots \right\rangle = 0 \qquad \psi_k \left| \cdots \underset{k}{\bullet} \cdots \right\rangle = (-1)^{\#\text{ particles to the right}} \left| \cdots \underset{k}{\circ} \cdots \right\rangle$$

$$\psi_k^\star \left| \cdots \underset{k}{\bullet} \cdots \right\rangle = 0 \qquad \psi_k^\star \left| \cdots \underset{k}{\circ} \cdots \right\rangle = (-1)^{\#\text{ particles to the right}} \left| \cdots \underset{k}{\bullet} \cdots \right\rangle$$

Example:

$$\psi_1^\star \begin{array}{cccc|cccc} \cdots\bullet & \bullet & \bullet & \bullet & \circ & \circ & \bullet & \circ\cdots \\ -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 \end{array} =$$

$$- \begin{array}{cccc|cccc} \cdots\bullet & \bullet & \bullet & \bullet & \bullet & \circ & \bullet & \circ\cdots \\ -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 \end{array}$$

Fock space

Start from $\mathbb{C}^2 = \langle \bullet, \circ \rangle$. Then the Fock space $\mathcal{F}$ is the subspace of $(\mathbb{C}^2)^{\otimes \mathbb{Z}}$ generated by the $(a_k) \in \{\bullet, \circ\}^{\mathbb{Z}}$ such that $a_k = \bullet$ (resp. $\circ$) sufficient far to the left (resp. right).

ψ_k and $\psi_k^\star$ act as annihilation/creation operators:

$$\psi_k \,|\cdots \underset{k}{\circ} \cdots\rangle = 0 \qquad \psi_k \,|\cdots \underset{k}{\bullet} \cdots\rangle = (-1)^{\#\text{ particles to the right}} |\cdots \underset{k}{\circ} \cdots\rangle$$

$$\psi_k^\star \,|\cdots \underset{k}{\bullet} \cdots\rangle = 0 \qquad \psi_k^\star \,|\cdots \underset{k}{\circ} \cdots\rangle = (-1)^{\#\text{ particles to the right}} |\cdots \underset{k}{\bullet} \cdots\rangle$$

Example:

$$\psi_1^\star \ \cdots \underset{-3}{\bullet} \underset{-2}{\bullet} \underset{-1}{\bullet} \underset{0}{\bullet} \,\vert\, \underset{1}{\circ} \underset{2}{\circ} \underset{3}{\bullet} \underset{4}{\circ} \cdots =$$

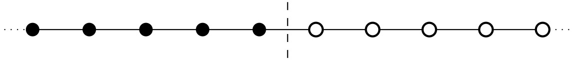
$$- \ \cdots \underset{-3}{\bullet} \underset{-2}{\bullet} \underset{-1}{\bullet} \underset{0}{\bullet} \,\vert\, \underset{1}{\bullet} \underset{2}{\circ} \underset{3}{\bullet} \underset{4}{\circ} \cdots$$

Vacua

Define the vacuum $|0\rangle$ as the only state (up to normalization) such that

$$\psi_k |0\rangle = 0 \quad k > 0, \qquad \psi_k^\star |0\rangle = 0 \quad k \leq 0$$

Explicitly,

$|0\rangle =$ 

More generally, one can define vacua $|\ell\rangle$, $\ell \in \mathbb{Z}$ (Fermi sea filled up to ℓ) by

$$|\ell\rangle = S^\ell |0\rangle$$

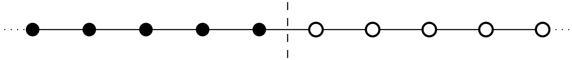
where S is the shift operator $S(a_k) = (a_{k-1})$.

Vacua

Define the vacuum $|0\rangle$ as the only state (up to normalization) such that

$$\psi_k |0\rangle = 0 \quad k > 0, \qquad \psi^\star_k |0\rangle = 0 \quad k \leq 0$$

Explicitly,

$|0\rangle =$ 

More generally, one can define vacua $|\ell\rangle$, $\ell \in \mathbb{Z}$ (Fermi sea filled up to ℓ) by

$$|\ell\rangle = S^\ell |0\rangle$$

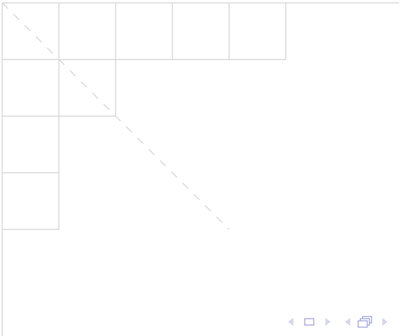
where S is the shift operator $S(a_k) = (a_{k-1})$.

Relation to Young diagrams

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, define

$$|\lambda\rangle = \psi^\star_{\lambda_1}\psi^\star_{\lambda_2-1}\cdots\psi^\star_{\lambda_n-n+1}\,|-n\rangle$$

Pictorially, if e.g. $\lambda = (5, 2, 1, 1)$,

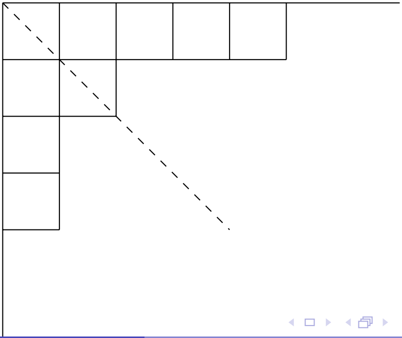


Relation to Young diagrams

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, define

$$|\lambda\rangle = \psi^\star_{\lambda_1}\psi^\star_{\lambda_2-1}\cdots\psi^\star_{\lambda_n-n+1}\,|-n\rangle$$

Pictorially, if e.g. $\lambda = (5, 2, 1, 1)$,

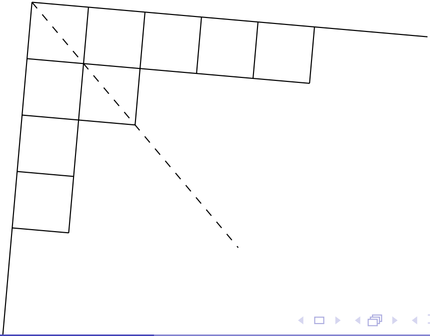


Relation to Young diagrams

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, define

$$|\lambda\rangle = \psi^\star_{\lambda_1} \psi^\star_{\lambda_2 - 1} \cdots \psi^\star_{\lambda_n - n + 1} |-n\rangle$$

Pictorially, if e.g. $\lambda = (5, 2, 1, 1)$,

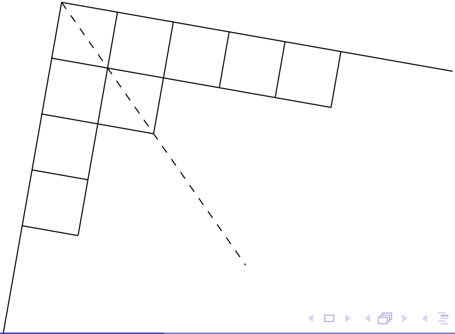


Relation to Young diagrams

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, define

$$|\lambda\rangle = \psi^\star_{\lambda_1} \psi^\star_{\lambda_2 - 1} \cdots \psi^\star_{\lambda_n - n + 1} \, |-n\rangle$$

Pictorially, if e.g. $\lambda = (5, 2, 1, 1)$,

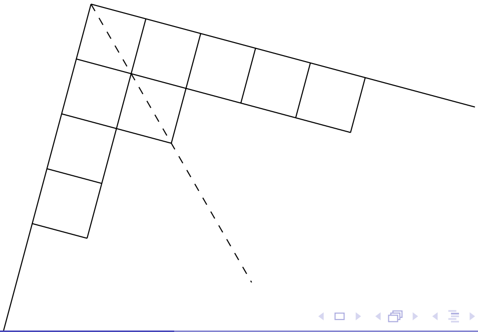


Relation to Young diagrams

Given a partition $\lambda = (\lambda_1, \dots, \lambda_n)$, define

$$|\lambda\rangle = \psi^\star_{\lambda_1} \psi^\star_{\lambda_2 - 1} \cdots \psi^\star_{\lambda_n - n + 1} \, |-n\rangle$$

Pictorially, if e.g. $\lambda = (5, 2, 1, 1)$,

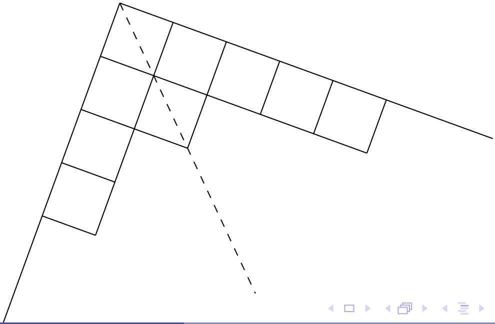


Relation to Young diagrams

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, define

$$|\lambda\rangle = \psi^\star_{\lambda_1} \psi^\star_{\lambda_2 - 1} \cdots \psi^\star_{\lambda_n - n + 1} \, |{-n}\rangle$$

Pictorially, if e.g. $\lambda = (5, 2, 1, 1)$,

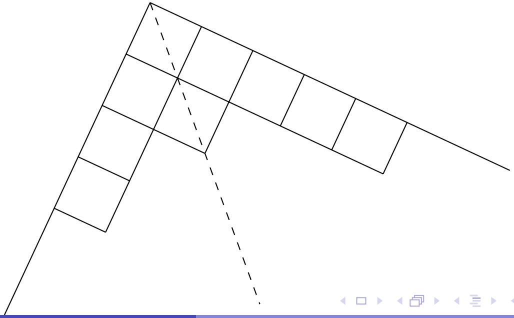


Relation to Young diagrams

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, define

$$|\lambda\rangle = \psi^\star_{\lambda_1}\psi^\star_{\lambda_2-1}\cdots\psi^\star_{\lambda_n-n+1}\,|-n\rangle$$

Pictorially, if e.g. $\lambda = (5, 2, 1, 1)$,

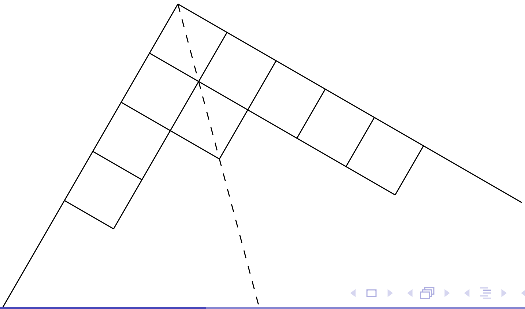


Relation to Young diagrams

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, define

$$|\lambda\rangle = \psi^\star_{\lambda_1} \psi^\star_{\lambda_2 - 1} \cdots \psi^\star_{\lambda_n - n + 1} \, |-n\rangle$$

Pictorially, if e.g. $\lambda = (5, 2, 1, 1)$,

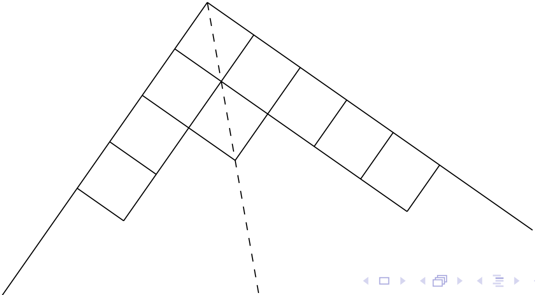


Relation to Young diagrams

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, define

$$|\lambda\rangle = \psi^\star_{\lambda_1} \psi^\star_{\lambda_2 - 1} \cdots \psi^\star_{\lambda_n - n + 1} |{-n}\rangle$$

Pictorially, if e.g. $\lambda = (5, 2, 1, 1)$,

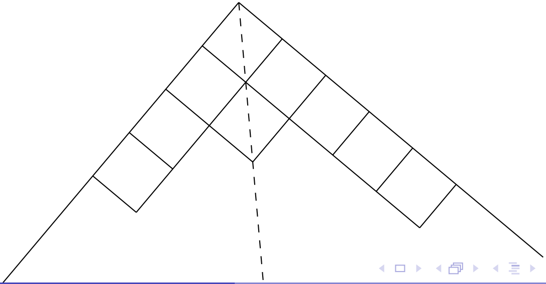


Relation to Young diagrams

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, define

$$|\lambda\rangle = \psi^\star_{\lambda_1} \psi^\star_{\lambda_2 - 1} \cdots \psi^\star_{\lambda_n - n + 1} |-n\rangle$$

Pictorially, if e.g. $\lambda = (5, 2, 1, 1)$,

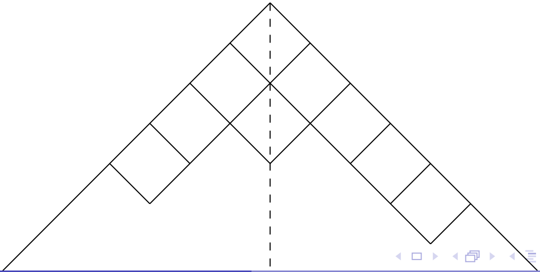


Relation to Young diagrams

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, define

$$|\lambda\rangle = \psi^\star_{\lambda_1} \psi^\star_{\lambda_2 - 1} \cdots \psi^\star_{\lambda_n - n + 1} \,|{-n}\rangle$$

Pictorially, if e.g. $\lambda = (5, 2, 1, 1)$,

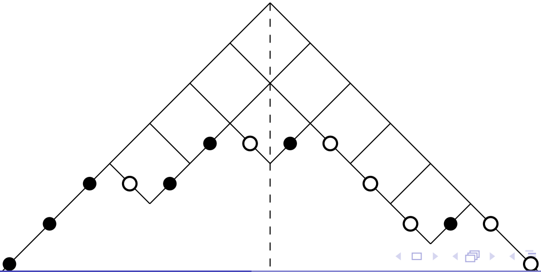


Relation to Young diagrams

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, define

$$|\lambda\rangle = \psi^\star_{\lambda_1} \psi^\star_{\lambda_2 - 1} \cdots \psi^\star_{\lambda_n - n + 1} \, |{-n}\rangle$$

Pictorially, if e.g. $\lambda = (5, 2, 1, 1)$,

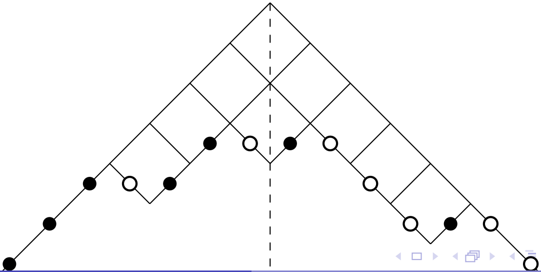


Relation to Young diagrams

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, define

$$|\lambda\rangle = \psi^\star_{\lambda_1} \psi^\star_{\lambda_2 - 1} \cdots \psi^\star_{\lambda_n - n + 1} \, |-n\rangle$$

Pictorially, if e.g. $\lambda = (5, 2, 1, 1)$,

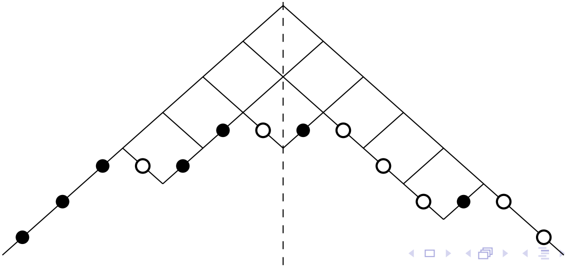


Relation to Young diagrams

Given a partition $\lambda = (\lambda_1, \dots, \lambda_n)$, define

$$|\lambda\rangle = \psi^\star_{\lambda_1}\psi^\star_{\lambda_2-1}\cdots\psi^\star_{\lambda_n-n+1}\,|-n\rangle$$

Pictorially, if e.g. $\lambda = (5, 2, 1, 1)$,

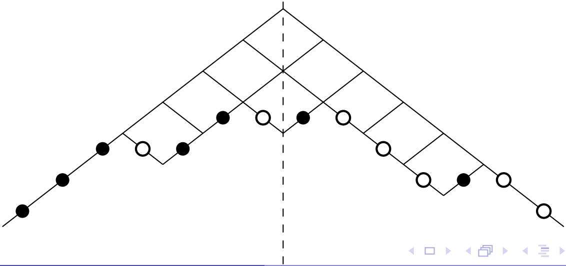


Relation to Young diagrams

Given a partition $\lambda = (\lambda_1, \dots, \lambda_n)$, define

$$|\lambda\rangle = \psi^\star_{\lambda_1}\psi^\star_{\lambda_2-1}\cdots\psi^\star_{\lambda_n-n+1}\,|-n\rangle$$

Pictorially, if e.g. $\lambda = (5, 2, 1, 1)$,

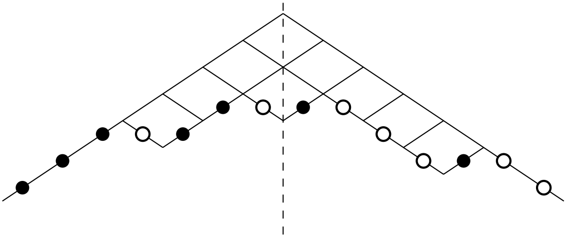


Relation to Young diagrams

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, define

$$|\lambda\rangle = \psi^\star_{\lambda_1}\psi^\star_{\lambda_2-1}\cdots\psi^\star_{\lambda_n-n+1}\,|-n\rangle$$

Pictorially, if e.g. $\lambda = (5,2,1,1)$,

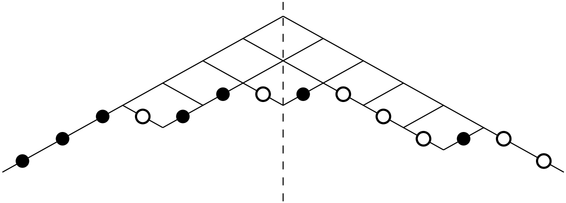


Relation to Young diagrams

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, define

$$|\lambda\rangle = \psi^\star_{\lambda_1}\psi^\star_{\lambda_2-1}\cdots\psi^\star_{\lambda_n-n+1}\,|-n\rangle$$

Pictorially, if e.g. $\lambda = (5, 2, 1, 1)$,

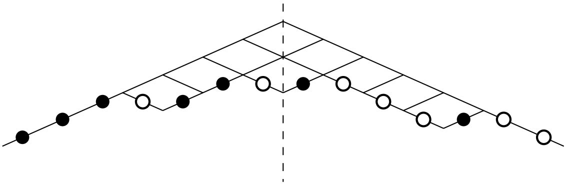


Relation to Young diagrams

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, define

$$|\lambda\rangle = \psi^\star_{\lambda_1} \psi^\star_{\lambda_2 - 1} \cdots \psi^\star_{\lambda_n - n + 1} |{-n}\rangle$$

Pictorially, if e.g. $\lambda = (5, 2, 1, 1)$,

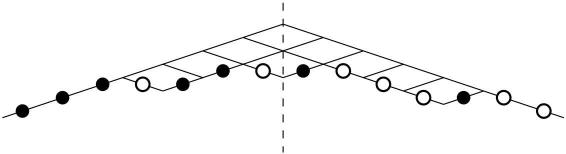


Relation to Young diagrams

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, define

$$|\lambda\rangle = \psi^\star_{\lambda_1} \psi^\star_{\lambda_2 - 1} \cdots \psi^\star_{\lambda_n - n + 1} \, |{-n}\rangle$$

Pictorially, if e.g. $\lambda = (5, 2, 1, 1)$,

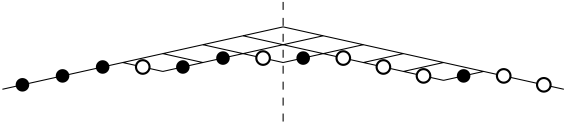


Relation to Young diagrams

Given a partition $\lambda = (\lambda_1, \dots, \lambda_n)$, define

$$|\lambda\rangle = \psi^\star_{\lambda_1}\psi^\star_{\lambda_2-1}\cdots\psi^\star_{\lambda_n-n+1}\,|-n\rangle$$

Pictorially, if e.g. $\lambda = (5,2,1,1)$,

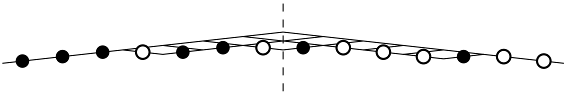


Relation to Young diagrams

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, define

$$|\lambda\rangle = \psi^\star_{\lambda_1} \psi^\star_{\lambda_2 - 1} \cdots \psi^\star_{\lambda_n - n + 1} \,|-n\rangle$$

Pictorially, if e.g. $\lambda = (5, 2, 1, 1)$,

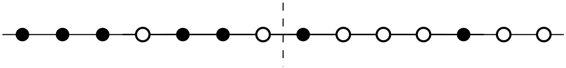


Relation to Young diagrams

Given a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, define

$$|\lambda\rangle = \psi^\star_{\lambda_1} \psi^\star_{\lambda_2 - 1} \cdots \psi^\star_{\lambda_n - n + 1} \, |-n\rangle$$

Pictorially, if e.g. $\lambda = (5, 2, 1, 1)$,

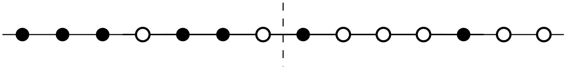


Relation to Young diagrams

Given a partition $\lambda = (\lambda_1, \dots, \lambda_n)$, define

$$|\lambda\rangle = \psi^\star_{\lambda_1} \psi^\star_{\lambda_2 - 1} \cdots \psi^\star_{\lambda_n - n + 1} \, |-n\rangle$$

Pictorially, if e.g. $\lambda = (5, 2, 1, 1)$,



More generally, one can define $|\lambda; \ell\rangle = S^\ell \, |\lambda\rangle$.

Introduce the normal ordering with respect to the vacuum $|0\rangle$:

$$:\psi_j^\star \psi_k: \, = - :\psi_k \psi_j^\star: \, = \begin{cases} \psi_j^\star \psi_k & j > 0 \\ -\psi_k \psi_j^\star & j \le 0 \end{cases}$$

The current

$$j(z) = \, :\psi^\star(z)\psi(z): \, = \sum_{n\in\mathbb{Z}} J_n z^{-n-1} \tag{1}$$

with $J_n = \sum_r :\psi^\star_{r-n}\psi_r:$ forms a $\widehat{\mathfrak{gl}(1)}$ (Heisenberg) Lie algebra:

$$[J_m, J_n] = m\delta_{m,-n}$$

$\mathcal{F} = \bigoplus_{\ell\in\mathbb{Z}} \mathcal{F}_\ell$ where $\mathcal{F}_\ell$ is the subspace where $J_0 = \sum_k :\psi_k^\star \psi_k:$ has eigenvalue ℓ, with basis the $|\lambda;\ell\rangle$.

Introduce the normal ordering with respect to the vacuum $|0\rangle$:

$$:\psi_j^\star \psi_k: \, = - :\psi_k \psi_j^\star: \, = \begin{cases} \psi_j^\star \psi_k & j > 0 \\ -\psi_k \psi_j^\star & j \le 0 \end{cases}$$

The current

$$j(z) = \, :\psi^\star(z)\psi(z): \, = \sum_{n\in\mathbb{Z}} J_n z^{-n-1} \qquad (1)$$

with $J_n = \sum_r :\psi_{r-n}^\star \psi_r:$ forms a $\widehat{\mathfrak{gl}(1)}$ (Heisenberg) Lie algebra:

$$[J_m, J_n] = m\delta_{m,-n}$$

$\mathcal{F} = \bigoplus_{\ell\in\mathbb{Z}} \mathcal{F}_\ell$ where $\mathcal{F}_\ell$ is the subspace where $J_0 = \sum_k :\psi_k^\star \psi_k:$ has eigenvalue ℓ, with basis the $|\lambda;\ell\rangle$.

Introduce the normal ordering with respect to the vacuum $|0\rangle$:

$$:\psi_j^\star \psi_k: \, = - :\psi_k \psi_j^\star: \, = \begin{cases} \psi_j^\star \psi_k & j > 0 \\ -\psi_k \psi_j^\star & j \le 0 \end{cases}$$

The current

$$j(z) = \, :\psi^\star(z)\psi(z): \, = \sum_{n\in\mathbb{Z}} J_n z^{-n-1} \tag{1}$$

with $J_n = \sum_r :\psi^\star_{r-n}\psi_r:$ forms a $\widehat{\mathfrak{gl}(1)}$ (Heisenberg) Lie algebra:

$$[J_m, J_n] = m\delta_{m,-n}$$

$\mathcal{F} = \bigoplus_{\ell\in\mathbb{Z}} \mathcal{F}_\ell$ where $\mathcal{F}_\ell$ is the subspace where $J_0 = \sum_k :\psi_k^\star \psi_k:$ has eigenvalue ℓ, with basis the $|\lambda;\ell\rangle$.

Hamiltonians

The subalgebra generated by the J_k, $k > 0$, is commutative; a generic element can be parameterized as

$$H[t] = \sum_{k=1}^{\infty} t_k J_k$$

where $t = (t_1, \ldots, t_k, \ldots)$ is a set of "times".
$H[t]$ is quadratic in the fermionic fields $\rightarrow$ free fermion Hamiltonian.

Similarly, we can define $H^\star[t] = \sum_{k=1}^{\infty} t_k J_{-k}$.

Define a scalar product $\langle \cdot | \cdot \rangle$ such that the standard basis is orthonormal. Then ψ_k and $\psi_k^\star$ are adjoint of each other; $J_k^\star = J_{-k}$.

Hamiltonians

The subalgebra generated by the J_k, $k > 0$, is commutative; a generic element can be parameterized as

$$H[t] = \sum_{k=1}^{\infty} t_k J_k$$

where $t = (t_1, \ldots, t_k, \ldots)$ is a set of "times".
$H[t]$ is quadratic in the fermionic fields $\rightarrow$ free fermion Hamiltonian.

Similarly, we can define $H^\star[t] = \sum_{k=1}^{\infty} t_k J_{-k}$.

Define a scalar product $\langle \cdot | \cdot \rangle$ such that the standard basis is orthonormal.
Then ψ_k and $\psi_k^\star$ are adjoint of each other; $J_k^\star = J_{-k}$.

Hamiltonians

The subalgebra generated by the J_k, $k > 0$, is commutative; a generic element can be parameterized as

$$H[t] = \sum_{k=1}^{\infty} t_k J_k$$

where $t = (t_1, \ldots, t_k, \ldots)$ is a set of "times".
$H[t]$ is quadratic in the fermionic fields $\rightarrow$ free fermion Hamiltonian.

Similarly, we can define $H^\star[t] = \sum_{k=1}^{\infty} t_k J_{-k}$.

Define a scalar product $\langle \cdot | \cdot \rangle$ such that the standard basis is orthonormal.
Then ψ_k and $\psi_k^\star$ are adjoint of each other; $J_k^\star = J_{-k}$.

Alternate definition of s_λ

Define

$$s_\lambda[t] = \langle 0|\, e^{H[t]}\, |\lambda\rangle$$

Example ($\lambda = \square$)

$$s_\square[t] = t_1$$

$s_\lambda[t]$ is a polynomial in the infinite set of variables t_k:

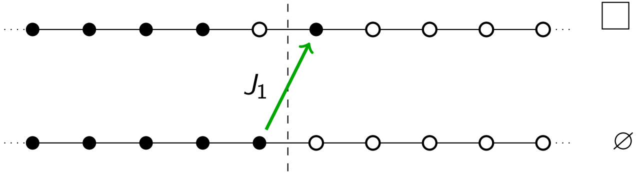
$$s_\lambda[t] = e^{\sum_{k\geq 1} t_k \sum_{i=1}^n z_i^k} \prod_{1\leq i<j\leq n} (z_i - z_j)\Big|_{z_1^{n+\lambda_1-1} z_2^{n+\lambda_2-2} \cdots z_n^{\lambda_n}}$$

Alternate definition of s_λ

Define

$$s_\lambda[t] = \langle 0| \, e^{H[t]} \, |\lambda\rangle$$

Example ($\lambda = \square$)



$$s_\square[t] = t_1$$

$s_\lambda[t]$ is a polynomial in the infinite set of variables t_k:

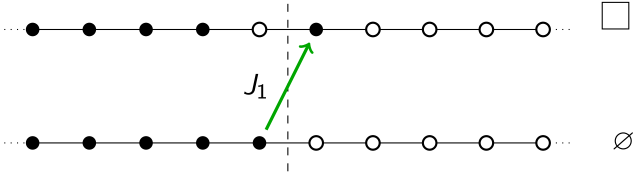
$$s_\lambda[t] = e^{\sum_{k\geq 1} t_k \sum_{i=1}^n z_i^k} \prod_{1\leq i<j\leq n} (z_i - z_j)\Big|_{z_1^{n+\lambda_1-1} z_2^{n+\lambda_2-2} \cdots z_n^{\lambda_n}}$$

Alternate definition of s_λ

Define

$$s_\lambda[t] = \langle 0|\, e^{H[t]}\, |\lambda\rangle$$

Example ($\lambda = \square$)

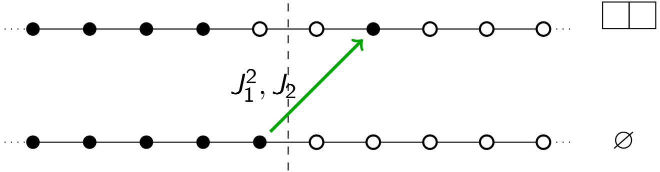


$$s_\square[t] = t_1$$

$s_\lambda[t]$ is a polynomial in the infinite set of variables t_k:

$$s_\lambda[t] = e^{\sum_{k\geq 1} t_k \sum_{i=1}^n z_i^k} \prod_{1\leq i<j\leq n} (z_i - z_j)\Big|_{z_1^{n+\lambda_1-1} z_2^{n+\lambda_2-2} \cdots z_n^{\lambda_n}}$$

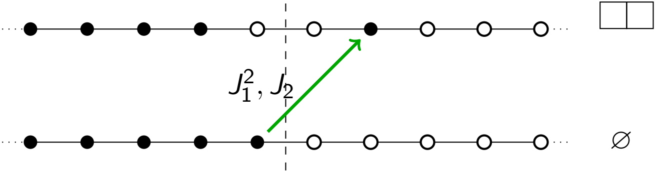
More examples



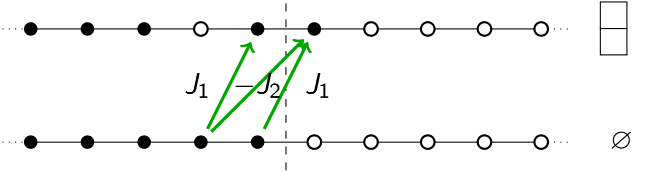
$$s_{\square\square}[t] = \frac{1}{2}t_1^2 + t_2$$

$$s_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}[t] = \frac{1}{2}t_1^2 - t_2$$

More examples

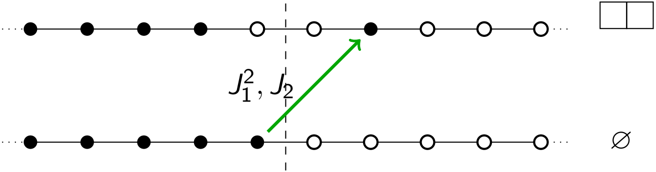


$$s_{\square\square}[t] = \frac{1}{2} t_1^2 + t_2$$

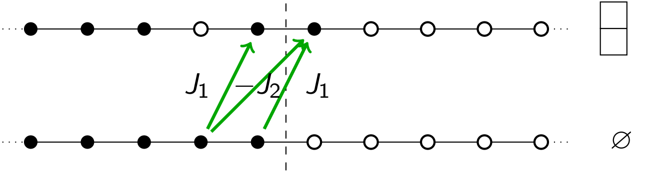


$$s_{\substack{\square\\ \square}}[t] = \frac{1}{2} t_1^2 - t_2$$

More examples



$$s_{\Box\Box}[t] = \frac{1}{2}t_1^2 + t_2$$



$$s_{\substack{\Box\\\Box}}[t] = \frac{1}{2}t_1^2 - t_2$$

Miwa transformation

This suggests the transformation

$$t_k = \frac{1}{k} p_k = \frac{1}{k} \sum_{i=1}^{n} x_i^k$$

which implies

$$e^{H[t]} = \prod_{i=1}^{n} e^{\varphi_+(x_i^{-1})} \qquad \varphi_+(z) = \sum_{k \geq 1} \frac{z^{-k}}{k} J_k$$

Similarly we have

$$e^{H^\star[t]} = \prod_{i=1}^{n} e^{\varphi_-(x_i)} \qquad \varphi_-(z) = \sum_{k \geq 1} \frac{z^k}{k} J_{-k}$$

Miwa transformation

This suggests the transformation

$$t_k = \frac{1}{k} p_k = \frac{1}{k} \sum_{i=1}^{n} x_i^k$$

which implies

$$e^{H[t]} = \prod_{i=1}^{n} e^{\varphi_+(x_i^{-1})} \qquad \varphi_+(z) = \sum_{k \geq 1} \frac{z^{-k}}{k} J_k$$

Similarly we have

$$e^{H^\star[t]} = \prod_{i=1}^{n} e^{\varphi_-(x_i)} \qquad \varphi_-(z) = \sum_{k \geq 1} \frac{z^k}{k} J_{-k}$$

Miwa transformation

This suggests the transformation

$$t_k = \frac{1}{k} p_k = \frac{1}{k} \sum_{i=1}^{n} x_i^k$$

which implies

$$e^{H[t]} = \prod_{i=1}^{n} e^{\varphi_+(x_i^{-1})} \qquad \varphi_+(z) = \sum_{k \geq 1} \frac{z^{-k}}{k} J_k$$

Similarly we have

$$e^{H^\star[t]} = \prod_{i=1}^{n} e^{\varphi_-(x_i)} \qquad \varphi_-(z) = \sum_{k \geq 1} \frac{z^{k}}{k} J_{-k}$$

Explicit formula

We then compute:

$$
\begin{aligned}
s_\lambda[t] &= e^{\sum_{k\geq 1} t_k \sum_{i=1}^n z_i^k} \prod_{1\leq i<j\leq n} (z_i - z_j)\Big|_{z_1^{n+\lambda_1-1} z_2^{n+\lambda_2-2} \ldots z_n^{\lambda_n}} \\
&= \prod_{i,j=1}^{n} \frac{1}{1 - z_i x_j} \prod_{1\leq i<j\leq n} (z_i - z_j)\Big|_{z_1^{n+\lambda_1-1} z_2^{n+\lambda_2-2} \ldots z_n^{\lambda_n}} \\
&= \frac{\det\limits_{1\leq i,j\leq n} \dfrac{1}{1 - x_i z_j}}{\prod_{i<j}(x_i - x_j)}\Big|_{z_1^{n+\lambda_1-1} z_2^{n+\lambda_2-2} \ldots z_n^{\lambda_n}} \\
&= \frac{\det_{1\leq i,j\leq n}(x_i^{\lambda_j+n-j})}{\prod_{i<j}(x_i - x_j)}
\end{aligned}
$$

Bosonization

Comparing the previous formula with the direct calculation:

$$\langle 0|\,\psi(z_1)\cdots\psi(z_n)\,|\lambda; n\rangle = \det(z_i^{-(\lambda_j+n-j)})$$

We are led to the identification

$$\psi(z) = S^{-1} z^{-J_0} e^{-\varphi_-(z)} e^{\varphi_+(z)}$$
$$= \text{“}e^{-\varphi(z)}\text{”}$$

with $\varphi(z) = -\frac{\partial}{\partial J_0} + J_0 \log z + \varphi_-(z) - \varphi_+(z)$.

$$\psi^\star(z) = S\, z^{J_0} e^{\varphi_-(z)} e^{-\varphi_+(z)}$$
$$= \text{“}e^{\varphi(z)}\text{”}$$

Bosonization

Comparing the previous formula with the direct calculation:

$$\langle 0|\,\psi(z_1)\cdots\psi(z_n)\,|\lambda; n\rangle = \det(z_i^{-(\lambda_j+n-j)})$$

We are led to the identification

$$\begin{aligned}\psi(z) &= S^{-1} z^{-J_0} e^{-\varphi_-(z)} e^{\varphi_+(z)}\\ &= \text{“}e^{-\varphi(z)}\text{”}\end{aligned}$$

with $\varphi(z) = -\frac{\partial}{\partial J_0} + J_0 \log z + \varphi_-(z) - \varphi_+(z)$.

$$\begin{aligned}\psi^\star(z) &= S\, z^{J_0} e^{\varphi_-(z)} e^{-\varphi_+(z)}\\ &= \text{“}e^{\varphi(z)}\text{”}\end{aligned}$$

Bosonization

Comparing the previous formula with the direct calculation:

$$\langle 0|\, \psi(z_1)\cdots\psi(z_n)\,|\lambda; n\rangle = \det(z_i^{-(\lambda_j+n-j)})$$

We are led to the identification

$$\begin{aligned}\psi(z) &= S^{-1} z^{-J_0} e^{-\varphi_-(z)} e^{\varphi_+(z)}\\ &= \text{“}e^{-\varphi(z)}\text{”}\end{aligned}$$

with $\varphi(z) = -\frac{\partial}{\partial J_0} + J_0 \log z + \varphi_-(z) - \varphi_+(z)$.

$$\begin{aligned}\psi^\star(z) &= S\, z^{J_0} e^{\varphi_-(z)} e^{-\varphi_+(z)}\\ &= \text{“}e^{\varphi(z)}\text{”}\end{aligned}$$

Schur–Weyl revisited

Admitting for now that $s_\lambda[t] = s_\lambda(x_1, \ldots, x_n, \ldots)$ with $t_k = \frac{1}{k} \sum_i x_i^k$, we expand the exponential:

$$s_\lambda[t] = \sum_{\alpha_k \geq 0} \prod_{k \geq 1} \frac{1}{\alpha_k!} t_k^{\alpha_k} \langle 0 | \prod_{k \geq 1} J_k^{\alpha_k} | \lambda \rangle$$

Comparing with , we conclude

$$\chi_\lambda(\alpha) = \langle 0 | \prod_{k \geq 1} J_k^{\alpha_k} | \lambda \rangle$$

(a form of the Murnaghan–Natayama formula for characters of the symmetric group)

Schur–Weyl revisited

Admitting for now that $s_\lambda[t] = s_\lambda(x_1, \ldots, x_n, \ldots)$ with $t_k = \frac{1}{k} \sum_i x_i^k$, we expand the exponential:

$$s_\lambda[t] = \sum_{\alpha_k \geq 0} \prod_{k \geq 1} \frac{1}{\alpha_k!} t_k^{\alpha_k} \langle 0| \prod_{k \geq 1} J_k^{\alpha_k} |\lambda\rangle$$

Comparing with ▸, we conclude

$$\chi_\lambda(\alpha) = \langle 0| \prod_{k \geq 1} J_k^{\alpha_k} |\lambda\rangle$$

(a form of the Murnaghan–Natayama formula for characters of the symmetric group)

Vertex operators as transfer matrices

Insert a basis of $\mathcal{F}_0$:

$$s_\lambda[t] = \sum_{0=\lambda_0,\lambda_1,\ldots,\lambda_n=\lambda} \prod_{i=1}^{n} \langle\lambda_{i-1}|\, T(x_i)\, |\lambda_i\rangle$$

with $T(x) = e^{\phi_+(x^{-1})} = e^{\sum_{k\geq 1} \frac{x^k}{k} J_k}$.

Moreover, if one defines $s_{\lambda/\mu}[t] = \langle\mu|\, e^{H[t]}\, |\lambda\rangle$, then is automatically satisfied, and one has

$$s_{\lambda/\mu}[t] = \sum_{\mu=\lambda_0,\lambda_1,\ldots,\lambda_n=\lambda} \prod_{i=1}^{n} \langle\lambda_{i-1}|\, T(x_i)\, |\lambda_i\rangle$$

All we need to do is compute $\langle\mu|\, T(x)\, |\lambda\rangle$ (i.e., skew-Schur function with a single argument!)

Vertex operators as transfer matrices

Insert a basis of $\mathcal{F}_0$:

$$s_\lambda[t] = \sum_{0=\lambda_0,\lambda_1,\ldots,\lambda_n=\lambda} \prod_{i=1}^{n} \langle\lambda_{i-1}|\, T(x_i)\, |\lambda_i\rangle$$

with $T(x) = e^{\phi_+(x^{-1})} = e^{\sum_{k\geq 1} \frac{x^k}{k} J_k}$.

Moreover, if one defines $s_{\lambda/\mu}[t] = \langle\mu|\, e^{H[t]}\, |\lambda\rangle$, then ▸ is automatically satisfied, and one has

$$s_{\lambda/\mu}[t] = \sum_{\mu=\lambda_0,\lambda_1,\ldots,\lambda_n=\lambda} \prod_{i=1}^{n} \langle\lambda_{i-1}|\, T(x_i)\, |\lambda_i\rangle$$

All we need to do is compute $\langle\mu|\, T(x)\, |\lambda\rangle$ (i.e., skew-Schur function with a single argument!)

Vertex operators as transfer matrices

Insert a basis of $\mathcal{F}_0$:

$$s_\lambda[t] = \sum_{0=\lambda_0,\lambda_1,\ldots,\lambda_n=\lambda} \prod_{i=1}^{n} \langle\lambda_{i-1}|\, T(x_i)\, |\lambda_i\rangle$$

with $T(x) = e^{\phi_+(x^{-1})} = e^{\sum_{k\geq 1} \frac{x^k}{k} J_k}$.

Moreover, if one defines $s_{\lambda/\mu}[t] = \langle\mu|\, e^{H[t]}\, |\lambda\rangle$, then is automatically satisfied, and one has

$$s_{\lambda/\mu}[t] = \sum_{\mu=\lambda_0,\lambda_1,\ldots,\lambda_n=\lambda} \prod_{i=1}^{n} \langle\lambda_{i-1}|\, T(x_i)\, |\lambda_i\rangle$$

All we need to do is compute $\langle\mu|\, T(x)\, |\lambda\rangle$ (i.e., skew-Schur function with a single argument!)

Digression: particle/hole duality

Let P be the involution which implements left-right reflection and $\bullet \leftrightarrow \circ$.

$$P\,|\lambda\rangle = |\lambda'\rangle$$

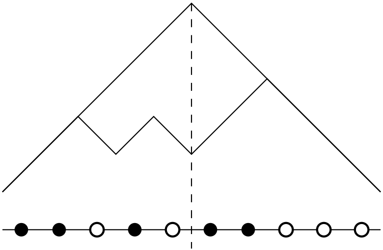
where λ' is the conjugate partition of λ.

Digression: particle/hole duality

Let P be the involution which implements left-right reflection and $\bullet \leftrightarrow \circ$.

$$P\left|\lambda\right\rangle = \left|\lambda'\right\rangle$$

where λ' is the conjugate partition of λ.

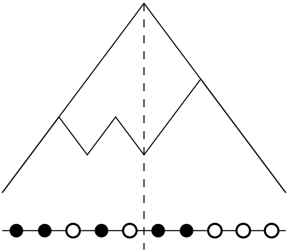


Digression: particle/hole duality

Let P be the involution which implements left-right reflection and $\bullet \leftrightarrow \circ$.

$$P\left|\lambda\right\rangle = \left|\lambda'\right\rangle$$

where λ' is the conjugate partition of λ.

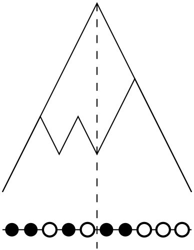


Digression: particle/hole duality

Let P be the involution which implements left-right reflection and $\bullet \leftrightarrow \circ$.

$$P\,|\lambda\rangle = |\lambda'\rangle$$

where λ' is the conjugate partition of λ.

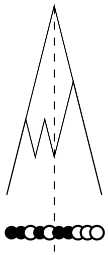


Digression: particle/hole duality

Let P be the involution which implements left-right reflection and $\bullet \leftrightarrow \circ$.

$$P\,|\lambda\rangle = |\lambda'\rangle$$

where λ' is the conjugate partition of λ.



Digression: particle/hole duality

Let P be the involution which implements left-right reflection and $\bullet \leftrightarrow \circ$.

$$P\left|\lambda\right\rangle = \left|\lambda'\right\rangle$$

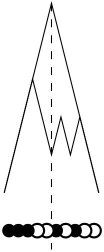
where λ' is the conjugate partition of λ.

Digression: particle/hole duality

Let P be the involution which implements left-right reflection and $\bullet \leftrightarrow \circ$.

$$P\left|\lambda\right\rangle = \left|\lambda'\right\rangle$$

where λ' is the conjugate partition of λ.

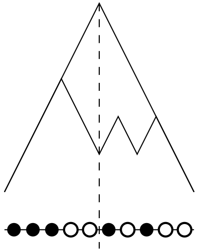


Digression: particle/hole duality

Let P be the involution which implements left-right reflection and $\bullet \leftrightarrow \circ$.

$$P\left|\lambda\right\rangle = \left|\lambda'\right\rangle$$

where λ' is the conjugate partition of λ.

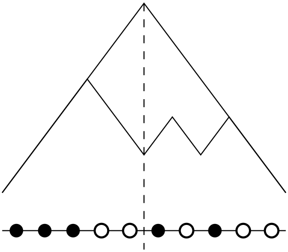


Digression: particle/hole duality

Let P be the involution which implements left-right reflection and $\bullet \leftrightarrow \circ$.

$$P\,|\lambda\rangle = |\lambda'\rangle$$

where λ' is the conjugate partition of λ.

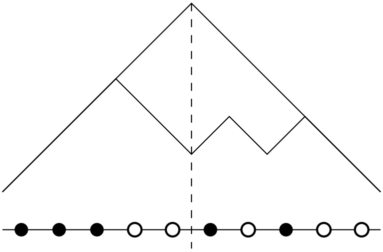


Digression: particle/hole duality

Let P be the involution which implements left-right reflection and $\bullet \leftrightarrow \circ$.

$$P\,|\lambda\rangle = |\lambda'\rangle$$

where λ' is the conjugate partition of λ.



Digression: particle/hole duality

Let P be the involution which implements left-right reflection and $\bullet \leftrightarrow \circ$.

$$P\left|\lambda\right\rangle = \left|\lambda'\right\rangle$$

where λ' is the conjugate partition of λ.

Noting that $PJ_kP^{-1} = (-1)^{k-1}J_k$, We conclude that

$$s_{\lambda'/\mu'}[t] = \left\langle\mu\right| e^{H[-\epsilon t]} \left|\lambda\right\rangle$$

where $-$ is plethystic negation, i.e., removing a variable x, and ϵ is ordinary negation, i.e., $x \to -x$:

$$e^{H[-\epsilon t]} = \prod_{i=1}^{n} T^{-1}(-x_i) \qquad t_k = \frac{1}{k}\sum_i x_i^k$$

... back to the one-step transfer matrix

Theorem

$$\langle\mu|\, T^{-1}(-x)\,|\lambda\rangle = \begin{cases} x^{|\lambda|-|\mu|} & \text{if } \lambda_i = \mu_i \text{ or } \lambda_i = \mu_i + 1 \le \mu_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

Proof: with the same type of calculation as before,

$$\langle\mu|\, T^{-1}(-x)\,|\lambda\rangle = e^{(-1)^{k-1}\frac{x^k}{k}\sum_i z_i^k} \det(z_i^{n+\mu_j-j})\big|_{z_1^{n+\lambda_1-1}\ldots z_n^{\lambda_n}}$$
$$= \det(z_i^{n+\mu_j-j}(1+z_i x))\big|_{z_1^{n+\lambda_1-1}\ldots z_n^{\lambda_n}}$$

Compare with ▸ Gelfand–Tseytlin. Here we add a vertical strip instead of a horizontal one.

. . . back to the one-step transfer matrix

Theorem

$$\langle\mu|\, T^{-1}(-x)\,|\lambda\rangle = \begin{cases} x^{|\lambda|-|\mu|} & \text{if } \lambda_i = \mu_i \text{ or } \lambda_i = \mu_i + 1 \leq \mu_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

Proof: with the same type of calculation as before,

$$\begin{aligned}\langle\mu|\, T^{-1}(-x)\,|\lambda\rangle &= e^{(-1)^{k-1}\frac{x^k}{k}\sum_i z_i^k} \det(z_i^{n+\mu_j-j})|_{z_1^{n+\lambda_1-1}\ldots z_n^{\lambda_n}} \\ &= \det(z_i^{n+\mu_j-j}(1+z_i x))|_{z_1^{n+\lambda_1-1}\ldots z_n^{\lambda_n}}\end{aligned}$$

Compare with Gelfand–Tseytlin. Here we add a vertical strip instead of a horizontal one.

… back to the one-step transfer matrix

Theorem

$$\langle\mu|\, T^{-1}(-x)\,|\lambda\rangle = \begin{cases} x^{|\lambda|-|\mu|} & \text{if } \lambda_i = \mu_i \text{ or } \lambda_i = \mu_i + 1 \le \mu_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

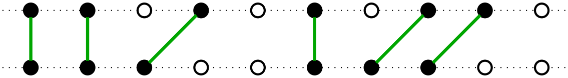
Proof: with the same type of calculation as before,

$$\begin{aligned}\langle\mu|\, T^{-1}(-x)\,|\lambda\rangle &= e^{(-1)^{k-1}\frac{x^k}{k}\sum_i z_i^k} \det(z_i^{n+\mu_j-j})|_{z_1^{n+\lambda_1-1}\dots z_n^{\lambda_n}} \\ &= \det(z_i^{n+\mu_j-j}(1+z_i x))|_{z_1^{n+\lambda_1-1}\dots z_n^{\lambda_n}}\end{aligned}$$

Compare with Gelfand–Tseytlin. Here we add a vertical strip instead of a horizontal one.

More on transfer matrices

Graphically, the action of $T^{-1}(-x)$ can be described as

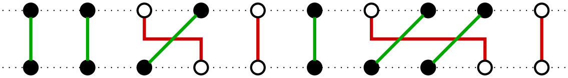
$T^{-1}(-x) =$ 

Therefore, dually,

$T(x) =$

More on transfer matrices

Graphically, the action of $T^{-1}(-x)$ can be described as

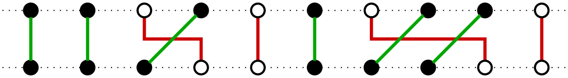
$T^{-1}(-x) =$ 

Therefore, dually,

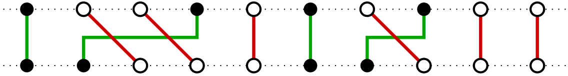
$T(x) =$

More on transfer matrices

Graphically, the action of $T^{-1}(-x)$ can be described as

$T^{-1}(-x) =$ 

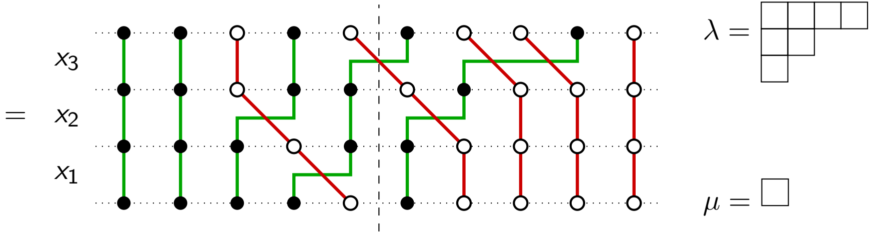
Therefore, dually,

$T(x) =$ 

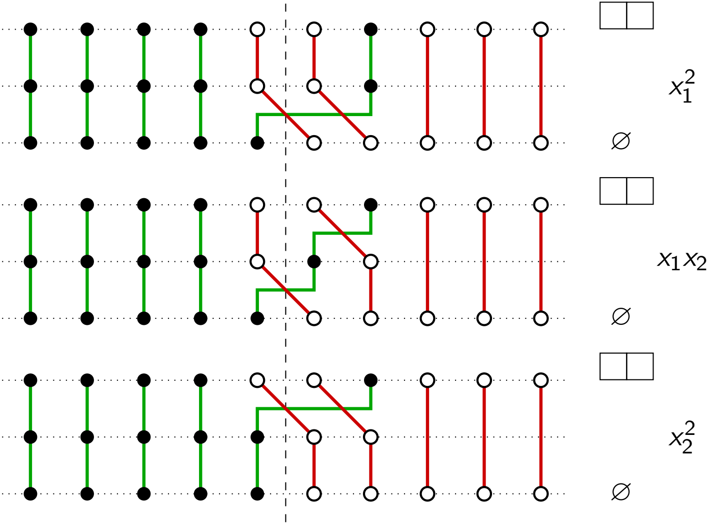
NILPs

Gluing together several transfer matrices produces Non-Intersecting Lattice Paths:

$$s_{\lambda/\mu}(x_1,\dots,x_n)$$
$$= \langle\mu|\, T(x_1)\dots T(x_n)\,|\lambda\rangle$$



Example



Lindström–Gessel–Viennot

The (weighted) enumeration of NILPs is given by a determinant:

$$N(i_1, \ldots, i_k; j_1, \ldots, j_k) = \det_{p,q} N(i_p; j_q)$$

This is known as the Lindström–Gessel–Viennot formula, which is a special case of the (fermionic) Wick theorem.

One can apply it to either holes or particles.

Lindström–Gessel–Viennot

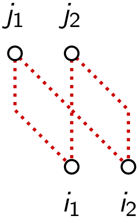
The (weighted) enumeration of NILPs is given by a determinant:

$$N(i_1, \dots, i_k; j_1, \dots, j_k) = \det_{p,q} N(i_p; j_q)$$

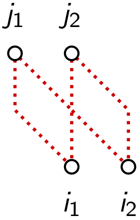
This is known as the Lindström–Gessel–Viennot formula, which is a special case of the (fermionic) Wick theorem.

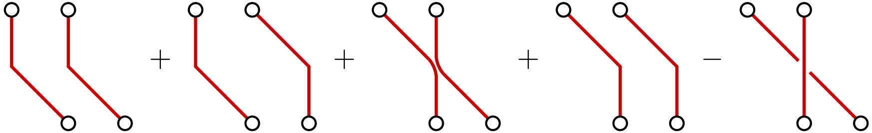
One can apply it to either holes or particles.

Lindström–Gessel–Viennot example



Lindström–Gessel–Viennot example





Lindström–Gessel–Viennot cont'd

Applying LGV to the particles, we find

$$s_{\lambda/\mu}(x_1,\ldots,x_n) = \det h_{\lambda_i-\mu_j-i+j}(x_1,\ldots,x_n)$$

where h_k is the (weighted) enumeration of one particle going n steps to the right $\Rightarrow \sum_{k\geq 0} h_k(x_1,\ldots,x_n)z^k = \prod_{i=1}^n \frac{1}{1-zx_i}$. (also, $h_k = s_{(k)}$)

Applying it to the holes, we find

$$s_{\lambda/\mu}(x_1,\ldots,x_n) = \det e_{\lambda'_i-\mu'_j-i+j}(x_1,\ldots,x_n)$$

where e_k is the (weighted) enumeration of one hole going k steps to the left $\rightarrow \sum_{k\geq 0} e_k(x_1,\ldots,x_n)z^k = \prod_{i=1}^n(1+zx_i)$. (also, $e_k = s_{(1,\ldots,1)}$)

Lindström–Gessel–Viennot cont'd

Applying LGV to the particles, we find

$$s_{\lambda/\mu}(x_1,\ldots,x_n) = \det h_{\lambda_i-\mu_j-i+j}(x_1,\ldots,x_n)$$

where h_k is the (weighted) enumeration of one particle going n steps to the right $\Rightarrow \sum_{k\geq 0} h_k(x_1,\ldots,x_n) z^k = \prod_{i=1}^n \frac{1}{1-zx_i}$. (also, $h_k = s_{(k)}$)

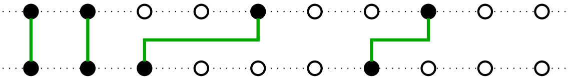
Applying it to the holes, we find

$$s_{\lambda/\mu}(x_1,\ldots,x_n) = \det e_{\lambda'_i-\mu'_j-i+j}(x_1,\ldots,x_n)$$

where e_k is the (weighted) enumeration of one hole going k steps to the left $\rightarrow \sum_{k\geq 0} e_k(x_1,\ldots,x_n) z^k = \prod_{i=1}^n (1+zx_i)$. (also, $e_k = s_{(1,\ldots,1)}$)

Transfer matrices and dualities

- The transfer matrix $T(x) = e^{\phi_+(x^{-1})}$:

$T(x) =$ 

- The adjoint transfer matrix $T^\star(x) = e^{\phi_-(x)}$:

$T^\star(x) =$

- The dual transfer matrix $\overline{T}(x) = e^{-\phi_+(-x^{-1})}$:

$\overline{T}(x) =$

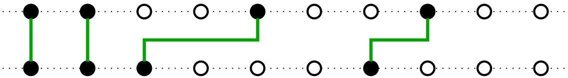
- The dual adjoint transfer matrix $\overline{T}^\star(x) = e^{-\phi_-(-x)}$:

$\overline{T}^\star(x) =$

Transfer matrices and dualities

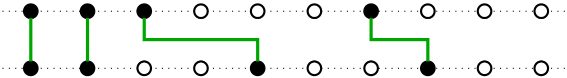
- The transfer matrix $T(x) = e^{\phi_+(x^{-1})}$:

$$T(x) =$$



- The adjoint transfer matrix $T^\star(x) = e^{\phi_-(x)}$:

$$T^\star(x) =$$



- The dual transfer matrix $\overline{T}(x) = e^{-\phi_+(-x^{-1})}$:

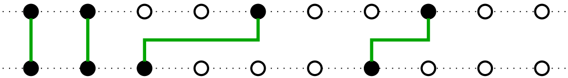
$$\overline{T}(x) =$$

- The dual adjoint transfer matrix $\overline{T}^\star(x) = e^{-\phi_-(-x)}$:

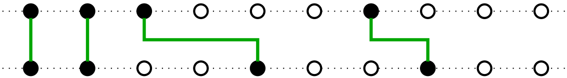
$$\overline{T}^\star(x) =$$

Transfer matrices and dualities

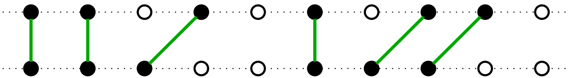
- The transfer matrix $T(x) = e^{\phi_+(x^{-1})}$:

$T(x) =$ 

- The adjoint transfer matrix $T^\star(x) = e^{\phi_-(x)}$:

$T^\star(x) =$ 

- The dual transfer matrix $\overline{T}(x) = e^{-\phi_+(-x^{-1})}$:

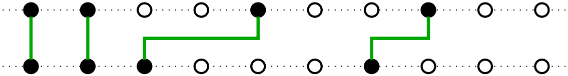
$\overline{T}(x) =$ 

- The dual adjoint transfer matrix $\overline{T}^\star(x) = e^{-\phi_-(-x)}$:

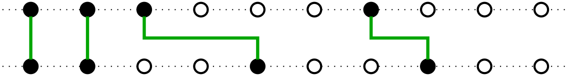
$\overline{T}^\star(x) =$

Transfer matrices and dualities

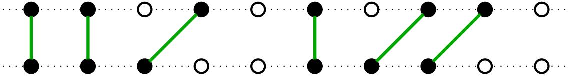
- The transfer matrix $T(x) = e^{\phi_+(x^{-1})}$:

$T(x) =$ 

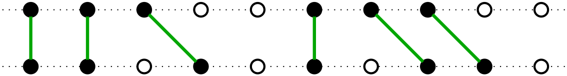
- The adjoint transfer matrix $T^\star(x) = e^{\phi_-(x)}$:

$T^\star(x) =$ 

- The dual transfer matrix $\overline{T}(x) = e^{-\phi_+(-x^{-1})}$:

$\overline{T}(x) =$ 

- The dual adjoint transfer matrix $\overline{T}^\star(x) = e^{-\phi_-(-x)}$:

$\overline{T}^\star(x) =$ 

Commutation relations

From the Heisenberg algebra relations of the J_k, one deduces:

$$
\begin{aligned}
T(x)T(w) &= T(w)T(x) \\
T(x)T^\star(w) &= \frac{1}{1-wx}T^\star(w)T(x) && |wx|<1 \\
T(x)\overline{T}(w) &= \overline{T}(w)T(x) \\
T(x)\overline{T}^\star(w) &= (1+wx)\overline{T}^\star(w)T(x) && |wx|<1
\end{aligned}
$$

Bijective interpretation $\to$ Fomin's growth diagrams / local rules. . .

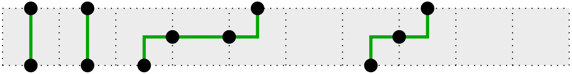
Commutation relations

From the Heisenberg algebra relations of the J_k, one deduces:

$$
\begin{aligned}
T(x)T(w) &= T(w)T(x) \\
T(x)T^\star(w) &= \frac{1}{1-wx}T^\star(w)T(x) && |wx|<1 \\
T(x)\overline{T}(w) &= \overline{T}(w)T(x) \\
T(x)\overline{T}^\star(w) &= (1+wx)\overline{T}^\star(w)T(x) && |wx|<1
\end{aligned}
$$

Bijective interpretation $\rightarrow$ Fomin's growth diagrams / local rules...

Building blocks of transfer matrices I

$T(x) =$ 

Each building block is of the form:

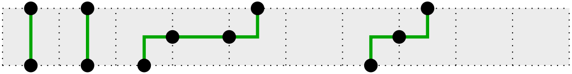
There are 5 possible inputs/outputs:

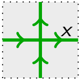
$a_1 = 0$ $a_2 = 1$ $b_1 = 1$ $b_2 = x$ $c_1 = x$ $c_2 = 1$

This is the five-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Building blocks of transfer matrices I

$T(x) =$ 

Each building block is of the form: 

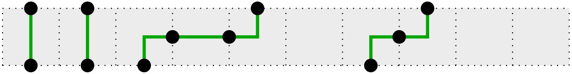
There are 5 possible inputs/outputs:

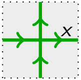
$a_1 = 0$ $a_2 = 1$ $b_1 = 1$ $b_2 = x$ $c_1 = x$ $c_2 = 1$

This is the five-vertex model, with the free fermionic condition:

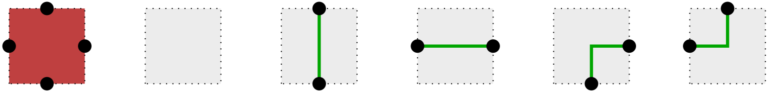
$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Building blocks of transfer matrices I

$T(x) =$ 

Each building block is of the form: 

There are 5 possible inputs/outputs:



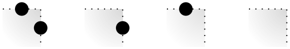
$a_1 = 0$ $a_2 = 1$ $b_1 = 1$ $b_2 = x$ $c_1 = x$ $c_2 = 1$

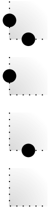
This is the five-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

L-matrix

Each building block is encoded into a 4×4 matrix





$$L(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & x & 1 & 0 \\ 0 & x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

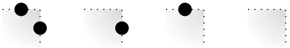
so that

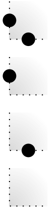
$$T(x) = \cdots L_{a,-1}(x) L_{a,0}(x) L_{a,1}(x) \cdots$$

where $L_{a,k}(x)$ acts on $(\mathbb{C}^2)_a \otimes (\mathbb{C}^2)_k$, and it is understood that sufficiently far to the left and right, the auxiliary space a is in the empty state.

L-matrix

Each building block is encoded into a 4×4 matrix





$$L(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & x & 1 & 0 \\ 0 & x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

so that

$$T(x) = \cdots L_{a,-1}(x) L_{a,0}(x) L_{a,1}(x) \cdots$$

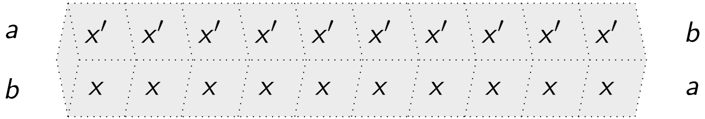
where $L_{a,k}(x)$ acts on $(\mathbb{C}^2)_a \otimes (\mathbb{C}^2)_k$, and it is understood that sufficiently far to the left and right, the auxiliary space a is in the empty state.

Yang–Baxter equation

If we can find a matrix $R(x, x')$ such that the "*RLL*" relation holds:

$$R_{a,b}(x, x')L_{a,k}(x)L_{b,k}(x') = L_{b,k}(x')L_{a,k}(x)R_{a,b}(x, x')$$

then $T(x)$ and $T(x')$ commute:



We find

$$R(x, x') = \begin{pmatrix} x' & 0 & 0 & 0 \\ 0 & x - x' & x' & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & x \end{pmatrix}$$

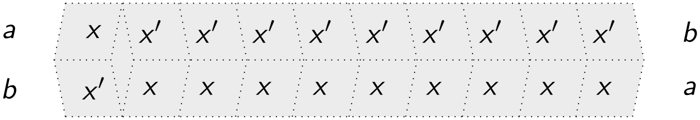
Consistency of "*RLL*" implies YBE and unitarity for R.

Yang–Baxter equation

If we can find a matrix $R(x,x')$ such that the "RLL" relation holds:

$$R_{a,b}(x,x')L_{a,k}(x)L_{b,k}(x') = L_{b,k}(x')L_{a,k}(x)R_{a,b}(x,x')$$

then $T(x)$ and $T(x')$ commute:



We find

$$R(x,x') = \begin{pmatrix} x' & 0 & 0 & 0 \\ 0 & x-x' & x' & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & x \end{pmatrix}$$

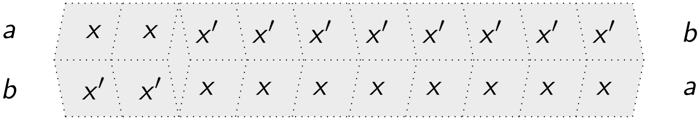
Consistency of "RLL" implies YBE and unitarity for R.

Yang–Baxter equation

If we can find a matrix $R(x,x')$ such that the "*RLL*" relation holds:

$$R_{a,b}(x,x')L_{a,k}(x)L_{b,k}(x') = L_{b,k}(x')L_{a,k}(x)R_{a,b}(x,x')$$

then $T(x)$ and $T(x')$ commute:



We find

$$R(x,x') = \begin{pmatrix} x' & 0 & 0 & 0 \\ 0 & x-x' & x' & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & x \end{pmatrix}$$

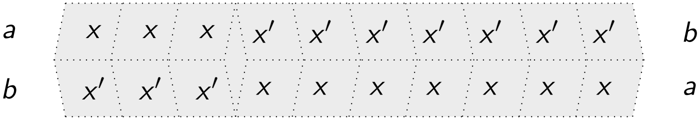
Consistency of "*RLL*" implies YBE and unitarity for R.

Yang–Baxter equation

If we can find a matrix $R(x, x')$ such that the "RLL" relation holds:

$$R_{a,b}(x, x')L_{a,k}(x)L_{b,k}(x') = L_{b,k}(x')L_{a,k}(x)R_{a,b}(x, x')$$

then $T(x)$ and $T(x')$ commute:



We find

$$R(x, x') = \begin{pmatrix} x' & 0 & 0 & 0 \\ 0 & x - x' & x' & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & x \end{pmatrix}$$

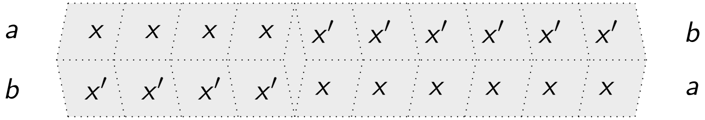
Consistency of "RLL" implies YBE and unitarity for R.

Yang–Baxter equation

If we can find a matrix $R(x,x')$ such that the "RLL" relation holds:

$$R_{a,b}(x,x')L_{a,k}(x)L_{b,k}(x') = L_{b,k}(x')L_{a,k}(x)R_{a,b}(x,x')$$

then $T(x)$ and $T(x')$ commute:



We find

$$R(x,x') = \begin{pmatrix} x' & 0 & 0 & 0 \\ 0 & x-x' & x' & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & x \end{pmatrix}$$

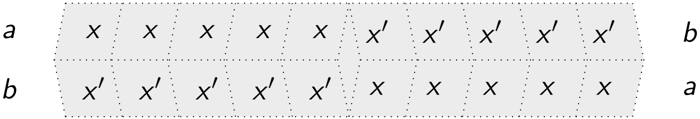
Consistency of "RLL" implies YBE and unitarity for R.

Yang–Baxter equation

If we can find a matrix $R(x, x')$ such that the "*RLL*" relation holds:

$$R_{a,b}(x, x')L_{a,k}(x)L_{b,k}(x') = L_{b,k}(x')L_{a,k}(x)R_{a,b}(x, x')$$

then $T(x)$ and $T(x')$ commute:



We find

$$R(x, x') = \begin{pmatrix} x' & 0 & 0 & 0 \\ 0 & x - x' & x' & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & x \end{pmatrix}$$

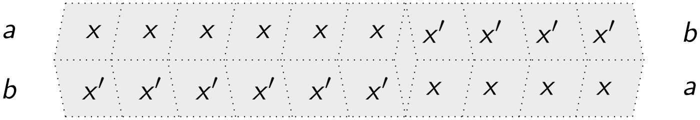
Consistency of "*RLL*" implies YBE and unitarity for R.

Yang–Baxter equation

If we can find a matrix $R(x, x')$ such that the "RLL" relation holds:

$$R_{a,b}(x, x')L_{a,k}(x)L_{b,k}(x') = L_{b,k}(x')L_{a,k}(x)R_{a,b}(x, x')$$

then $T(x)$ and $T(x')$ commute:



We find

$$R(x, x') = \begin{pmatrix} x' & 0 & 0 & 0 \\ 0 & x - x' & x' & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & x \end{pmatrix}$$

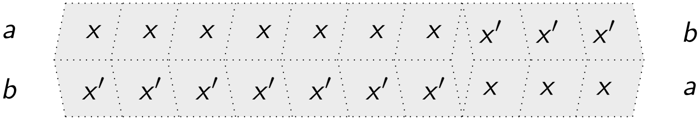
Consistency of "RLL" implies YBE and unitarity for R.

Yang–Baxter equation

If we can find a matrix $R(x, x')$ such that the "RLL" relation holds:

$$R_{a,b}(x, x')L_{a,k}(x)L_{b,k}(x') = L_{b,k}(x')L_{a,k}(x)R_{a,b}(x, x')$$

then $T(x)$ and $T(x')$ commute:



We find

$$R(x, x') = \begin{pmatrix} x' & 0 & 0 & 0 \\ 0 & x - x' & x' & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & x \end{pmatrix}$$

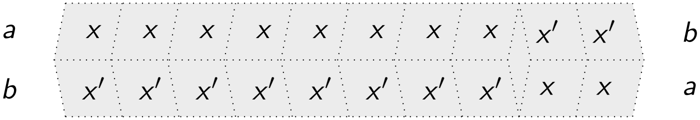
Consistency of "RLL" implies YBE and unitarity for R.

Yang–Baxter equation

If we can find a matrix $R(x,x')$ such that the "*RLL*" relation holds:

$$R_{a,b}(x,x')L_{a,k}(x)L_{b,k}(x') = L_{b,k}(x')L_{a,k}(x)R_{a,b}(x,x')$$

then $T(x)$ and $T(x')$ commute:



We find

$$R(x,x') = \begin{pmatrix} x' & 0 & 0 & 0 \\ 0 & x-x' & x' & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & x \end{pmatrix}$$

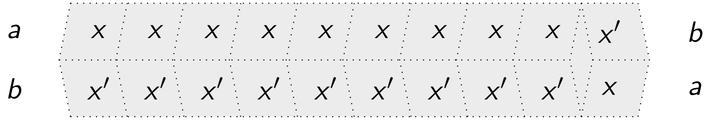
Consistency of "*RLL*" implies YBE and unitarity for R.

Yang–Baxter equation

If we can find a matrix $R(x, x')$ such that the “RLL” relation holds:

$$R_{a,b}(x, x')L_{a,k}(x)L_{b,k}(x') = L_{b,k}(x')L_{a,k}(x)R_{a,b}(x, x')$$

then $T(x)$ and $T(x')$ commute:



We find

$$R(x, x') = \begin{pmatrix} x' & 0 & 0 & 0 \\ 0 & x - x' & x' & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & x \end{pmatrix}$$

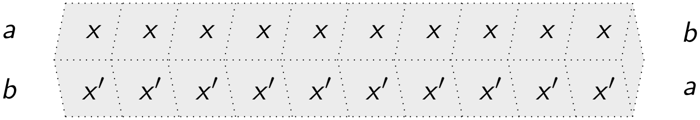
Consistency of “RLL” implies YBE and unitarity for R.

Yang–Baxter equation

If we can find a matrix $R(x,x')$ such that the "*RLL*" relation holds:

$$R_{a,b}(x,x')L_{a,k}(x)L_{b,k}(x') = L_{b,k}(x')L_{a,k}(x)R_{a,b}(x,x')$$

then $T(x)$ and $T(x')$ commute:



We find

$$R(x,x') = \begin{pmatrix} x' & 0 & 0 & 0 \\ 0 & x-x' & x' & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & x \end{pmatrix}$$

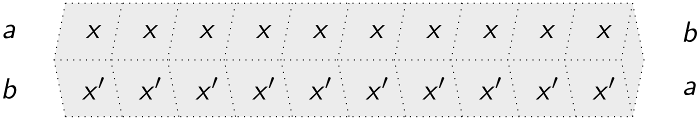
Consistency of "*RLL*" implies YBE and unitarity for R.

Yang–Baxter equation

If we can find a matrix $R(x, x')$ such that the "*RLL*" relation holds:

$$R_{a,b}(x, x')L_{a,k}(x)L_{b,k}(x') = L_{b,k}(x')L_{a,k}(x)R_{a,b}(x, x')$$

then $T(x)$ and $T(x')$ commute:



We find

$$R(x, x') = \begin{pmatrix} x' & 0 & 0 & 0 \\ 0 & x - x' & x' & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & x \end{pmatrix}$$

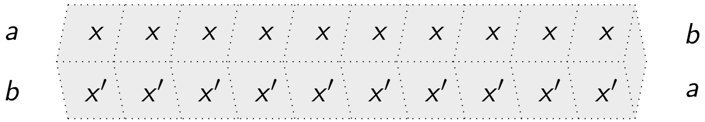
Consistency of "*RLL*" implies YBE and unitarity for R.

Yang–Baxter equation

If we can find a matrix $R(x, x')$ such that the "*RLL*" relation holds:

$$R_{a,b}(x, x')L_{a,k}(x)L_{b,k}(x') = L_{b,k}(x')L_{a,k}(x)R_{a,b}(x, x')$$

then $T(x)$ and $T(x')$ commute:

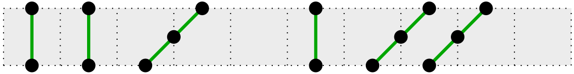


We find

$$R(x, x') = \begin{pmatrix} x' & 0 & 0 & 0 \\ 0 & x - x' & x' & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & x \end{pmatrix}$$

Consistency of "*RLL*" implies YBE and unitarity for R...

Building blocks of transfer matrices II

$\overline{T}(x) =$ 

Each building block is of the form:

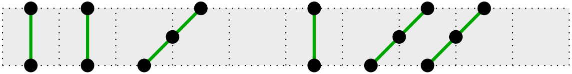
There are 5 possible inputs/outputs:

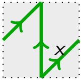
$a_1 = x$ $\quad a_2 = 1$ $\quad b_1 = 1$ $\quad b_2 = 0$ $\quad c_1 = x$ $\quad c_2 = 1$

This is the five-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Building blocks of transfer matrices II

$\overline{T}(x) =$ 

Each building block is of the form: 

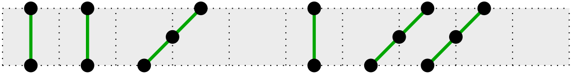
There are 5 possible inputs/outputs:

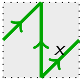
$a_1 = x$ $a_2 = 1$ $b_1 = 1$ $b_2 = 0$ $c_1 = x$ $c_2 = 1$

This is the five-vertex model, with the free fermionic condition:

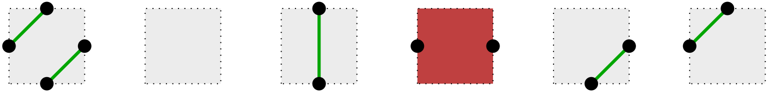
$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Building blocks of transfer matrices II

$\overline{T}(x) =$ 

Each building block is of the form: 

There are 5 possible inputs/outputs:

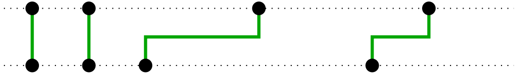


$a_1 = x$ $a_2 = 1$ $b_1 = 1$ $b_2 = 0$ $c_1 = x$ $c_2 = 1$

This is the five-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Building blocks of transfer matrices III

$T(x)S^{-1} =$ 

Each building block is of the form:

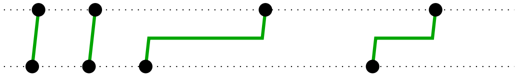
There are 5 possible inputs/outputs:

$a_1 = 1$ $\quad a_2 = 1$ $\quad b_1 = 0$ $\quad b_2 = x$ $\quad c_1 = 1$ $\quad c_2 = 1$

This is the five-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Building blocks of transfer matrices III

$T(x)S^{-1} =$ 

Each building block is of the form:

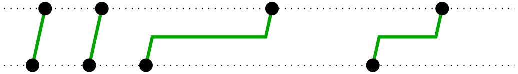
There are 5 possible inputs/outputs:

$a_1 = 1$ $a_2 = 1$ $b_1 = 0$ $b_2 = x$ $c_1 = 1$ $c_2 = 1$

This is the five-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Building blocks of transfer matrices III

$T(x)S^{-1} =$ 

Each building block is of the form:

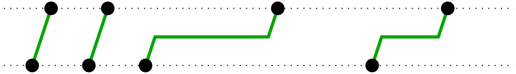
There are 5 possible inputs/outputs:

$a_1 = 1$ $\quad a_2 = 1$ $\quad b_1 = 0$ $\quad b_2 = x$ $\quad c_1 = 1$ $\quad c_2 = 1$

This is the five-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Building blocks of transfer matrices III

$T(x)S^{-1} =$ 

Each building block is of the form:

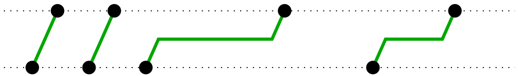
There are 5 possible inputs/outputs:

$a_1 = 1$ $a_2 = 1$ $b_1 = 0$ $b_2 = x$ $c_1 = 1$ $c_2 = 1$

This is the five-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Building blocks of transfer matrices III

$T(x)S^{-1} =$ 

Each building block is of the form:

There are 5 possible inputs/outputs:

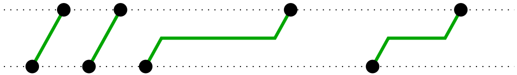
$a_1 = 1$ $\quad a_2 = 1$ $\quad b_1 = 0$ $\quad b_2 = x$ $\quad c_1 = 1$ $\quad c_2 = 1$

This is the five-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Building blocks of transfer matrices III

$$T(x)S^{-1} =$$



Each building block is of the form:

There are 5 possible inputs/outputs:

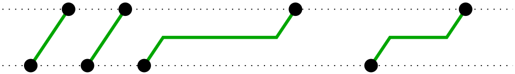
$a_1 = 1$ $a_2 = 1$ $b_1 = 0$ $b_2 = x$ $c_1 = 1$ $c_2 = 1$

This is the five-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Building blocks of transfer matrices III

$$T(x)S^{-1} =$$



Each building block is of the form:

There are 5 possible inputs/outputs:

$a_1 = 1$ $\quad a_2 = 1$ $\quad b_1 = 0$ $\quad b_2 = x$ $\quad c_1 = 1$ $\quad c_2 = 1$

This is the five-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Building blocks of transfer matrices III

$T(x)S^{-1} =$ 

Each building block is of the form:

There are 5 possible inputs/outputs:

$a_1 = 1$ $\quad a_2 = 1$ $\quad b_1 = 0$ $\quad b_2 = x$ $\quad c_1 = 1$ $\quad c_2 = 1$

This is the five-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Building blocks of transfer matrices III

$T(x)S^{-1} =$ 

Each building block is of the form:

There are 5 possible inputs/outputs:

$a_1 = 1$ $a_2 = 1$ $b_1 = 0$ $b_2 = x$ $c_1 = 1$ $c_2 = 1$

This is the five-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Building blocks of transfer matrices III

$T(x)S^{-1} =$ 

Each building block is of the form:

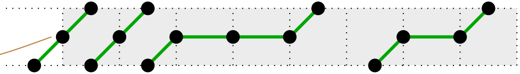
There are 5 possible inputs/outputs:

$a_1 = 1$	$a_2 = 1$	$b_1 = 0$	$b_2 = x$	$c_1 = 1$	$c_2 = 1$

This is the five-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Building blocks of transfer matrices III

$T(x)S^{-1} =$ 

DWBC

Each building block is of the form:

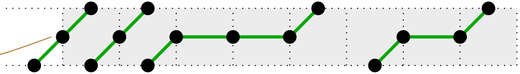
There are 5 possible inputs/outputs:

$a_1 = 1$ $\quad a_2 = 1$ $\quad b_1 = 0$ $\quad b_2 = x$ $\quad c_1 = 1$ $\quad c_2 = 1$

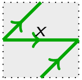
This is the five-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Building blocks of transfer matrices III

$T(x)S^{-1} =$ 

DWBC

Each building block is of the form: 

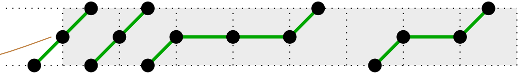
There are 5 possible inputs/outputs:

$a_1 = 1$ $a_2 = 1$ $b_1 = 0$ $b_2 = x$ $c_1 = 1$ $c_2 = 1$

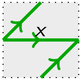
This is the five-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

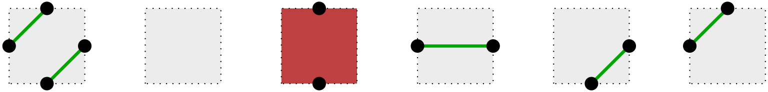
Building blocks of transfer matrices III

$T(x)S^{-1} =$ 

DWBC

Each building block is of the form: 

There are 5 possible inputs/outputs:



$a_1 = 1$ $\quad a_2 = 1$ $\quad b_1 = 0$ $\quad b_2 = x$ $\quad c_1 = 1$ $\quad c_2 = 1$

This is the five-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

L-matrix

Each building block is encoded into a 4×4 matrix

$$L(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

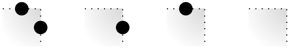
so that

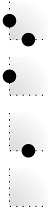
$$T(x)S^{-1} = \cdots L_{a,-1}(x)L_{a,0}(x)L_{a,1}(x)\cdots$$

where $L_{a,k}(x)$ acts on $(\mathbb{C}^2)_a \otimes (\mathbb{C}^2)_k$, and sufficiently far to the left (resp. right), the auxiliary space a is in the occupied (resp. empty) state.

L-matrix

Each building block is encoded into a 4×4 matrix





$$L(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

so that

$$T(x)S^{-1} = \cdots L_{a,-1}(x)L_{a,0}(x)L_{a,1}(x)\cdots$$

where $L_{a,k}(x)$ acts on $(\mathbb{C}^2)_a \otimes (\mathbb{C}^2)_k$, and sufficiently far to the left (resp. right), the auxiliary space a is in the occupied (resp. empty) state.

Yang–Baxter equation

If we can find a matrix $R(x,x')$ such that

$$R_{a,b}(x,x')L_{a,k}(x)L_{b,k}(x') = L_{b,k}(x')L_{a,k}(x)R_{a,b}(x,x')$$

and  = , then $T(x)S^{-1}$ and $T(x')S^{-1}$ commute as before.

We find

$$R(x,x') = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x-x' & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Note that $R(x,x') = L(x-x') = R(x-x')$, $R_{12}(x)R_{21}(-x) = 1$, $R_{12}(x)R_{13}(x+y)R_{23}(y) = R_{23}(y)R_{13}(x+y)R_{12}(x)$, and $R(0)$ is the permutation of factors of the tensor product.

Yang–Baxter equation

If we can find a matrix $R(x,x')$ such that

$$R_{a,b}(x,x')L_{a,k}(x)L_{b,k}(x') = L_{b,k}(x')L_{a,k}(x)R_{a,b}(x,x')$$

and  $=$ , then $T(x)S^{-1}$ and $T(x')S^{-1}$ commute as before.

We find

$$R(x,x') = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x-x' & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Note that $R(x,x') = L(x-x') = R(x-x')$, $R_{12}(x)R_{21}(-x) = 1$, $R_{12}(x)R_{13}(x+y)R_{23}(y) = R_{23}(y)R_{13}(x+y)R_{12}(x)$, and $R(0)$ is the permutation of factors of the tensor product.

Building blocks of transfer matrices IV

$\overline{T}^{\star}(x)S^{-1} =$ 

Each building block is of the form:

There are 5 possible inputs/outputs:

$a_1 = 1$ $a_2 = 1$ $b_1 = x$ $b_2 = 0$ $c_1 = 1$ $c_2 = 1$

This is the five-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Building blocks of transfer matrices IV

$\overline{T}^{\star}(x)S^{-1} =$ 

Each building block is of the form:

There are 5 possible inputs/outputs:

$a_1 = 1$ $a_2 = 1$ $b_1 = x$ $b_2 = 0$ $c_1 = 1$ $c_2 = 1$

This is the five-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Building blocks of transfer matrices IV

$\overline{T}^{\star}(x)S^{-1} =$ 

Each building block is of the form:

There are 5 possible inputs/outputs:

$a_1 = 1$ $\quad a_2 = 1$ $\quad b_1 = x$ $\quad b_2 = 0$ $\quad c_1 = 1$ $\quad c_2 = 1$

This is the five-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Building blocks of transfer matrices IV

$\overline{T}^{\star}(x)S^{-1} =$ 

Each building block is of the form:

There are 5 possible inputs/outputs:

$a_1 = 1$	$a_2 = 1$	$b_1 = x$	$b_2 = 0$	$c_1 = 1$	$c_2 = 1$

This is the five-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Building blocks of transfer matrices IV

$\overline{T}^{\star}(x)S^{-1} =$ 

Each building block is of the form:

There are 5 possible inputs/outputs:

$a_1 = 1$ $\quad a_2 = 1$ $\quad b_1 = x$ $\quad b_2 = 0$ $\quad c_1 = 1$ $\quad c_2 = 1$

This is the five-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Building blocks of transfer matrices IV

$\overline{T}^{\star}(x)S^{-1} =$ 

Each building block is of the form:

There are 5 possible inputs/outputs:

$a_1 = 1$ $\quad a_2 = 1$ $\quad b_1 = x$ $\quad b_2 = 0$ $\quad c_1 = 1$ $\quad c_2 = 1$

This is the five-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Building blocks of transfer matrices IV

$\overline{T}^{\star}(x)S^{-1} =$ 

Each building block is of the form:

There are 5 possible inputs/outputs:

$a_1 = 1$ $\quad a_2 = 1$ $\quad b_1 = x$ $\quad b_2 = 0$ $\quad c_1 = 1$ $\quad c_2 = 1$

This is the five-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Building blocks of transfer matrices IV

$$\overline{T}^{\star}(x)S^{-1} =$$ 

Each building block is of the form:

There are 5 possible inputs/outputs:

$a_1 = 1$ $\quad a_2 = 1$ $\quad b_1 = x$ $\quad b_2 = 0$ $\quad c_1 = 1$ $\quad c_2 = 1$

This is the five-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Building blocks of transfer matrices IV

$\overline{T}^{\star}(x)S^{-1} =$ 

Each building block is of the form:

There are 5 possible inputs/outputs:

$a_1 = 1$ $\quad a_2 = 1$ $\quad b_1 = x$ $\quad b_2 = 0$ $\quad c_1 = 1$ $\quad c_2 = 1$

This is the five-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Building blocks of transfer matrices IV

$\overline{T}^{\star}(x)S^{-1} =$ 

Each building block is of the form:

There are 5 possible inputs/outputs:

$a_1 = 1$ $\quad a_2 = 1$ $\quad b_1 = x$ $\quad b_2 = 0$ $\quad c_1 = 1$ $\quad c_2 = 1$

This is the five-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Building blocks of transfer matrices IV

$\overline{T}^{\star}(x)S^{-1} =$ 

DWBC

Each building block is of the form:

There are 5 possible inputs/outputs:

$a_1 = 1$ $\quad a_2 = 1$ $\quad b_1 = x$ $\quad b_2 = 0$ $\quad c_1 = 1$ $\quad c_2 = 1$

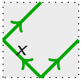
This is the five-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Building blocks of transfer matrices IV

$$\overline{T}^\star(x)S^{-1} =$$ 

DWBC

Each building block is of the form: 

There are 5 possible inputs/outputs:

$a_1 = 1$ $a_2 = 1$ $b_1 = x$ $b_2 = 0$ $c_1 = 1$ $c_2 = 1$

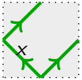
This is the five-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

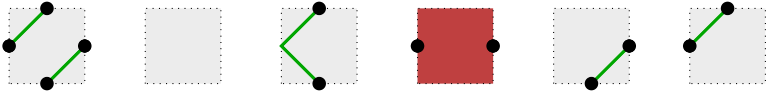
Building blocks of transfer matrices IV

$\overline{T}^{\star}(x)S^{-1} =$ 

DWBC

Each building block is of the form: 

There are 5 possible inputs/outputs:



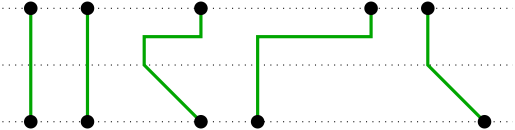
$a_1 = 1$ $\quad a_2 = 1$ $\quad b_1 = x$ $\quad b_2 = 0$ $\quad c_1 = 1$ $\quad c_2 = 1$

This is the five-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Building blocks of transfer matrices V

$$\overline{T}^{\star}(w)T(x)S^{-1} =$$



Each building block is of the form:

There are 6 possible inputs/outputs:

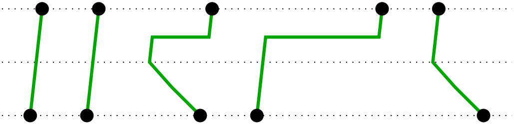
$a_1 = 1$ $\quad a_2 = 1$ $\quad b_1 = w$ $\quad b_2 = x$ $\quad c_1 = 1 + wx$ $\quad c_2 = 1$

This is the six-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Building blocks of transfer matrices V

$$\overline{T}^{\star}(w)\,T(x)S^{-1} =$$



Each building block is of the form:

There are 6 possible inputs/outputs:

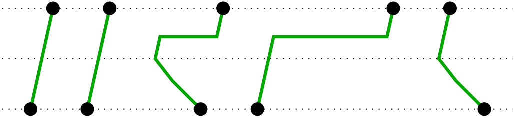
$a_1 = 1$ $a_2 = 1$ $b_1 = w$ $b_2 = x$ $c_1 = 1 + wx$ $c_2 = 1$

This is the six-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Building blocks of transfer matrices V

$$\overline{T}^{\star}(w)T(x)S^{-1} =$$



Each building block is of the form:

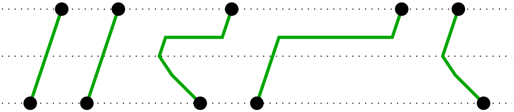
There are 6 possible inputs/outputs:

$a_1 = 1$ $\quad a_2 = 1$ $\quad b_1 = w$ $\quad b_2 = x$ $\quad c_1 = 1 + wx$ $\quad c_2 = 1$

This is the six-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Building blocks of transfer matrices V

$\overline{T}^\star(w)T(x)S^{-1} =$ 

Each building block is of the form:

There are 6 possible inputs/outputs:

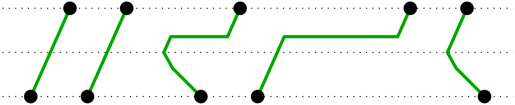
$a_1 = 1 \quad a_2 = 1 \quad b_1 = w \quad b_2 = x \quad c_1 = 1 + wx \quad c_2 = 1$

This is the six-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Building blocks of transfer matrices V

$$\overline{T}^{\star}(w)\,T(x)\,S^{-1} =$$



Each building block is of the form:

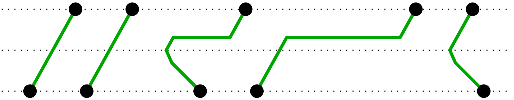
There are 6 possible inputs/outputs:

$a_1 = 1$ $\quad a_2 = 1$ $\quad b_1 = w$ $\quad b_2 = x$ $\quad c_1 = 1 + wx$ $\quad c_2 = 1$

This is the six-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Building blocks of transfer matrices V

$$\overline{T}^{\star}(w)T(x)S^{-1} =$$ 

Each building block is of the form:

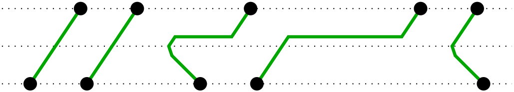
There are 6 possible inputs/outputs:

$a_1 = 1$ $a_2 = 1$ $b_1 = w$ $b_2 = x$ $c_1 = 1 + wx$ $c_2 = 1$

This is the six-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Building blocks of transfer matrices V

$\overline{T}^{\star}(w)\,T(x)\,S^{-1} =$ 

Each building block is of the form:

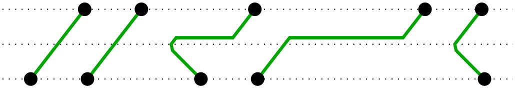
There are 6 possible inputs/outputs:

$a_1 = 1$ $\quad a_2 = 1$ $\quad b_1 = w$ $\quad b_2 = x$ $\quad c_1 = 1 + wx$ $\quad c_2 = 1$

This is the six-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Building blocks of transfer matrices V

$$\overline{T}^\star(w)T(x)S^{-1} =$$ 

Each building block is of the form:

There are 6 possible inputs/outputs:

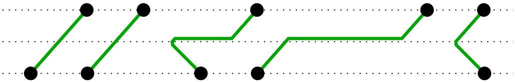
$a_1 = 1$ $a_2 = 1$ $b_1 = w$ $b_2 = x$ $c_1 = 1 + wx$ $c_2 = 1$

This is the six-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Building blocks of transfer matrices V

$$\overline{T}^{\star}(w)\,T(x)\,S^{-1} =$$



Each building block is of the form:

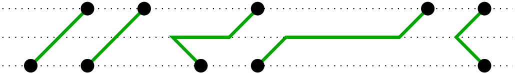
There are 6 possible inputs/outputs:

$a_1 = 1$ $\quad a_2 = 1$ $\quad b_1 = w$ $\quad b_2 = x$ $\quad c_1 = 1 + wx$ $\quad c_2 = 1$

This is the six-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Building blocks of transfer matrices V

$$\overline{T}^{\star}(w)T(x)S^{-1} =$$ 

Each building block is of the form:

There are 6 possible inputs/outputs:

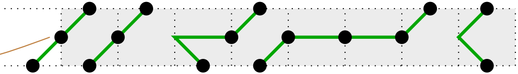
$a_1 = 1$ $a_2 = 1$ $b_1 = w$ $b_2 = x$ $c_1 = 1 + wx$ $c_2 = 1$

This is the six-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Building blocks of transfer matrices V

$$\overline{T}^{\star}(w)\,T(x)\,S^{-1} = $$



DWBC

Each building block is of the form:

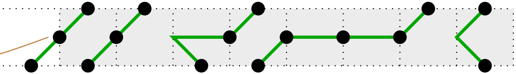
There are 6 possible inputs/outputs:

$a_1 = 1$ $\quad a_2 = 1$ $\quad b_1 = w$ $\quad b_2 = x$ $\quad c_1 = 1 + wx$ $\quad c_2 = 1$

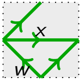
This is the six-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

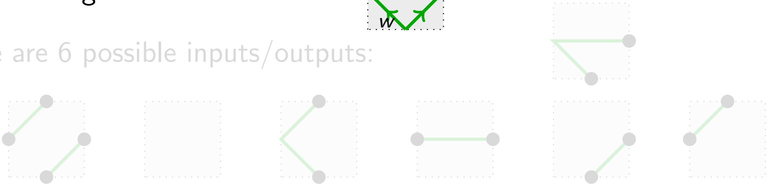
Building blocks of transfer matrices V

$\overline{T}^{\star}(w)T(x)S^{-1} =$ 

DWBC

Each building block is of the form: 

There are 6 possible inputs/outputs:



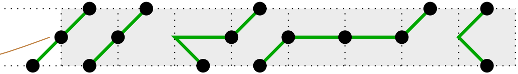
$a_1 = 1$ $a_2 = 1$ $b_1 = w$ $b_2 = x$ $c_1 = 1 + wx$ $c_2 = 1$

This is the six-vertex model, with the free fermionic condition:

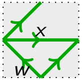
$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Building blocks of transfer matrices V

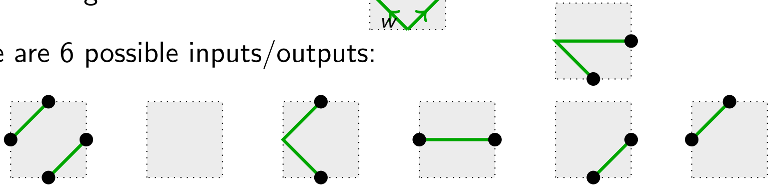
$$\overline{T}^\star(w)\,T(x)\,S^{-1} =$$



DWBC

Each building block is of the form: 

There are 6 possible inputs/outputs:



$a_1 = 1$ $\quad a_2 = 1$ $\quad b_1 = w$ $\quad b_2 = x$ $\quad c_1 = 1 + wx$ $\quad c_2 = 1$

This is the six-vertex model, with the free fermionic condition:

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

Commuting transfer matrices

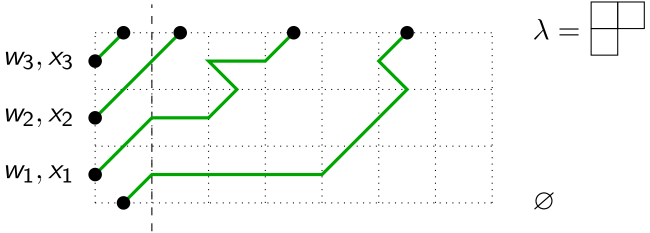
The transfer matrices $T_{6v}(w,x) = \overline{T}^{\star}(w)\,T(x)S^{-1}$ satisfy the commutation relations

$$T_{6v}(w,x)\,T_{6v}(w',x') = \frac{1+w'x}{1+wx'}\,T_{6v}(w',x')\,T_{6v}(w,x)$$

In particular they commute if $w/x = w'/x'$. (as a consequence of YBE...)
In the language of the six-vertex model, w/x is an electric field (or twist).
x is (a function of) the *spectral parameter*.

Partial DWBC partition function

$$\langle 0|\, T_{6v}(w_1,x_1)\ldots T_{6v}(w_n,x_n)\,|\lambda;n\rangle$$

$=$ 

$$= \langle 0|\, \overline{T}^{\star}(w_1)T(x_1)\ldots\overline{T}^{\star}(w_n)T(x_n)\,|\lambda\rangle$$

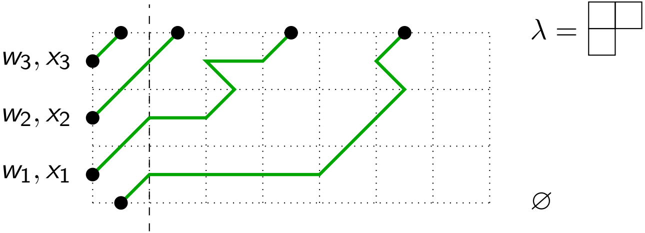
$$= \prod_{1\le i<j\le n}(1+x_iw_j)\; s_\lambda(x_1,\ldots,x_n)$$

This formula was obtained for $w_i = t/x_i$ in [Brubaker, Bump, Friedberg, 2011; see also Tokuyama, 1988]. It interpolates between $t=0$: $\langle 0|\prod_i T(x_i)\,|\lambda\rangle$ and $t=-1$: $\langle 0|\prod_i \psi(x_i^{-1})\,|\lambda;n\rangle$.

Partial DWBC partition function

$$\langle 0|\, T_{6v}(w_1,x_1)\ldots T_{6v}(w_n,x_n)\,|\lambda;n\rangle$$

$=$



$$= \langle 0|\, \overline{T}^{\star}(w_1)\,T(x_1)\ldots \overline{T}^{\star}(w_n)\,T(x_n)\,|\lambda\rangle$$

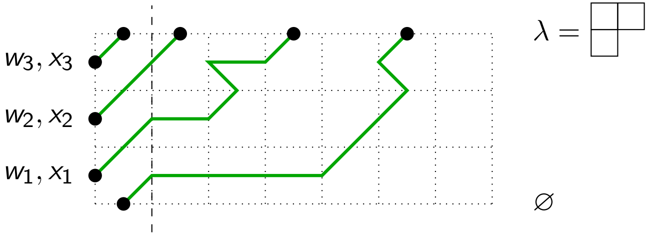
$$= \prod_{1\le i<j\le n} (1+x_i w_j)\; s_\lambda(x_1,\ldots,x_n)$$

This formula was obtained for $w_i = t/x_i$ in [Brubaker, Bump, Friedberg, 2011; see also Tokuyama, 1988]. It interpolates between $t=0$: $\langle 0|\prod_i T(x_i)\,|\lambda\rangle$ and $t=-1$: $\langle 0|\prod_i \psi(x_i^{-1})\,|\lambda;n\rangle$.

Partial DWBC partition function

$$\langle 0|\, T_{6v}(w_1,x_1)\ldots T_{6v}(w_n,x_n)\,|\lambda; n\rangle$$

$=$



$$= \langle 0|\, \overline{T}^{\star}(w_1)\,T(x_1)\ldots \overline{T}^{\star}(w_n)\,T(x_n)\,|\lambda\rangle$$

$$= \prod_{1\le i<j\le n} (1+x_i w_j)\; s_\lambda(x_1,\ldots,x_n)$$

This formula was obtained for $w_i = t/x_i$ in [Brubaker, Bump, Friedberg, 2011; see also Tokuyama, 1988]. It interpolates between $t=0$: $\langle 0|\prod_i T(x_i)\,|\lambda\rangle$ and $t=-1$: $\langle 0|\prod_i \psi(x_i^{-1})\,|\lambda; n\rangle$.

Supersymmetric Schur functions

We define the supersymmetric Schur function

$$s_\lambda[t_1,\ldots,t_k\ldots|u_1,\ldots,u_k\ldots] = s_\lambda[t_1+u_1,\ldots,t_k-(-1)^k u_k\ldots]$$

where $t_k = \frac{1}{k}\sum_i x_i^k$, $u_k = \frac{1}{k}\sum_i y_i^k$.

In terms of two finite sets of variables (supercharacter of $GL(n|m)$), we have:

$$s_\lambda(x_1,\ldots,x_n|y_1,\ldots,y_m) = \langle 0|\prod_{i=1}^{n} T(x_i)\prod_{i=1}^{m}\overline{T}(y_i)\,|\lambda\rangle$$

Supersymmetric Schur functions

We define the supersymmetric Schur function

$$s_\lambda[t_1,\ldots,t_k\ldots|u_1,\ldots,u_k\ldots] = s_\lambda[t_1+u_1,\ldots,t_k-(-1)^k u_k\ldots]$$

where $t_k = \frac{1}{k}\sum_i x_i^k$, $u_k = \frac{1}{k}\sum_i y_i^k$.

In terms of two finite sets of variables (supercharacter of $GL(n|m)$), we have:

$$s_\lambda(x_1,\ldots,x_n|y_1,\ldots,y_m) = \langle 0|\prod_{i=1}^{n} T(x_i)\prod_{i=1}^{m}\overline{T}(y_i)\,|\lambda\rangle$$

Factorial Schur polynomials

Natural generalization: inhomogeneous statistical model. Define the inhomogeneous five-vertex transfer matrix:

$$T(x;\ldots,y_{-1},y_0,y_1,\ldots)S^{-1} = \begin{array}{|c|c|c|c|c|c|} \hline x-y_{-2} & x-y_{-1} & x-y_0 & x-y_1 & x-y_2 & x-y_3 \\ \hline \end{array}$$

with same weights as ▸ before, except $x \to x - y_i$.

The YBE still implies that $[T(x;\ldots,y_{-1},y_0,y_1,\ldots),T(x';\ldots,y_{-1},y_0,y_1,\ldots)] = 0$.

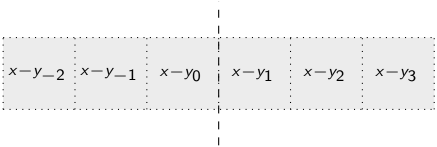
$$s_\lambda(x_1,\ldots,x_n;y_1,\ldots) = \langle 0| \prod_{i=1}^{n} T(x_i;\ldots,y_{-1},y_0,y_1,\ldots)\,|\lambda\rangle$$

is the factorial Schur polynomial.

Factorial Schur polynomials

Natural generalization: inhomogeneous statistical model. Define the inhomogeneous five-vertex transfer matrix:

$$T(x; \dots, y_{-1}, y_0, y_1, \dots) S^{-1} =$$



with same weights as ▸ before, except $x \to x - y_i$.

The YBE still implies that
$[T(x; \dots, y_{-1}, y_0, y_1, \dots), T(x'; \dots, y_{-1}, y_0, y_1, \dots)] = 0$.

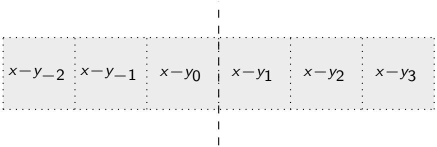
$$s_\lambda(x_1, \dots, x_n; y_1, \dots) = \langle 0 | \prod_{i=1}^{n} T(x_i; \dots, y_{-1}, y_0, y_1, \dots) | \lambda \rangle$$

is the factorial Schur polynomial.

Factorial Schur polynomials

Natural generalization: inhomogeneous statistical model. Define the inhomogeneous five-vertex transfer matrix:

$$T(x;\ldots,y_{-1},y_0,y_1,\ldots)S^{-1} =$$



with same weights as before, except $x \to x - y_i$.

The YBE still implies that
$[T(x;\ldots,y_{-1},y_0,y_1,\ldots),T(x';\ldots,y_{-1},y_0,y_1,\ldots)] = 0$.

$$s_\lambda(x_1,\ldots,x_n;y_1,\ldots) = \langle 0| \prod_{i=1}^{n} T(x_i;\ldots,y_{-1},y_0,y_1,\ldots)\,|\lambda\rangle$$

is the factorial Schur polynomial.

More generalizations

- Schur's Q-functions correspond to neutral fermions, i.e., living on a half-line...
- k-Schur functions related to the affine Grassmannian...
- Schubert and Grothendieck polynomials related to the flag variety...
- Hall–Littlewood polynomials related to Calogero model (nonlocal interaction) and generalizations (Jack/Macdonald polynomials?)
- Cotangent bundle of the Grassmannian...

More generalizations

- Schur's Q-functions correspond to neutral fermions, i.e., living on a half-line...
- k-Schur functions related to the affine Grassmannian...
- Schubert and Grothendieck polynomials related to the flag variety...
- Hall–Littlewood polynomials related to Calogero model (nonlocal interaction) and generalizations (Jack/Macdonald polynomials?)
- Cotangent bundle of the Grassmannian...

More generalizations

- Schur's Q-functions correspond to neutral fermions, i.e., living on a half-line. . .
- k-Schur functions related to the affine Grassmannian. . .
- Schubert and Grothendieck polynomials related to the flag variety. . .
- Hall–Littlewood polynomials related to Calogero model (nonlocal interaction) and generalizations (Jack/Macdonald polynomials?)
- Cotangent bundle of the Grassmannian. . .

More generalizations

- Schur's Q-functions correspond to neutral fermions, i.e., living on a half-line. . .
- k-Schur functions related to the affine Grassmannian. . .
- Schubert and Grothendieck polynomials related to the flag variety. . .
- Hall–Littlewood polynomials related to Calogero model (nonlocal interaction) and generalizations (Jack/Macdonald polynomials?)
- Cotangent bundle of the Grassmannian. . .

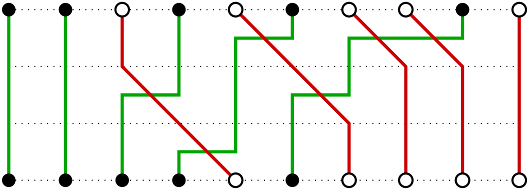
More generalizations

- Schur's Q-functions correspond to neutral fermions, i.e., living on a half-line...
- k-Schur functions related to the affine Grassmannian...
- Schubert and Grothendieck polynomials related to the flag variety...
- Hall–Littlewood polynomials related to Calogero model (nonlocal interaction) and generalizations (Jack/Macdonald polynomials?)
- Cotangent bundle of the Grassmannian...

1. Representation theory
2. Geometry
3. 1D Free fermions
4. 2D Lattice models
5. Tilings
   - Lozenge tilings
   - Domino tilings

Lozenge tilings

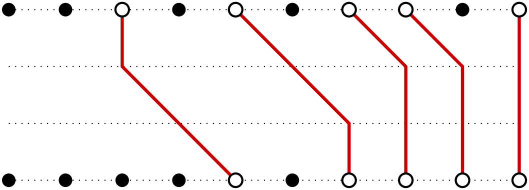
Tilt in yet another way $T(x)$ and focus on holes:



A weight of x is given to each light pink lozenge.

Lozenge tilings

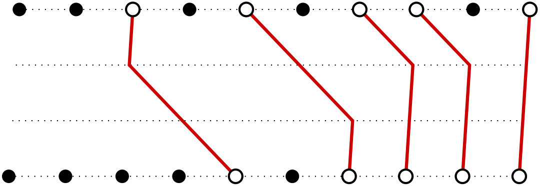
Tilt in yet another way $T(x)$ and focus on holes:



A weight of x is given to each light pink lozenge.

Lozenge tilings

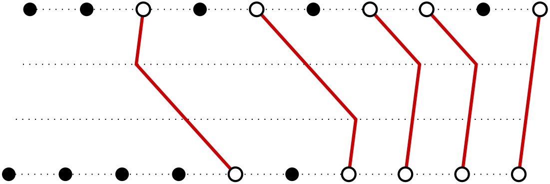
Tilt in yet another way $T(x)$ and focus on holes:



A weight of x is given to each light pink lozenge.

Lozenge tilings

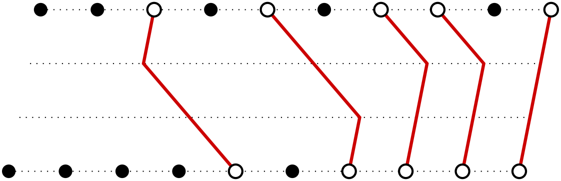
Tilt in yet another way $T(x)$ and focus on holes:



A weight of x is given to each light pink lozenge.

Lozenge tilings

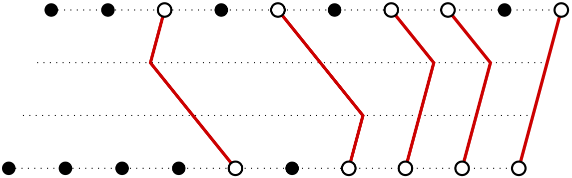
Tilt in yet another way $T(x)$ and focus on holes:



A weight of x is given to each light pink lozenge.

Lozenge tilings

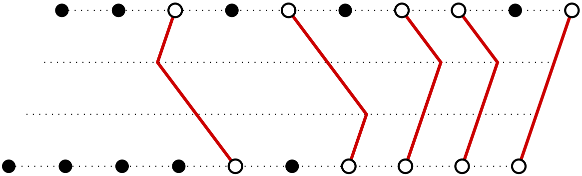
Tilt in yet another way $T(x)$ and focus on holes:



A weight of x is given to each light pink lozenge.

Lozenge tilings

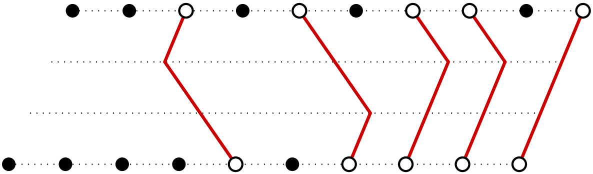
Tilt in yet another way $T(x)$ and focus on holes:



A weight of x is given to each light pink lozenge.

Lozenge tilings

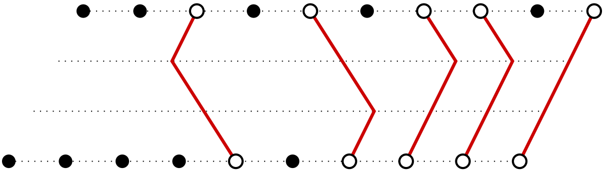
Tilt in yet another way $T(x)$ and focus on holes:



A weight of x is given to each light pink lozenge.

Lozenge tilings

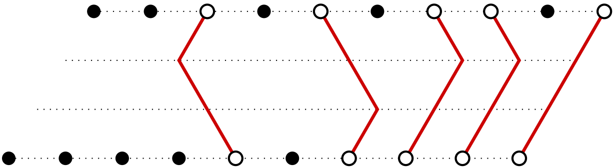
Tilt in yet another way $T(x)$ and focus on holes:



A weight of x is given to each light pink lozenge.

Lozenge tilings

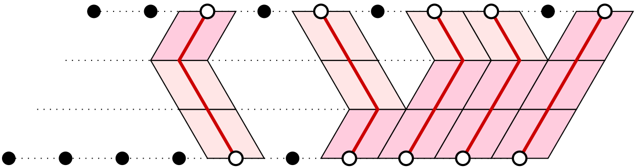
Tilt in yet another way $T(x)$ and focus on holes:



A weight of x is given to each light pink lozenge.

Lozenge tilings

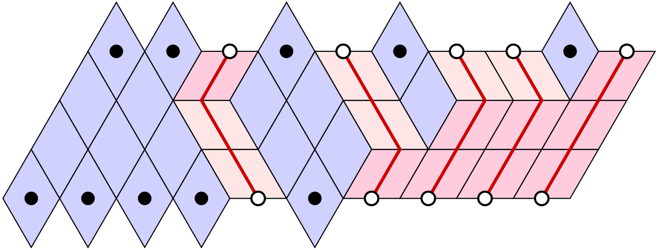
Tilt in yet another way $T(x)$ and focus on holes:



A weight of x is given to each light pink lozenge.

Lozenge tilings

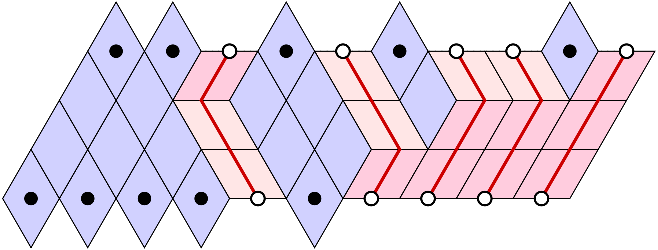
Tilt in yet another way $T(x)$ and focus on holes:



A weight of x is given to each light pink lozenge.

Lozenge tilings

Tilt in yet another way $T(x)$ and focus on holes:

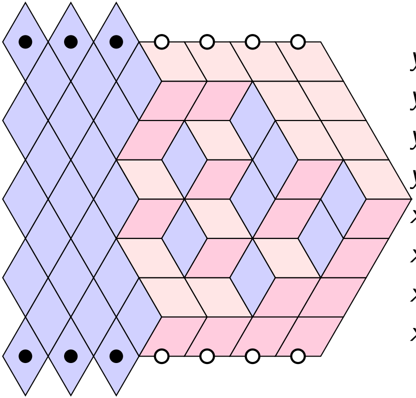


A weight of x is given to each light pink lozenge.

Application: plane partitions

$$\langle 0|\, T(x_1)\dots T(x_n)\, T^\star(y_1)\, T^\star(y_m)\, |0\rangle = \prod_{i,j} \frac{1}{1 - x_i y_j}$$

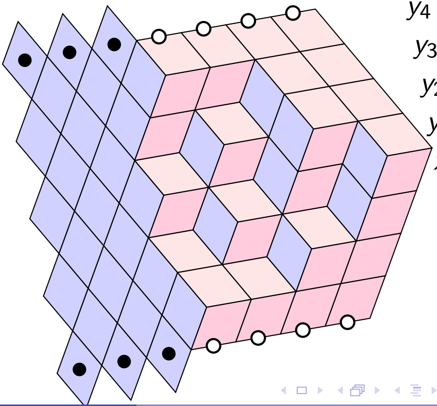
$=$



Application: plane partitions

$$\langle 0|\, T(x_1)\dots T(x_n)\, T^\star(y_1)\, T^\star(y_m)\, |0\rangle = \prod_{i,j} \frac{1}{1-x_i y_j}$$

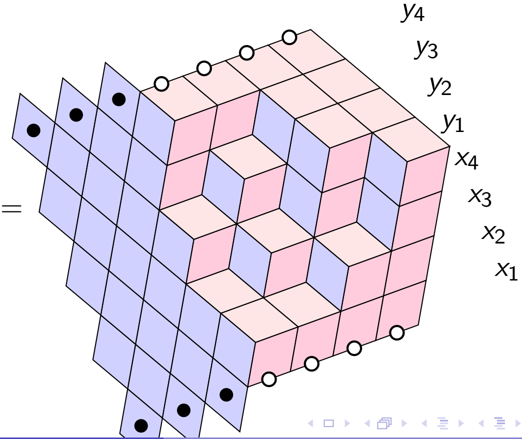
$=$



Application: plane partitions

$$\langle 0|\, T(x_1)\dots T(x_n)\, T^\star(y_1)\, T^\star(y_m)\, |0\rangle = \prod_{i,j} \frac{1}{1 - x_i y_j}$$

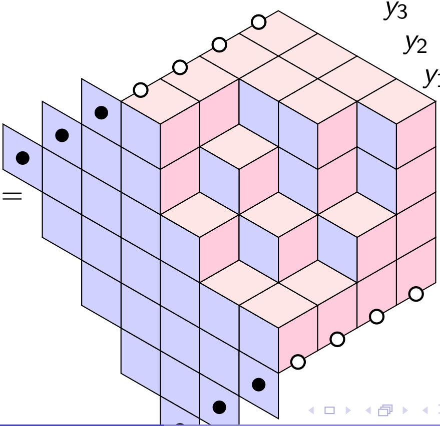
$=$



Application: plane partitions

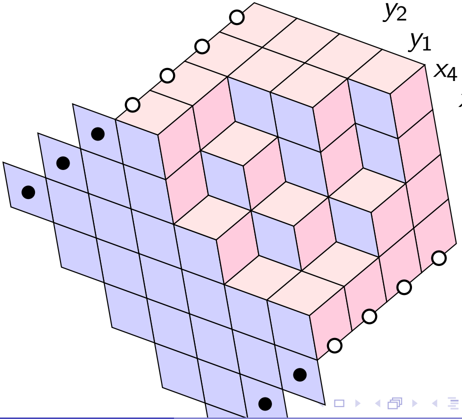
$$\langle 0|\, T(x_1)\dots T(x_n)\, T^\star(y_1)\, T^\star(y_m)\,|0\rangle = \prod_{i,j} \frac{1}{1-x_i y_j}$$

$=$



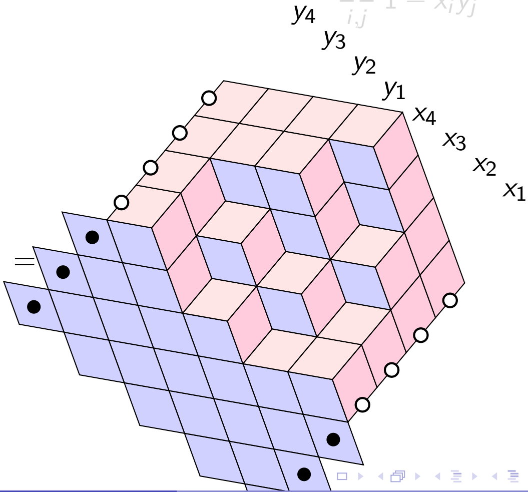
Application: plane partitions

$$\langle 0|\, T(x_1)\dots T(x_n)\, T^\star(y_1)\, T^\star(y_m)\, |0\rangle = \prod_{i,j} \frac{1}{1-x_i y_j}$$



Application: plane partitions

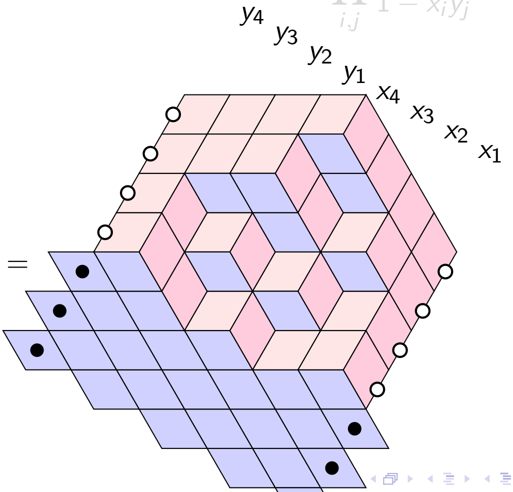
$$\langle 0|\, T(x_1)\dots T(x_n)\, T^\star(y_1)\, T^\star(y_m)\, |0\rangle = \prod_{i,j} \frac{1}{1 - x_i y_j}$$



Application: plane partitions

$$\langle 0|\, T(x_1)\dots T(x_n)\, T^\star(y_1)\, T^\star(y_m)\, |0\rangle = \prod_{i,j} \frac{1}{1-x_i y_j}$$

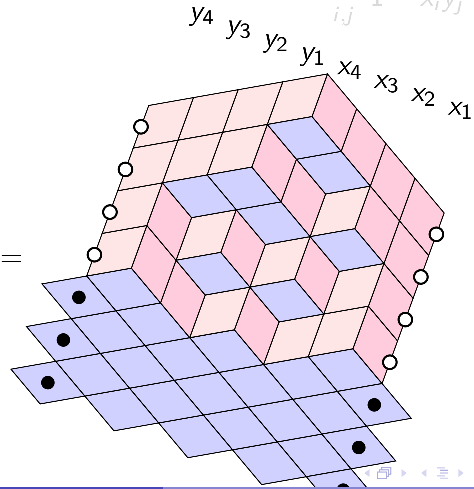
$=$



Application: plane partitions

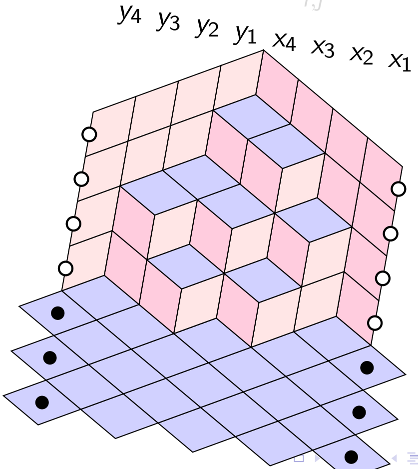
$$\langle 0|\, T(x_1)\dots T(x_n)\, T^\star(y_1)\, T^\star(y_m)\, |0\rangle = \prod_{i,j} \frac{1}{1-x_i y_j}$$

$=$



Application: plane partitions

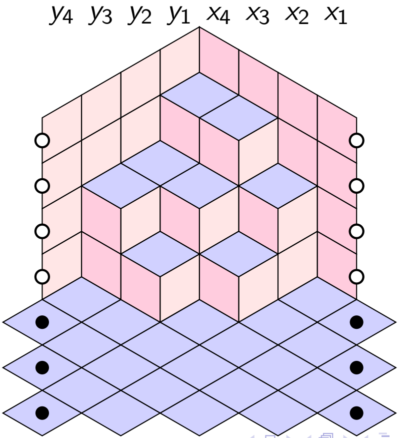
$$\langle 0|\, T(x_1)\ldots T(x_n)\, T^\star(y_1)\, T^\star(y_m)\, |0\rangle = \prod_{i,j} \frac{1}{1 - x_i y_j}$$

$=$ 

Application: plane partitions

$$\langle 0|\, T(x_1)\dots T(x_n)\, T^\star(y_1)\, T^\star(y_m)\, |0\rangle = \prod_{i,j} \frac{1}{1 - x_i y_j}$$

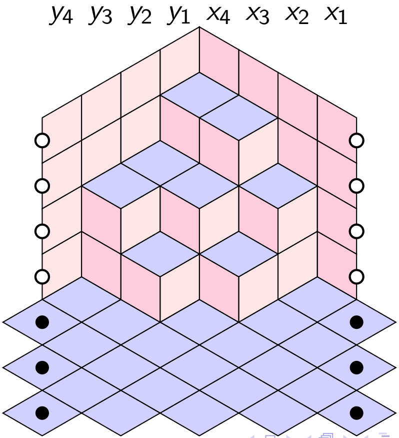
$=$



Application: plane partitions

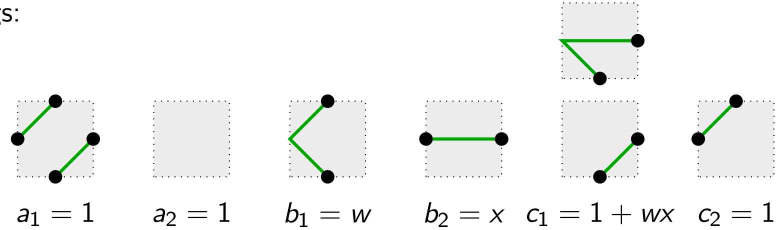
$$\langle 0|\, T(x_1)\dots T(x_n)\, T^\star(y_1)\, T^\star(y_m)\, |0\rangle \;=\prod_{i,j}\frac{1}{1-x_i y_j}$$

$=$



Domino tilings

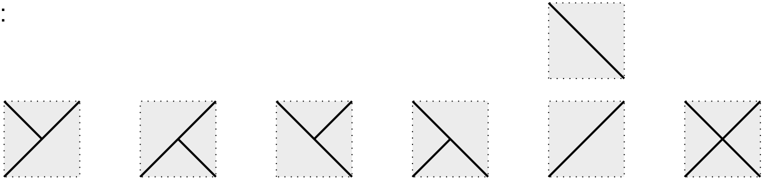
Similarly, the free-fermionic six-vertex model is equivalent to domino tilings:



On north and west edges, occupied = small piece, empty = large. the opposite for south and east edges. One large piece + 2 small pieces = one domino.

Domino tilings

Similarly, the free-fermionic six-vertex model is equivalent to domino tilings:

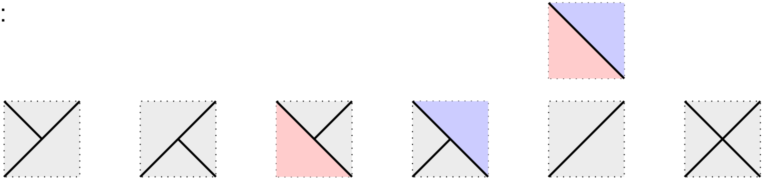


$a_1 = 1$ $a_2 = 1$ $b_1 = w$ $b_2 = x$ $c_1 = 1 + wx$ $c_2 = 1$

On north and west edges, occupied = small piece, empty = large. the opposite for south and east edges. One large piece + 2 small pieces = one domino.

Domino tilings

Similarly, the free-fermionic six-vertex model is equivalent to domino tilings:



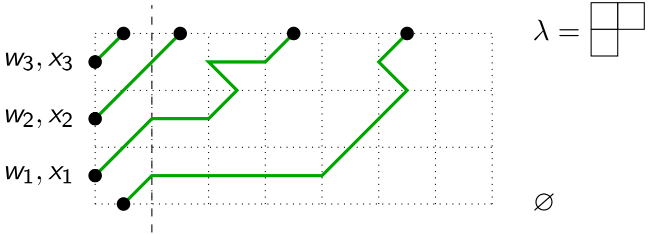
$a_1 = 1 \quad a_2 = 1 \quad b_1 = w \quad b_2 = x \quad c_1 = 1 + wx \quad c_2 = 1$

On north and west edges, occupied = small piece, empty = large. the opposite for south and east edges. One large piece + 2 small pieces = one domino.

Partial DWBC revisited

$$\langle 0|\, T_{6v}(w_1,x_1)\dots T_{6v}(w_n,x_n)\,|\lambda;n\rangle$$

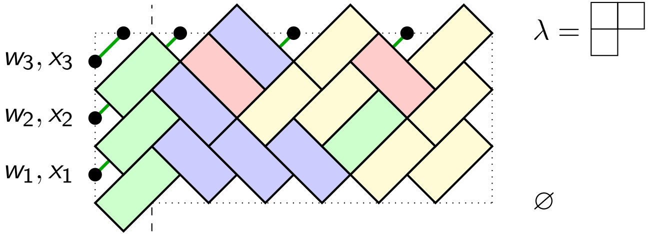
$=$



Partial DWBC revisited

$$\langle 0|\, T_{6v}(w_1, x_1) \dots T_{6v}(w_n, x_n)\, |\lambda; n\rangle$$

$=$



Part II

The Littlewood–Richardson rule

6 Various formulations of the rule
- Littlewood–Richardson coefficients
- The original Littlewood–Richardson rule
- Honeycombs
- Puzzles

7 Relation to tilings and projection method

8 Integrability of the square-triangle tiling model

Littlewood–Richardson coefficients

The ring of symmetric functions, equipped with its basis of Schur functions s_λ, possesses structure constants:

$$s_\lambda \, s_\mu = \sum_\nu c^\nu_{\lambda,\mu} s_\nu$$

called Littlewood–Richardson coefficients.

We are looking for combinatorial rules to compute them.

Littlewood–Richardson coefficients

The ring of symmetric functions, equipped with its basis of Schur functions s_λ, possesses structure constants:

$$s_\lambda\, s_\mu = \sum_\nu c^\nu_{\lambda,\mu} s_\nu$$

called Littlewood–Richardson coefficients.

We are looking for combinatorial rules to compute them.

Representation theory interpretation

- In the representation theory of the general linear group $G = GL(n)$,

$$V_\lambda \otimes V_\mu = \bigoplus_{\nu \text{ at most } n \text{ parts}} \mathbb{C}^{c^{\nu}_{\lambda,\mu}} \otimes V_\nu$$

- By Schur–Weyl duality: $(r = |\nu|, p + q = r)$

$$\mathrm{Res}^{\mathcal{S}_r}_{\mathcal{S}_p \times \mathcal{S}_q} W_\nu = \bigoplus_{\lambda,\mu : |\lambda| = p, |\mu| = q} \mathbb{C}^{c^{\nu}_{\lambda,\mu}} \otimes W_\lambda \otimes W_\mu$$

Representation theory interpretation

- In the representation theory of the general linear group $G = GL(n)$,

$$V_\lambda \otimes V_\mu = \bigoplus_{\nu \text{ at most } n \text{ parts}} \mathbb{C}^{c^\nu_{\lambda,\mu}} \otimes V_\nu$$

- By Schur–Weyl duality: $(r = |\nu|, p + q = r)$

$$\mathrm{Res}^{\mathcal{S}_r}_{\mathcal{S}_p \times \mathcal{S}_q} W_\nu = \bigoplus_{\lambda,\mu : |\lambda|=p, |\mu|=q} \mathbb{C}^{c^\nu_{\lambda,\mu}} \otimes W_\lambda \otimes W_\mu$$

Representation theory interpretation cont'd

- An *a priori* distinct problem: restriction from $G = GL(n)$ to $GL(\ell) \times GL(m)$, $n = \ell + m$:

$$V_\nu = \bigoplus_{\substack{\lambda \text{ at most } \ell \text{ parts} \\ \mu \text{ at most } m \text{ parts}}} \mathbb{C}^{c^\nu_{\lambda,\mu}} \otimes V_\lambda \otimes V_\mu$$

At the level of Schur polynomials, this gives the coproduct formula:

$$s_\nu(x_1, \ldots, x_\ell, y_1, \ldots, y_m) = \sum_{\lambda,\mu} c^\nu_{\lambda,\mu} s_\lambda(x_1, \ldots, x_\ell) s_\mu(y_1, \ldots, y_m)$$

- By Schur–Weyl duality: $(|\lambda| = p, |\mu| = q, p + q = r)$

$$\mathrm{Ind}^{\mathcal{S}_r}_{\mathcal{S}_p \times \mathcal{S}_q} W_\lambda \otimes W_\mu = \bigoplus_{\nu : |\nu| = r} \mathbb{C}^{c^\nu_{\lambda,\mu}} \otimes W_\nu$$

- We conclude by Frobenius duality that these coefficients are the same as on the previous slide.

Representation theory interpretation cont'd

- An *a priori* distinct problem: restriction from $G = GL(n)$ to $GL(\ell) \times GL(m)$, $n = \ell + m$:

$$V_\nu = \bigoplus_{\substack{\lambda \text{ at most } \ell \text{ parts} \\ \mu \text{ at most } m \text{ parts}}} \mathbb{C}^{c^\nu_{\lambda,\mu}} \otimes V_\lambda \otimes V_\mu$$

  At the level of Schur polynomials, this gives the coproduct formula:

$$s_\nu(x_1,\ldots,x_\ell,y_1,\ldots,y_m) = \sum_{\lambda,\mu} c^\nu_{\lambda,\mu} s_\lambda(x_1,\ldots,x_\ell) s_\mu(y_1,\ldots,y_m)$$

- By Schur–Weyl duality: $(|\lambda| = p, |\mu| = q, p + q = r)$

$$\mathrm{Ind}^{\mathcal{S}_r}_{\mathcal{S}_p \times \mathcal{S}_q} W_\lambda \otimes W_\mu = \bigoplus_{\nu : |\nu| = r} \mathbb{C}^{c^\nu_{\lambda,\mu}} \otimes W_\nu$$

- We conclude by Frobenius duality that these coefficients are the same as on the previous slide.

Representation theory interpretation cont'd

- An *a priori* distinct problem: restriction from $G = GL(n)$ to $GL(\ell) \times GL(m)$, $n = \ell + m$:

$$V_\nu = \bigoplus_{\substack{\lambda \text{ at most } \ell \text{ parts} \\ \mu \text{ at most } m \text{ parts}}} \mathbb{C}^{c^\nu_{\lambda,\mu}} \otimes V_\lambda \otimes V_\mu$$

  At the level of Schur polynomials, this gives the coproduct formula:

$$s_\nu(x_1, \ldots, x_\ell, y_1, \ldots, y_m) = \sum_{\lambda,\mu} c^\nu_{\lambda,\mu} s_\lambda(x_1, \ldots, x_\ell) s_\mu(y_1, \ldots, y_m)$$

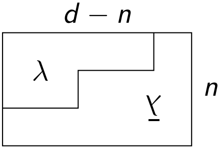
- By Schur–Weyl duality: $(|\lambda| = p, |\mu| = q, p + q = r)$

$$\mathrm{Ind}_{\mathcal{S}_p \times \mathcal{S}_q}^{\mathcal{S}_r} W_\lambda \otimes W_\mu = \bigoplus_{\nu : |\nu| = r} \mathbb{C}^{c^\nu_{\lambda,\mu}} \otimes W_\nu$$

- We conclude by Frobenius duality that these coefficients are the same as on the previous slide.

Geometric interpretation

If Young diagrams are restricted to be inside a given rectangle $n \times (d-n)$, there is an additional symmetry $\lambda \mapsto \bar{\lambda}$:



For Schubert varieties, this symmetry is Poincaré duality, i.e.,

$$[S_\lambda][S_\mu] = \delta_{\lambda,\bar{\mu}} \qquad |\lambda| + |\mu| = n(d-n)$$

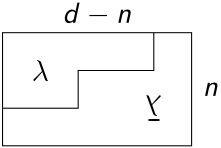
This gives an "intersection theory" interpretation of the LR coefficients:

$$c^{\nu}_{\lambda,\mu} = c_{\bar{\nu},\lambda,\mu} = [S_{\bar{\nu}}][S_\lambda][S_\mu] \qquad |\lambda| + |\mu| + |\bar{\nu}| = n(d-n)$$

Remark: equivariant analogues. . . cf [Knutson, Tao].

Geometric interpretation

If Young diagrams are restricted to be inside a given rectangle $n \times (d-n)$, there is an additional symmetry $\lambda \mapsto \bar{\lambda}$:



For Schubert varieties, this symmetry is Poincaré duality, i.e.,

$$[S_\lambda][S_\mu] = \delta_{\lambda,\bar{\mu}} \qquad |\lambda| + |\mu| = n(d-n)$$

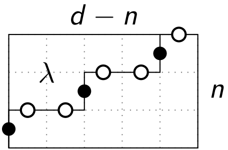
This gives an "intersection theory" interpretation of the LR coefficients:

$$c^{\nu}_{\lambda,\mu} = c_{\bar{\nu},\lambda,\mu} = [S_{\bar{\nu}}][S_\lambda][S_\mu] \qquad |\lambda| + |\mu| + |\bar{\nu}| = n(d-n)$$

Remark: equivariant analogues. . . cf [Knutson, Tao].

Binary strings

Young diagrams inside a rectangle $n \times (d-n)$ are in bijection with finite binary strings of n $\bullet$ and $d-n$ $\circ$:



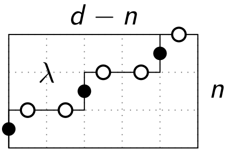
Various symmetries:

- $\lambda \mapsto \lambda'$: conjugation corresponds as before to reading right to left plus hole $\leftrightarrow$ particle:

- $\lambda \mapsto \bar{\lambda}$: complementation is only reading right to left:

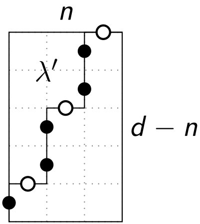
Binary strings

Young diagrams inside a rectangle $n \times (d-n)$ are in bijection with finite binary strings of n ● and $d-n$ ○:



Various symmetries:

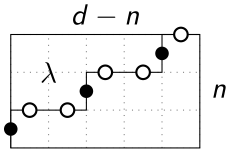
- $\lambda \mapsto \lambda'$: conjugation corresponds as before to reading right to left plus hole $\leftrightarrow$ particle:



- $\lambda \mapsto \bar{\lambda}$: complementation is only reading right to left:

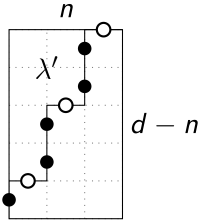
Binary strings

Young diagrams inside a rectangle $n \times (d-n)$ are in bijection with finite binary strings of n ● and $d-n$ ○:

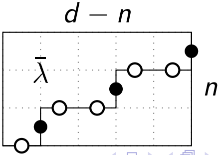


Various symmetries:

- $\lambda \mapsto \lambda'$: conjugation corresponds as before to reading right to left plus hole $\leftrightarrow$ particle:



- $\lambda \mapsto \bar{\lambda}$: complementation is only reading right to left:



Symmetries

Summary of symmetries:

$$c_{\nu,\lambda,\mu} = c_{\nu,\mu,\lambda}$$

$$c_{\nu,\lambda,\mu} = c_{\lambda,\mu,\nu}$$

$$c_{\nu,\lambda,\mu} = c_{\nu',\mu',\lambda'}$$

Symmetries

Summary of symmetries:

$$c_{\nu,\lambda,\mu} = c_{\nu,\mu,\lambda}$$

$$c_{\nu,\lambda,\mu} = c_{\lambda,\mu,\nu}$$

$$c_{\nu,\lambda,\mu} = c_{\nu',\mu',\lambda'} \qquad \checkmark$$

Symmetries

Summary of symmetries:

$$c_{\nu,\lambda,\mu} = c_{\nu,\mu,\lambda}$$

$$c_{\nu,\lambda,\mu} = c_{\lambda,\mu,\nu} \qquad \checkmark$$

$$c_{\nu,\lambda,\mu} = c_{\nu',\mu',\lambda'} \qquad \checkmark$$

Symmetries

Summary of symmetries:

$$c_{\nu,\lambda,\mu} = c_{\nu,\mu,\lambda} \qquad ?$$

$$c_{\nu,\lambda,\mu} = c_{\lambda,\mu,\nu} \qquad \checkmark$$

$$c_{\nu,\lambda,\mu} = c_{\nu',\mu',\lambda'} \qquad \checkmark$$

Symmetries

Summary of symmetries:

$$c_{\nu,\lambda,\mu} = c_{\nu,\mu,\lambda} \qquad ?$$

$$c_{\nu,\lambda,\mu} = c_{\lambda,\mu,\nu} \qquad \checkmark$$

$$c_{\nu,\lambda,\mu} = c_{\nu',\mu',\lambda'} \qquad \checkmark$$

see however [Thomas, Yong].

The original Littlewood–Richardson rule

Given three Young diagrams λ, μ, ν, $|\lambda| + |\mu| = |\nu|$, a Littlewood–Richardson tableau is a filling of the boxes of ν/μ with λ_1 1's, λ_2 2's, . . . , in such a way that

1. The rows are weakly increasing.
2. The columns are strictly increasing.
3. The word obtained by reading the filling from right to left, top to bottom is such that any initial subword has more i's than $i+1$'s.

Theorem

$c^{\nu}_{\lambda,\mu}$ is the number of such tableaux.

The original Littlewood–Richardson rule

Given three Young diagrams λ, μ, ν, $|\lambda| + |\mu| = |\nu|$, a Littlewood–Richardson tableau is a filling of the boxes of ν/μ with λ_1 1's, λ_2 2's, ..., in such a way that

1. The rows are weakly increasing.
2. The columns are strictly increasing.
3. The word obtained by reading the filling from right to left, top to bottom is such that any initial subword has more i's than $i+1$'s.

Theorem

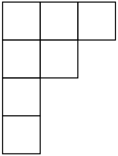
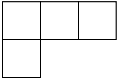
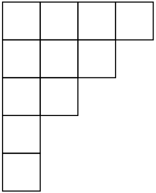
$c^{\nu}_{\lambda,\mu}$ is the number of such tableaux.

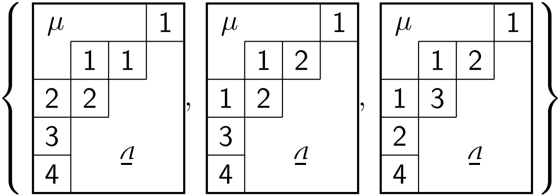
Example

$$\lambda = \ydiagram{3,2,1,1}\,, \quad \mu = \ydiagram{3,1}\,, \quad \nu = \ydiagram{4,3,2,1,1}\,.$$

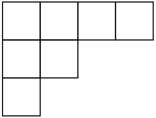
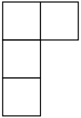
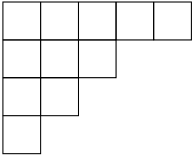
$$c^{\nu}_{\lambda,\mu} = c_{\bar\nu,\lambda,\mu} = \# \left\{ \begin{ytableau} \none & \none & \none & 1 \\ \none & 1 & 1 \\ 2 & 2 \\ 3 \\ 4 \end{ytableau}\,, \quad \begin{ytableau} \none & \none & \none & 1 \\ \none & 1 & 2 \\ 1 & 2 \\ 3 \\ 4 \end{ytableau}\,, \quad \begin{ytableau} \none & \none & \none & 1 \\ \none & 1 & 2 \\ 1 & 3 \\ 2 \\ 4 \end{ytableau} \right\}$$

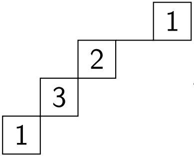
Example

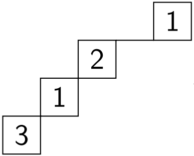
$\lambda =$ , $\mu =$ , $\nu =$ .

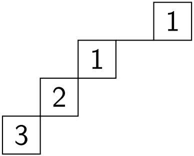
$$c^{\nu}_{\lambda,\mu} = c_{\bar{\nu},\lambda,\mu} = \#$$ 

Example

$\lambda' =$  , $\mu' =$  , $\nu' =$  .

$c^{\nu'}_{\mu',\lambda'} = c_{\bar{\nu}',\mu',\lambda'} = \#\Bigg\{$ 

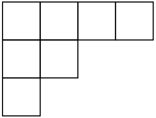
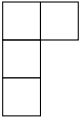
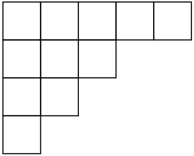
, 

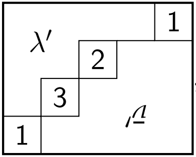
, 

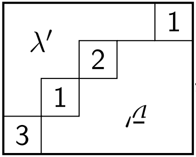
$\Bigg\}$

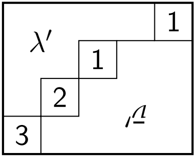
Remark: by conjugating the diagrams, we *do not* obtain the LR rule for $c^{\nu}_{\mu,\lambda}$.

Example

$\lambda' =$ , $\mu' =$ , $\nu' =$ .

$c^{\nu'}_{\mu',\lambda'} = c_{\bar{\nu}',\mu',\lambda'} = \#\Bigg\{$ 

, 

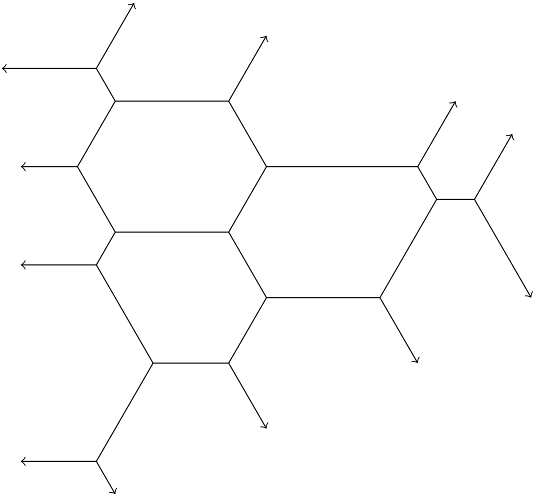
, 

$\Bigg\}$

Remark: by conjugating the diagrams, we *do not* obtain the LR rule for $c^{\nu}_{\mu,\lambda}$.

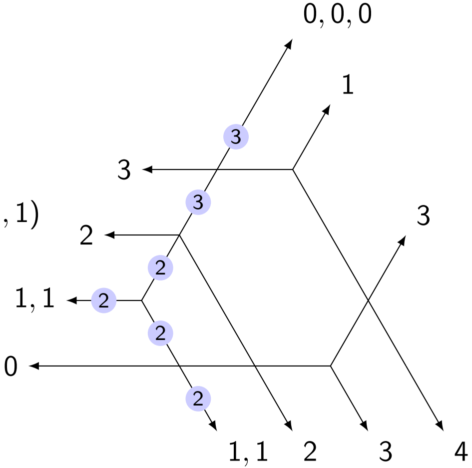
Honeycombs

A nondegenerate honeycomb:



Honeycomb example

$\lambda = (3, 2, 1, 1)$

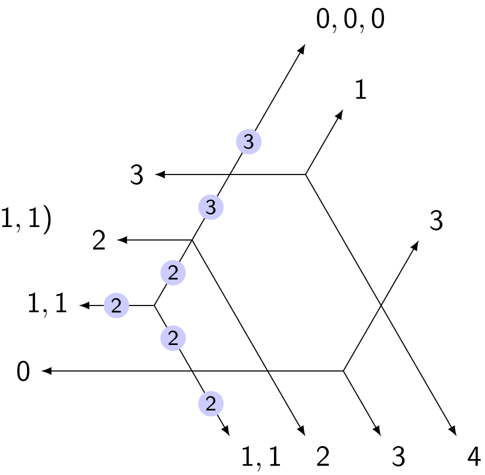


$\mu = (3, 1)$

plus two more...

$\nu = (4, 3, 2, 1, 1)$

Honeycomb example



$\lambda = (3, 2, 1, 1)$

$\mu = (3, 1)$

$\nu = (4, 3, 2, 1, 1)$

plus two more...

The three states

Introduce three states $\textcolor{green}{\bullet}$, $\textcolor{red}{\bullet}$, $\textcolor{lightblue}{\bullet}$. Substitute them to the usual two states according to the rule:

λ :	$\bullet = \textcolor{green}{\bullet}$	$\circ = \textcolor{lightblue}{\bullet}$
μ :	$\bullet = \textcolor{lightblue}{\bullet}$	$\circ = \textcolor{red}{\bullet}$
ν or $\bar{\nu}$:	$\bullet = \textcolor{green}{\bullet}$	$\circ = \textcolor{red}{\bullet}$

Red particles will be called right-movers, green particles left-movers (gauche-movers!).

The three states

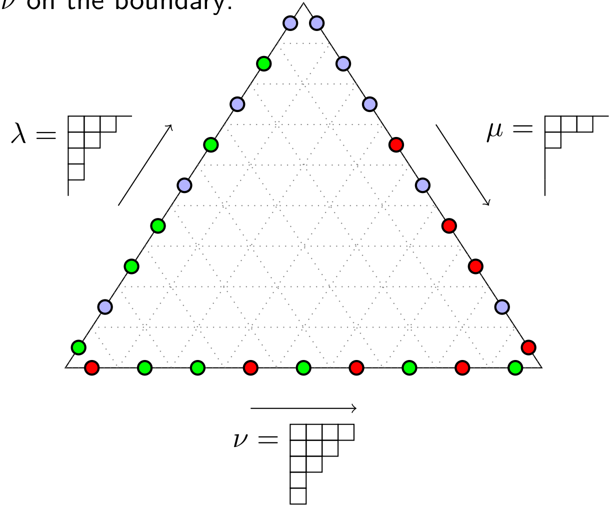
Introduce three states ${\color{green}\bullet}$, ${\color{red}\bullet}$, ${\color{lavender}\bullet}$. Substitute them to the usual two states according to the rule:

$$\begin{array}{rcc}
\lambda: & \bullet = {\color{green}\bullet} & \circ = {\color{lavender}\bullet} \\
\mu: & \bullet = {\color{lavender}\bullet} & \circ = {\color{red}\bullet} \\
\nu \text{ or } \bar{\nu}: & \bullet = {\color{green}\bullet} & \circ = {\color{red}\bullet}
\end{array}$$

Red particles will be called right-movers, green particles left-movers (gauche-movers!).

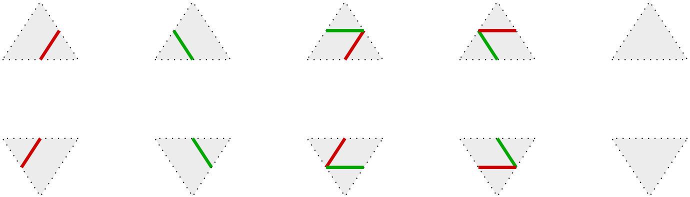
Puzzles: boundaries

In a triangle of size d on the triangular lattice, draw the three binary strings λ, μ, ν on the boundary:



Puzzles: inside

Fill the inside of the triangle with the following tiles:

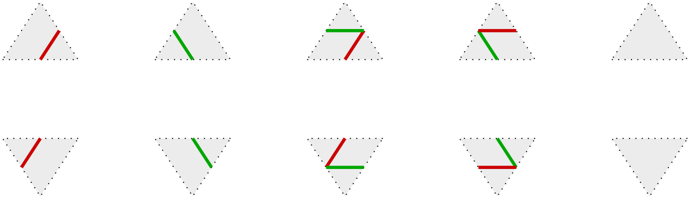


so red and green lines are continuous.

On the boundary, ● (resp. ●) are the entering/exiting locations of red (resp. green) lines.

Puzzles: inside

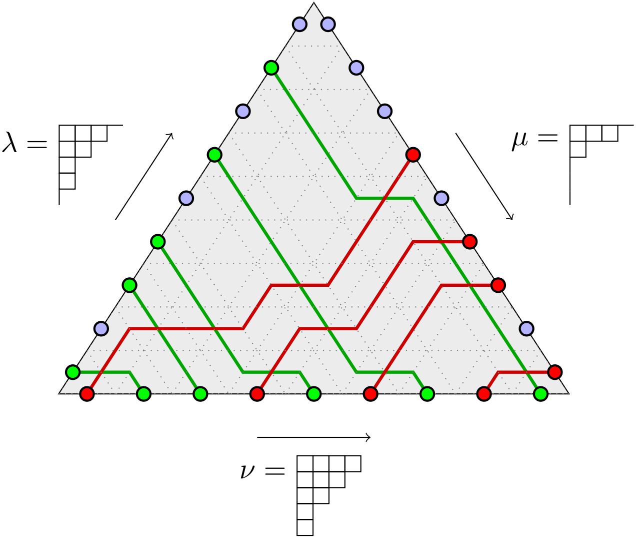
Fill the inside of the triangle with the following tiles:



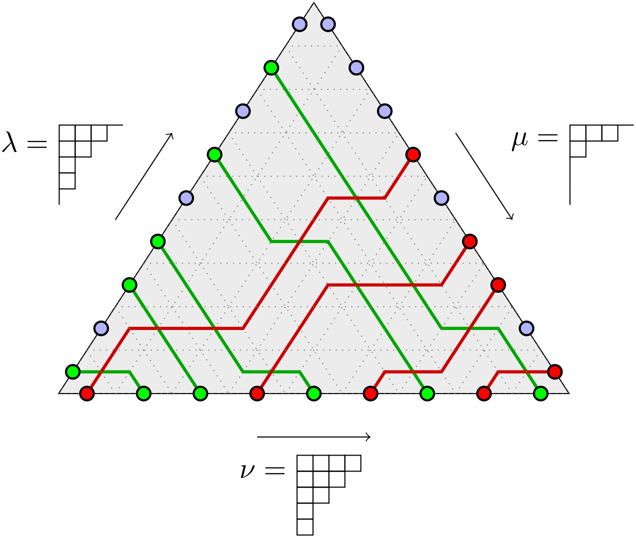
so red and green lines are continuous.

On the boundary, (resp.) are the entering/exiting locations of red (resp. green) lines.

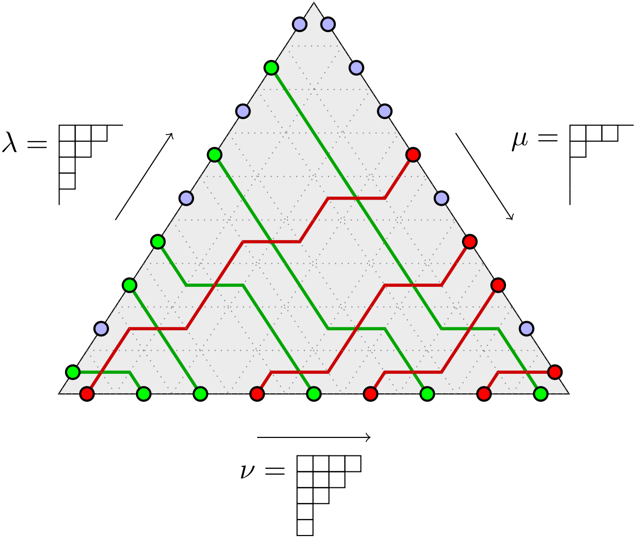
Puzzle example I



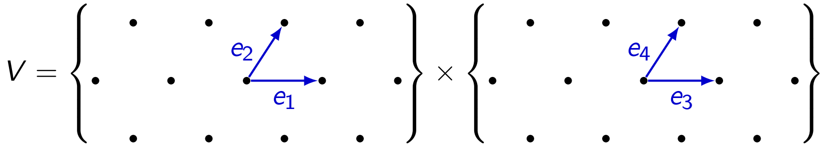
Puzzle example II



Puzzle example III



Let V be the lattice inside Euclidean space $\mathbb{R}^4$ obtained as the Cartesian product of two triangular lattices:



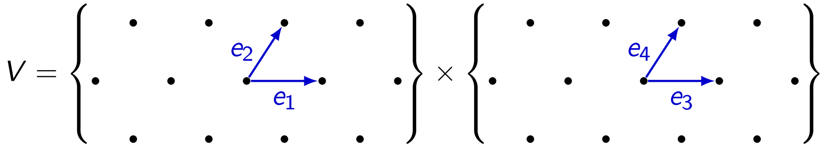
Define E to be edges of equilateral triangles of either triangular lattice:

$$E = \{[x, x+e], x \in V, e \in \{e_1, e_2, e_2 - e_1, e_3, e_4, e_4 - e_3\}\}$$

Define F to be triangles of either triangular lattice and certain rhombi:

$$F = V + \Big\{[0, e_1, e_2], [0, e_2, e_2 - e_1], [0, e_3, e_4], [0, e_4, e_4 - e_3],$$
$$[0, e_1, e_4 - e_3, e_1 + e_4 - e_3], [0, e_4, e_2 - e_1, e_4 + e_2 - e_1], [0, e_2, e_3, e_2 + e_3]\Big\}$$

Let V be the lattice inside Euclidean space $\mathbb{R}^4$ obtained as the Cartesian product of two triangular lattices:



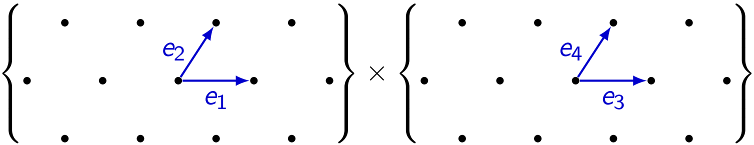
Define E to be edges of equilateral triangles of either triangular lattice:

$$E = \{[x, x+e], x \in V, e \in \{e_1, e_2, e_2 - e_1, e_3, e_4, e_4 - e_3\}\}$$

Define F to be triangles of either triangular lattice and certain rhombi:

$$F = V + \Big\{[0, e_1, e_2], [0, e_2, e_2 - e_1], [0, e_3, e_4], [0, e_4, e_4 - e_3],$$
$$[0, e_1, e_4 - e_3, e_1 + e_4 - e_3], [0, e_4, e_2 - e_1, e_4 + e_2 - e_1], [0, e_2, e_3, e_2 + e_3]\Big\}$$

Let V be the lattice inside Euclidean space $\mathbb{R}^4$ obtained as the Cartesian product of two triangular lattices:

$V =$ 

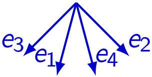
Define E to be edges of equilateral triangles of either triangular lattice:

$$E = \{[x, x+e], x \in V, e \in \{e_1, e_2, e_2 - e_1, e_3, e_4, e_4 - e_3\}\}$$

Define F to be triangles of either triangular lattice and certain rhombi:

$$F = V + \Big\{[0, e_1, e_2], [0, e_2, e_2 - e_1], [0, e_3, e_4], [0, e_4, e_4 - e_3],$$
$$[0, e_1, e_4 - e_3, e_1 + e_4 - e_3], [0, e_4, e_2 - e_1, e_4 + e_2 - e_1], [0, e_2, e_3, e_2 + e_3]\Big\}$$

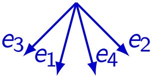
We are interested in surfaces inside (V, E, F) with a certain transversality condition. For that we need a canonical projection $p_\parallel : \mathbb{R}^4 \to \mathbb{R}^2$:



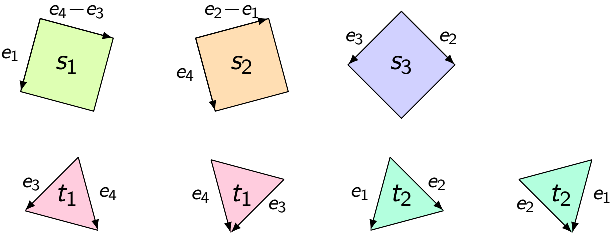
Elements of F, once projected, become tiles:

We define a surface Σ to be a tiling if the map $p_\parallel : \Sigma \to p_\parallel(\Sigma)$ is one-to-one.

We are interested in surfaces inside (V, E, F) with a certain transversality condition. For that we need a canonical projection $p_{\|} : \mathbb{R}^4 \to \mathbb{R}^2$:

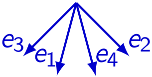


Elements of F, once projected, become tiles:

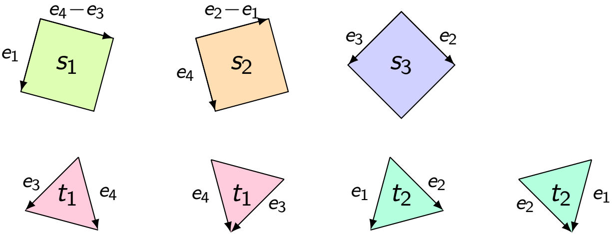


We define a surface Σ to be a tiling if the map $p_{\|} : \Sigma \to p_{\|}(\Sigma)$ is one-to-one.

We are interested in surfaces inside (V, E, F) with a certain transversality condition. For that we need a canonical projection $p_\parallel : \mathbb{R}^4 \to \mathbb{R}^2$:



Elements of F, once projected, become tiles:



We define a surface Σ to be a tiling if the map $p_\parallel : \Sigma \to p_\parallel(\Sigma)$ is one-to-one.

Boundary conditions

We require Σ to be connected, with a boundary as follows: Given $\lambda, \mu, \nu \subset n \times (d-n)$, we translate them into binary strings, convert them to vectors according to

$$
\begin{array}{lll}
\lambda : & \bullet = \textcolor{green}{\bullet} \to -e_1, & \circ = \textcolor{lightblue}{\bullet} \to -e_3 \\
\mu : & \bullet = \textcolor{lightblue}{\bullet} \to e_2, & \circ = \textcolor{red}{\bullet} \to e_4 \\
\bar\nu : & \bullet = \textcolor{green}{\bullet} \to e_1 - e_2, & \circ = \textcolor{red}{\bullet} \to e_3 - e_4
\end{array}
$$

and then concatenate them (preserving the cyclic order $\lambda, \mu, \bar\nu$). This is the required sequence of edges at the boundary.

In projection: [puzzle viewer]
Such tilings were introduced essentially independently in [Purbhoo, Puzzles, Tableaux and Mosaics] and [PZJ, Littlewood–Richardson coefficients and integrable tilings].

Boundary conditions

We require Σ to be connected, with a boundary as follows: Given $\lambda, \mu, \nu \subset n \times (d-n)$, we translate them into binary strings, convert them to vectors according to

$$\begin{array}{llll} \lambda: & \bullet = \textcolor{green}{\bullet} \to -e_1, & \circ = \textcolor{lavender}{\bullet} \to -e_3 \\ \mu: & \bullet = \textcolor{lavender}{\bullet} \to e_2, & \circ = \textcolor{red}{\bullet} \to e_4 \\ \bar\nu: & \bullet = \textcolor{green}{\bullet} \to e_1 - e_2, & \circ = \textcolor{red}{\bullet} \to e_3 - e_4 \end{array}$$

and then concatenate them (preserving the cyclic order $\lambda, \mu, \bar\nu$). This is the required sequence of edges at the boundary.

In projection: [puzzle viewer]
Such tilings were introduced essentially independently in [Purbhoo, Puzzles, Tableaux and Mosaics] and [PZJ, Littlewood–Richardson coefficients and integrable tilings].

Theorem

The number of tilings Σ with boundaries of type $\lambda, \mu, \bar{\nu}$ is $c_{\bar{\nu},\lambda,\mu}$.

As a corollary, we obtain for free all other forms of the Littlewood–Richardson that were mentioned before! (and bijections between them)
(to get the content of the tableau simply number the red or green lines)
[puzzle viewer]

Theorem

The number of tilings Σ with boundaries of type $\lambda, \mu, \bar{\nu}$ is $c_{\bar{\nu},\lambda,\mu}$.

As a corollary, we obtain for free all other forms of the Littlewood–Richardson that were mentioned before! (and bijections between them)
(to get the content of the tableau simply number the red or green lines)
[puzzle viewer]

- Random tilings are simple models whose main purpose is to describe quasi-crystals. They can always be thought as (fluctuating) surfaces obtained by projection from a higher-dimensional space.
  Remark: a general surface in (V, E) will project to squares, triangles, and thin rhombi.
- They typically correspond to a high-temperature limit where entropy considerations dominate – as opposed to deterministic tilings, i.e., quasi-periodic tilings, which correspond to a low-temperature limit where energy dominates. In fact in 2D, no phase transition.
- One can still hope for a form of quasi-periodicity, in the sense that "typical" configurations may have forbidden symmetries. For example, the square/triangle model can have 12-fold symmetry!
- Question: is the square-triangle tiling integrable? [Widom, Kalugin, de Gier–Nienhuis]

- Random tilings are simple models whose main purpose is to describe quasi-crystals. They can always be thought as (fluctuating) surfaces obtained by projection from a higher-dimensional space.
  Remark: a general surface in (V, E) will project to squares, triangles, and thin rhombi.
- They typically correspond to a high-temperature limit where entropy considerations dominate – as opposed to deterministic tilings, i.e., quasi-periodic tilings, which correspond to a low-temperature limit where energy dominates. In fact in 2D, no phase transition.
- One can still hope for a form of quasi-periodicity, in the sense that "typical" configurations may have forbidden symmetries. For example, the square/triangle model can have 12-fold symmetry!
- Question: is the square-triangle tiling integrable? [Widom, Kalugin, de Gier–Nienhuis]

- Random tilings are simple models whose main purpose is to describe quasi-crystals. They can always be thought as (fluctuating) surfaces obtained by projection from a higher-dimensional space.
  Remark: a general surface in (V, E) will project to squares, triangles, and thin rhombi.
- They typically correspond to a high-temperature limit where entropy considerations dominate – as opposed to deterministic tilings, i.e., quasi-periodic tilings, which correspond to a low-temperature limit where energy dominates. In fact in 2D, no phase transition.
- One can still hope for a form of quasi-periodicity, in the sense that "typical" configurations may have forbidden symmetries. For example, the square/triangle model can have 12-fold symmetry!
- Question: is the square-triangle tiling integrable? [Widom, Kalugin, de Gier–Nienhuis]

- Random tilings are simple models whose main purpose is to describe quasi-crystals. They can always be thought as (fluctuating) surfaces obtained by projection from a higher-dimensional space.
  Remark: a general surface in (V, E) will project to squares, triangles, and thin rhombi.
- They typically correspond to a high-temperature limit where entropy considerations dominate – as opposed to deterministic tilings, i.e., quasi-periodic tilings, which correspond to a low-temperature limit where energy dominates. In fact in 2D, no phase transition.
- One can still hope for a form of quasi-periodicity, in the sense that "typical" configurations may have forbidden symmetries. For example, the square/triangle model can have 12-fold symmetry!
- Question: is the square-triangle tiling integrable? [Widom, Kalugin, de Gier–Nienhuis]

- Random tilings are simple models whose main purpose is to describe quasi-crystals. They can always be thought as (fluctuating) surfaces obtained by projection from a higher-dimensional space.
  Remark: a general surface in (V, E) will project to squares, triangles, and thin rhombi.
- They typically correspond to a high-temperature limit where entropy considerations dominate – as opposed to deterministic tilings, i.e., quasi-periodic tilings, which correspond to a low-temperature limit where energy dominates. In fact in 2D, no phase transition.
- One can still hope for a form of quasi-periodicity, in the sense that "typical" configurations may have forbidden symmetries. For example, the square/triangle model can have 12-fold symmetry!
- Question: is the square-triangle tiling integrable? [Widom, Kalugin, de Gier–Nienhuis]

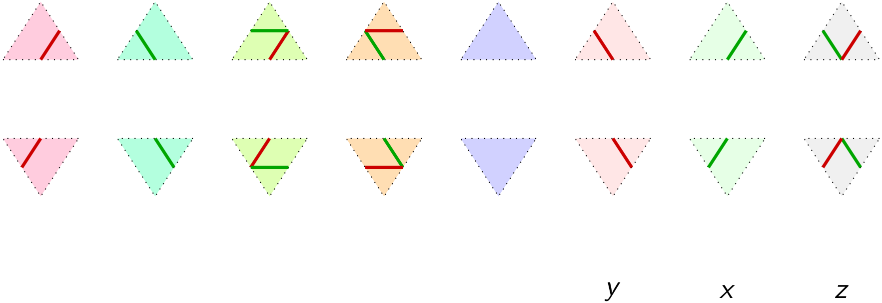
6 Various formulations of the rule

7 Relation to tilings and projection method

8 Integrability of the square-triangle tiling model
- Squares, triangles and thin rhombi
- Transfer matrices
- An integrable proof of the LR rule
- Inhomogeneities and equivariant puzzles
- Solving random tiling models

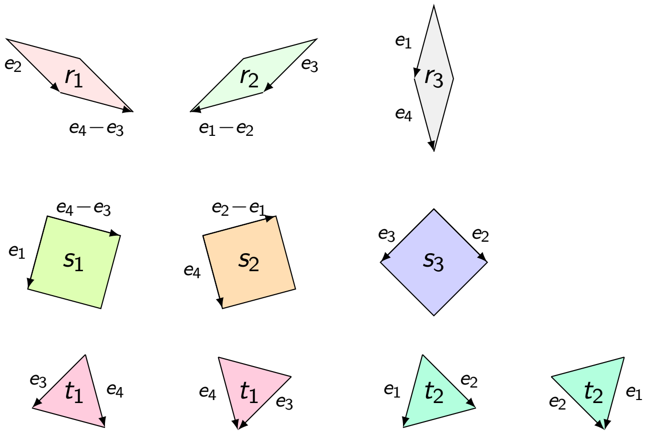
More tiles!

To make sense of the (co)product of two Schur functions, we want two independent NILPs (red and green lines) → introduce extra triangles for puzzles: [de Gier, Nienhuis; PZJ]



Thin rhombi

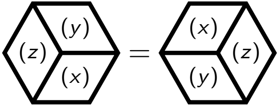
It amounts to adding thin rhombi to square-triangle tilings:



Yang–Baxter equation

Theorem

If $x + y + z = 0$, then



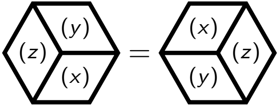
for any fixed boundaries and where tile x (resp. y, z) is only allowed where marked.

Example:

Yang–Baxter equation

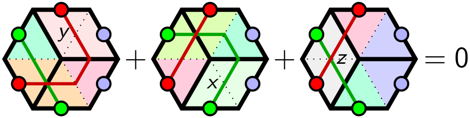
Theorem

If $x + y + z = 0$*, then*



for any fixed boundaries and where tile x *(resp.* y*,* z*) is only allowed where marked.*

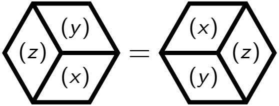
Example:



Yang–Baxter equation

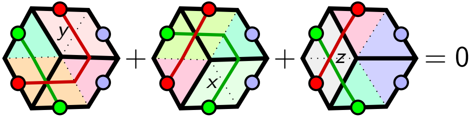
Theorem

If $x + y + z = 0$, then



for any fixed boundaries and where tile x (resp. y, z) is only allowed where marked.

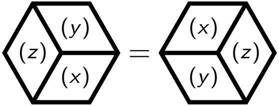
Example:



Yang–Baxter equation

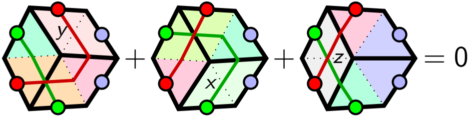
Theorem

If $x + y + z = 0$, then



for any fixed boundaries and where tile x (resp. y, z) is only allowed where marked.

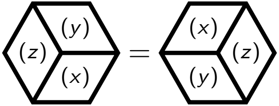
Example:



Yang–Baxter equation

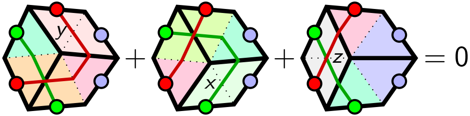
Theorem

If $x + y + z = 0$, *then*



for any fixed boundaries and where tile x *(resp.* y, z*) is only allowed where marked.*

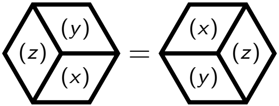
Example:



Yang–Baxter equation

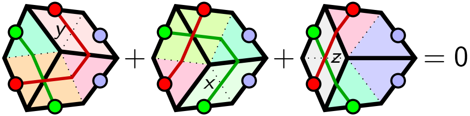
Theorem

If $x + y + z = 0$, then



for any fixed boundaries and where tile x (resp. y, z) is only allowed where marked.

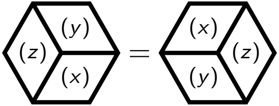
Example:



Yang–Baxter equation

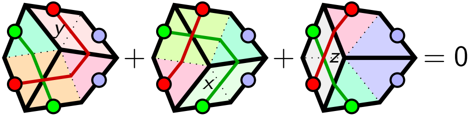
Theorem

If $x + y + z = 0$, then



for any fixed boundaries and where tile x (resp. y, z) is only allowed where marked.

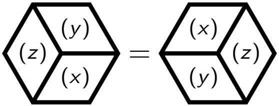
Example:



Yang–Baxter equation

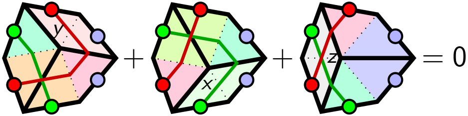
Theorem

If $x + y + z = 0$, then



for any fixed boundaries and where tile x (resp. y, z) is only allowed where marked.

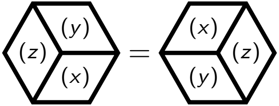
Example:



Yang–Baxter equation

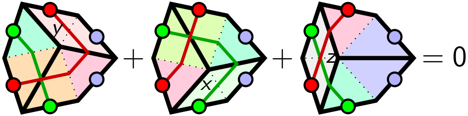
Theorem

If $x + y + z = 0$, then



for any fixed boundaries and where tile x (resp. y, z) is only allowed where marked.

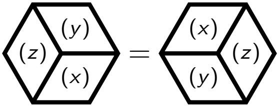
Example:



Yang–Baxter equation

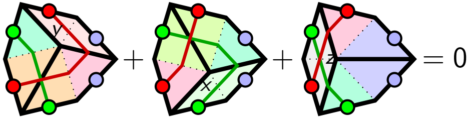
Theorem

If $x + y + z = 0$, then



for any fixed boundaries and where tile x (resp. y, z) is only allowed where marked.

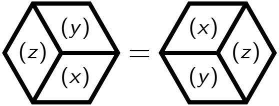
Example:



Yang–Baxter equation

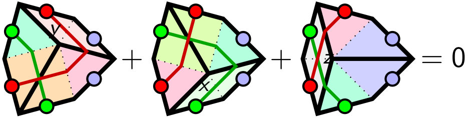
Theorem

If $x + y + z = 0$*, then*



for any fixed boundaries and where tile x *(resp.* y*,* z*) is only allowed where marked.*

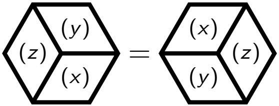
Example:



Yang–Baxter equation

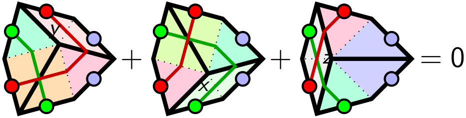
Theorem

If $x + y + z = 0$, then



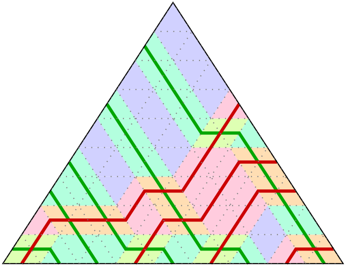
for any fixed boundaries and where tile x (resp. y, z) is only allowed where marked.

Example:



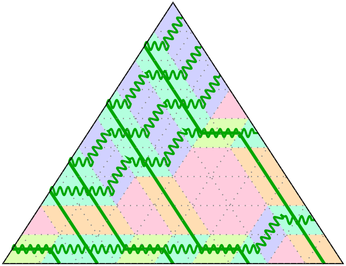
Extending puzzles

There are various ways to extend puzzles. A honeycomb-inspired extension:



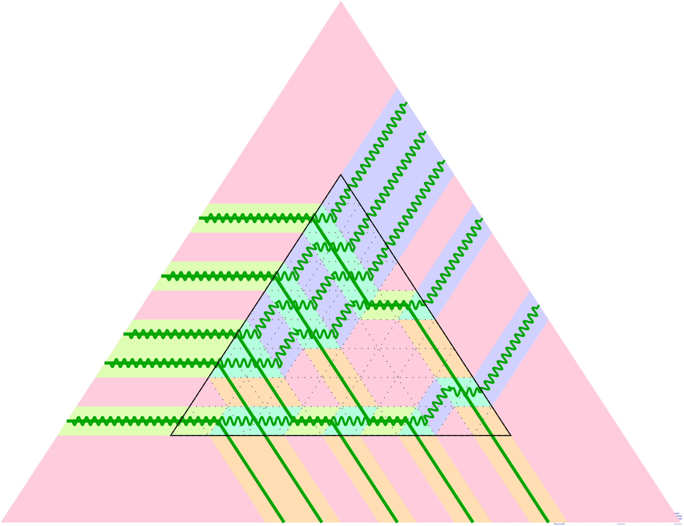
Extending puzzles

There are various ways to extend puzzles. A honeycomb-inspired extension:



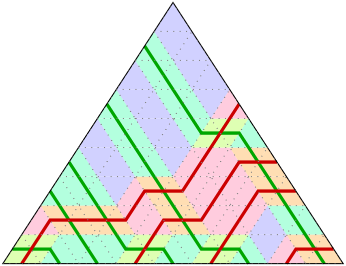
Extending puzzles

There are various ways to extend puzzles. A honeycomb-inspired extension:



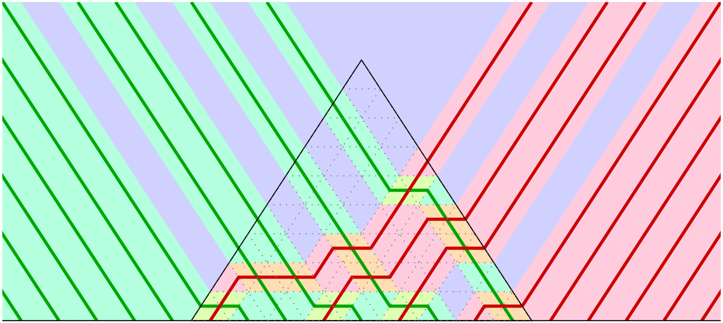
Extending puzzles cont'd

A red/green line extension to the half-plane:



Extending puzzles cont'd

A red/green line extension to the half-plane:



Fock space and Transfer matrix

The Fock space $\mathcal{G}$ has a natural basis indexed by sequences $(a_k) \in \{\textcolor{green}{\bullet}, \textcolor{lavender}{\bullet}, \textcolor{red}{\bullet}\}^{\mathbb{Z}}$ such that sufficiently far to the left (resp. right), $a_k = \textcolor{green}{\bullet}$ (resp. $\textcolor{red}{\bullet}$).

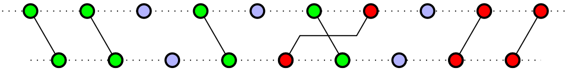
The row transfer matrix T evolves the system without the use of extra pieces, i.e., $x = y = z = 0$, and such that sufficiently far to the left (resp. right), green lines move up/one half step to the left (resp. right):

$T =$

Fock space and Transfer matrix

The Fock space $\mathcal{G}$ has a natural basis indexed by sequences $(a_k) \in \{\textcolor{green}{\bullet}, \textcolor{lavender}{\bullet}, \textcolor{red}{\bullet}\}^{\mathbb{Z}}$ such that sufficiently far to the left (resp. right), $a_k = \textcolor{green}{\bullet}$ (resp. $\textcolor{red}{\bullet}$).

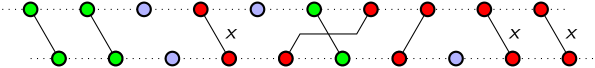
The row transfer matrix T evolves the system without the use of extra pieces, i.e., $x = y = z = 0$, and such that sufficiently far to the left (resp. right), green lines move up/one half step to the left (resp. right):

$T =$ 

More transfer matrices!

$T_r(x)$ evolves the system with the extra piece x, such that sufficiently far lines move up/left:

$$T_r(x) = \lim \frac{1}{x^{\#\ \text{red}}}$$



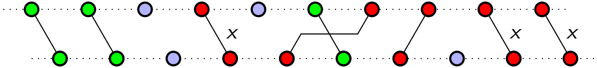
$T_g(y)$ evolves the system with the extra piece y, such that sufficiently far lines move up/right:

$$T_g(y) = \lim \frac{1}{y^{\#\ \text{green}}}$$

More transfer matrices!

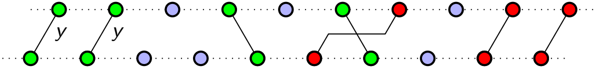
$T_r(x)$ evolves the system with the extra piece x, such that sufficiently far lines move up/left:

$$T_r(x) = \lim \frac{1}{x^{\#\ \text{red}}}$$



$T_g(y)$ evolves the system with the extra piece y, such that sufficiently far lines move up/right:

$$T_g(y) = \lim \frac{1}{y^{\#\ \text{green}}}$$



Commutation relations

As a consequence of the Yang–Baxter equation (unzipping argument),

$$[T, T_r(x)] = [T, T_g(y)] = [T_r(x), T_g(y)] = 0$$

We're going to expand the equality

$$\langle \nu | T^k \prod_{i=1}^{n} T_r(x_i) \prod_{i=1}^{m} T_g(y_i) | \cdots {\color{green}\bullet} \underbrace{\circ \cdots \circ}_{k} {\color{red}\bullet} \cdots \rangle$$

$$= \langle \nu | \prod_{i=1}^{n} T_r(x_i) \prod_{i=1}^{m} T_g(y_i) T^k | \cdots {\color{green}\bullet} \underbrace{\circ \cdots \circ}_{k} {\color{red}\bullet} \cdots \rangle$$

where the initial state is the encoding of ν as a binary string made of $\{{\color{green}\bullet}, {\color{red}\bullet}\}$.

Commutation relations

As a consequence of the Yang–Baxter equation (unzipping argument),

$$[T, T_r(x)] = [T, T_g(y)] = [T_r(x), T_g(y)] = 0$$

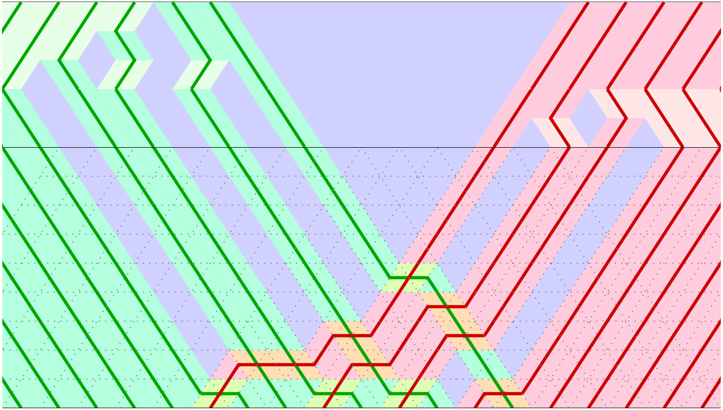
We're going to expand the equality

$$\langle \nu | \, T^k \prod_{i=1}^{n} T_r(x_i) \prod_{i=1}^{m} T_g(y_i) \Big| \cdots \textcolor{green}{\bullet} \underbrace{\textcolor{lavender}{\bullet} \cdots \textcolor{lavender}{\bullet}}_{k} \textcolor{red}{\bullet} \cdots \Big\rangle$$

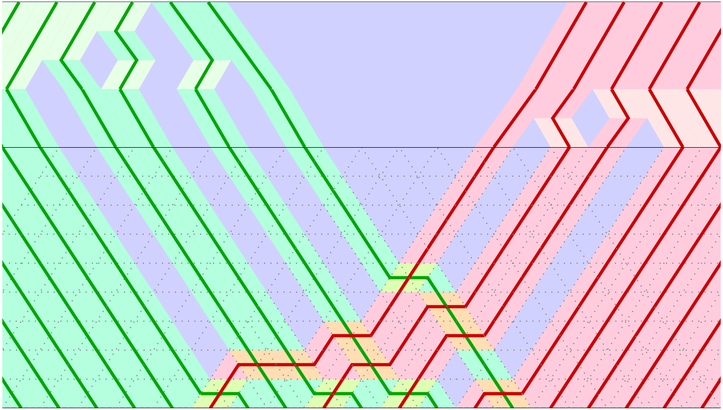
$$= \langle \nu | \prod_{i=1}^{n} T_r(x_i) \prod_{i=1}^{m} T_g(y_i) \, T^k \Big| \cdots \textcolor{green}{\bullet} \underbrace{\textcolor{lavender}{\bullet} \cdots \textcolor{lavender}{\bullet}}_{k} \textcolor{red}{\bullet} \cdots \Big\rangle$$

where the initial state is the encoding of ν as a binary string made of $\{\textcolor{green}{\bullet}, \textcolor{red}{\bullet}\}$.

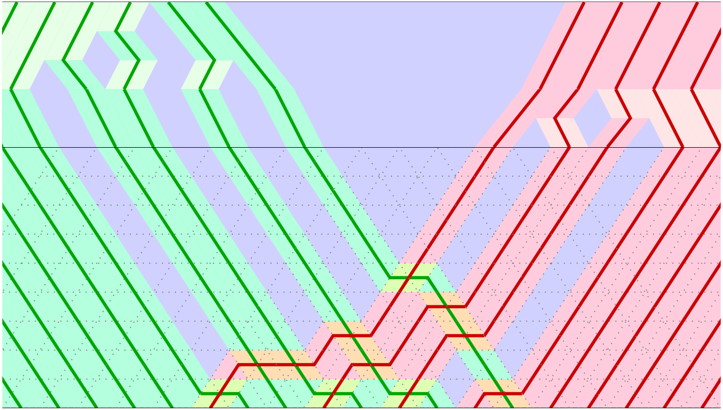
The left hand side



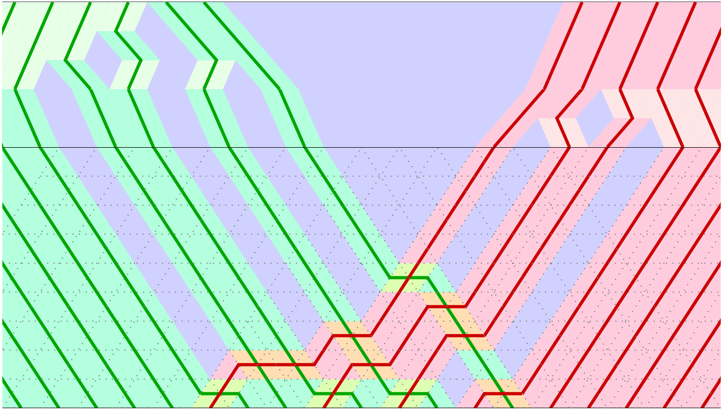
The left hand side



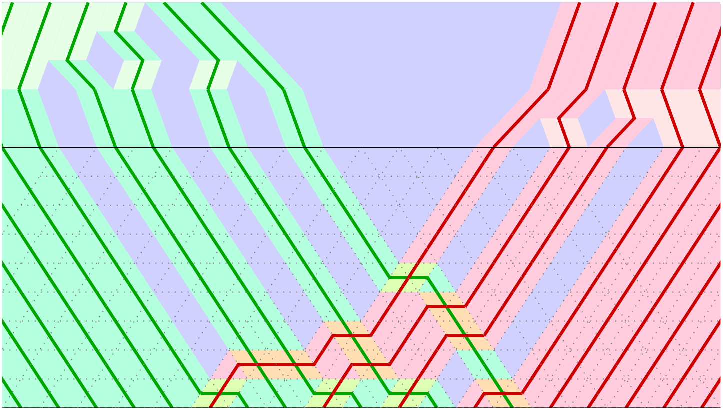
The left hand side



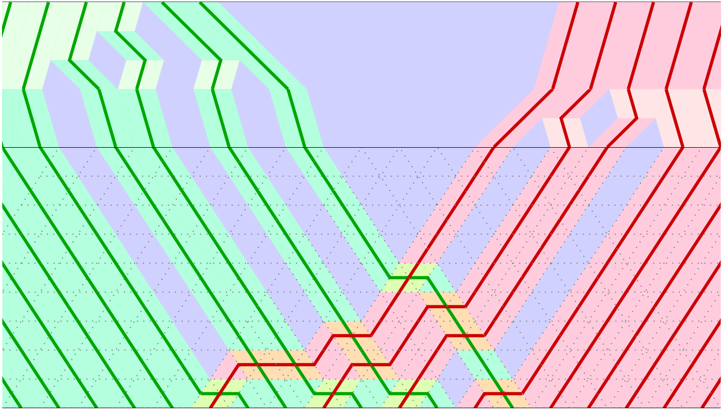
The left hand side



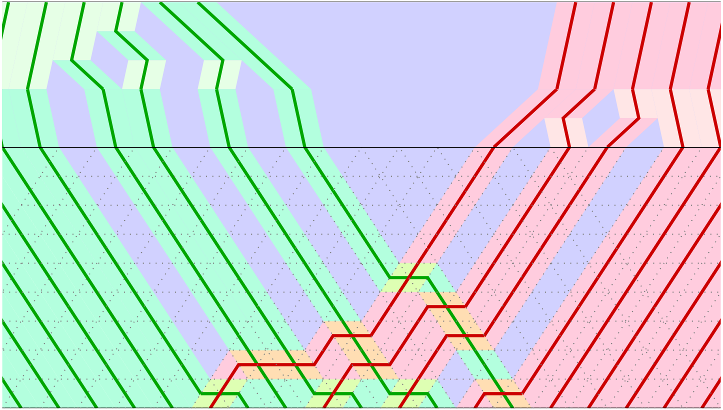
The left hand side



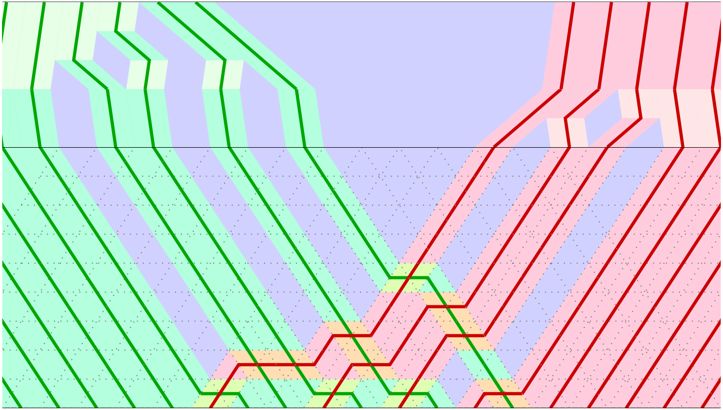
The left hand side



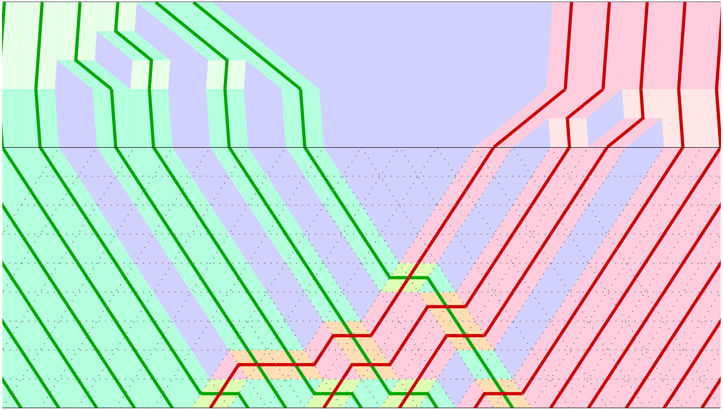
The left hand side



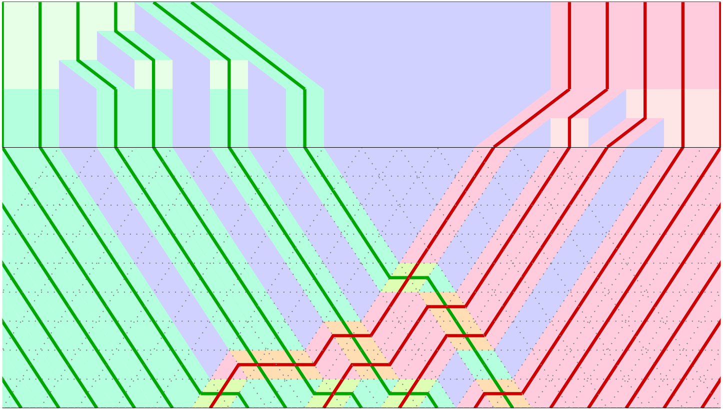
The left hand side



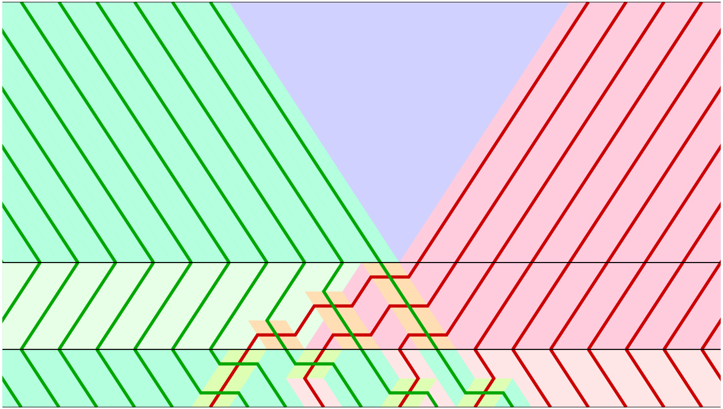
The left hand side



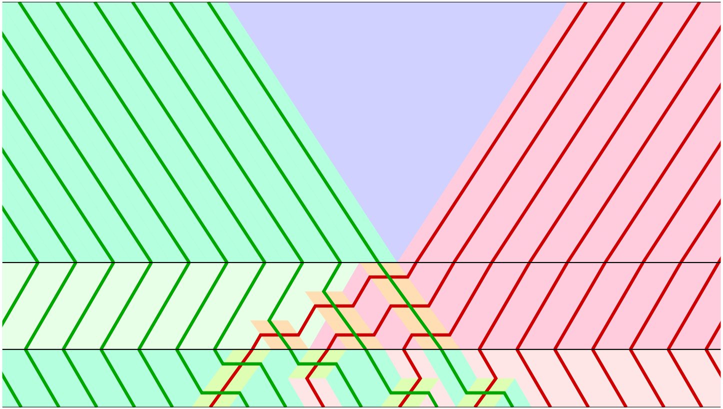
The left hand side



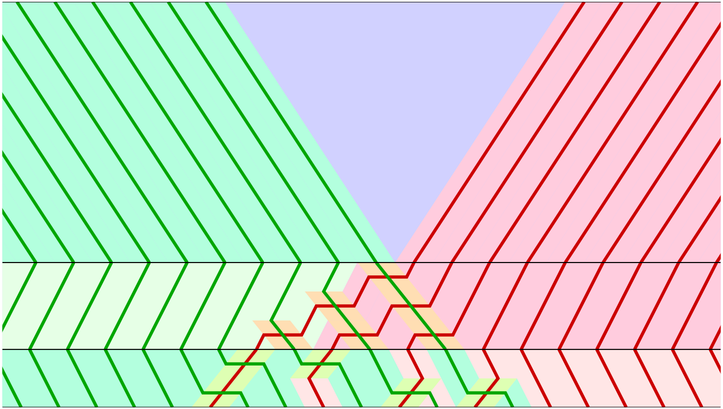
The right hand side



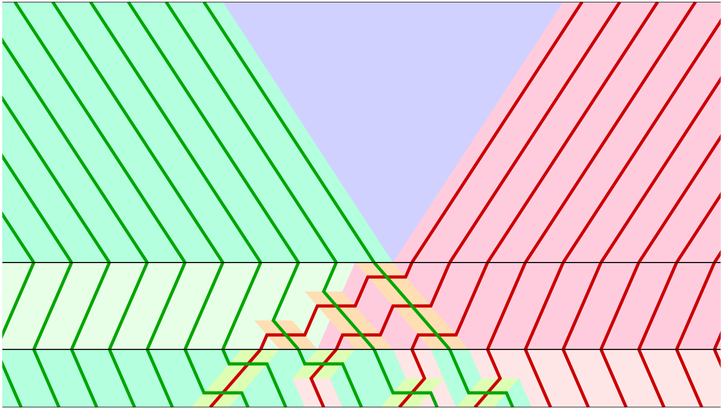
The right hand side



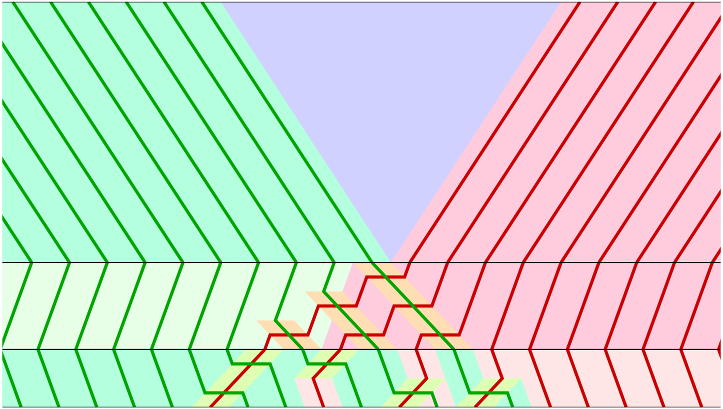
The right hand side



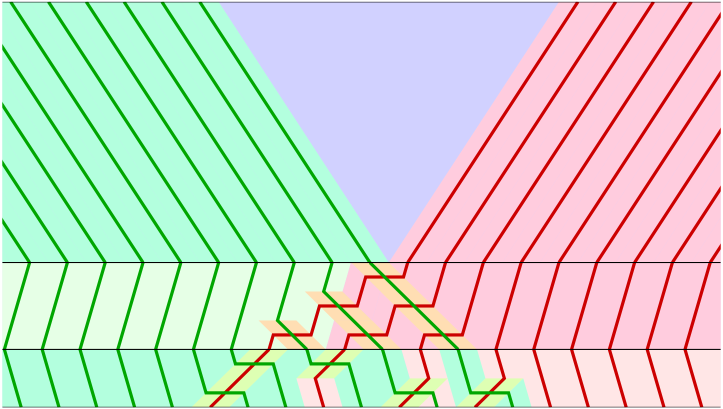
The right hand side



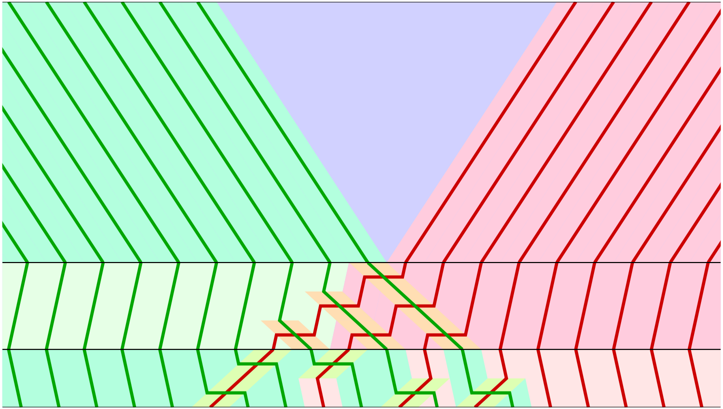
The right hand side



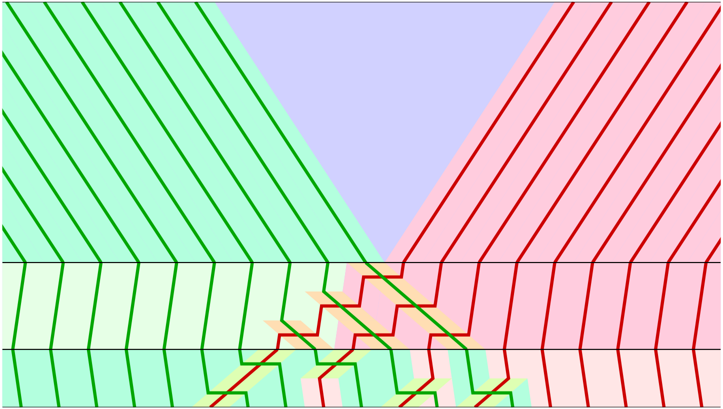
The right hand side



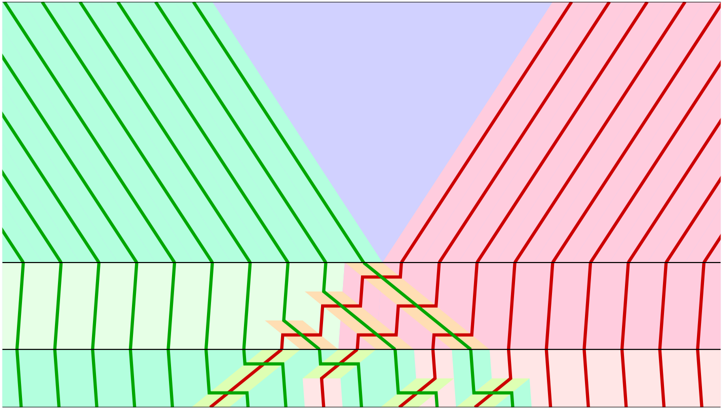
The right hand side



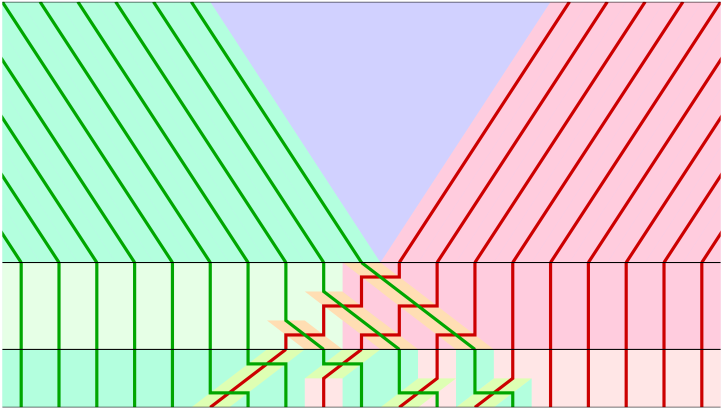
The right hand side



The right hand side



The right hand side



- Knutson–Tao problem:

$$s_\lambda(x_1,\ldots,x_k|z_1,\ldots,z_n)s_\mu(x_1,\ldots,x_k|z_1,\ldots,z_n) = \sum_\nu c^\nu_{\mu,\lambda}(z_1,\ldots,z_n)s_\nu(x_1,\ldots,x_k|z_1,\ldots,z_n)$$

- Molev–Sagan problem:

$$s_\lambda(x_1,\ldots,x_k|z_1,\ldots,z_n)s_\mu(x_1,\ldots,x_k|y_1,\ldots,y_n) = \sum_\nu e^\nu_{\lambda,\mu}(y_1,\ldots,y_n|z_1,\ldots,z_n)s_\nu(x_1,\ldots,x_k|y_1,\ldots,y_n)$$

- Unifying solution of these two problems in [PZJ, Littlewood–Richardson coefficients and integrable tilings] using the third extra tile (z).

- Knutson–Tao problem:

$$s_\lambda(x_1,\ldots,x_k|z_1,\ldots,z_n)s_\mu(x_1,\ldots,x_k|z_1,\ldots,z_n) = \sum_\nu c^\nu_{\mu,\lambda}(z_1,\ldots,z_n)s_\nu(x_1,\ldots,x_k|z_1,\ldots,z_n)$$

- Molev–Sagan problem:

$$s_\lambda(x_1,\ldots,x_k|z_1,\ldots,z_n)s_\mu(x_1,\ldots,x_k|y_1,\ldots,y_n) = \sum_\nu e^\nu_{\lambda,\mu}(y_1,\ldots,y_n|z_1,\ldots,z_n)s_\nu(x_1,\ldots,x_k|y_1,\ldots,y_n)$$

- Unifying solution of these two problems in [PZJ, Littlewood–Richardson coefficients and integrable tilings] using the third extra tile (z).

- Knutson–Tao problem:

$$s_\lambda(x_1,\ldots,x_k|z_1,\ldots,z_n)s_\mu(x_1,\ldots,x_k|z_1,\ldots,z_n) = \sum_\nu c^\nu_{\mu,\lambda}(z_1,\ldots,z_n)s_\nu(x_1,\ldots,x_k|z_1,\ldots,z_n)$$

- Molev–Sagan problem:

$$s_\lambda(x_1,\ldots,x_k|z_1,\ldots,z_n)s_\mu(x_1,\ldots,x_k|y_1,\ldots,y_n) = \sum_\nu e^\nu_{\lambda,\mu}(y_1,\ldots,y_n|z_1,\ldots,z_n)s_\nu(x_1,\ldots,x_k|y_1,\ldots,y_n)$$

- Unifying solution of these two problems in [PZJ, Littlewood–Richardson coefficients and integrable tilings] using the third extra tile (z).

Cancelled due to lack of time.