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The Brauer Loop Scheme and Orbital Varieties

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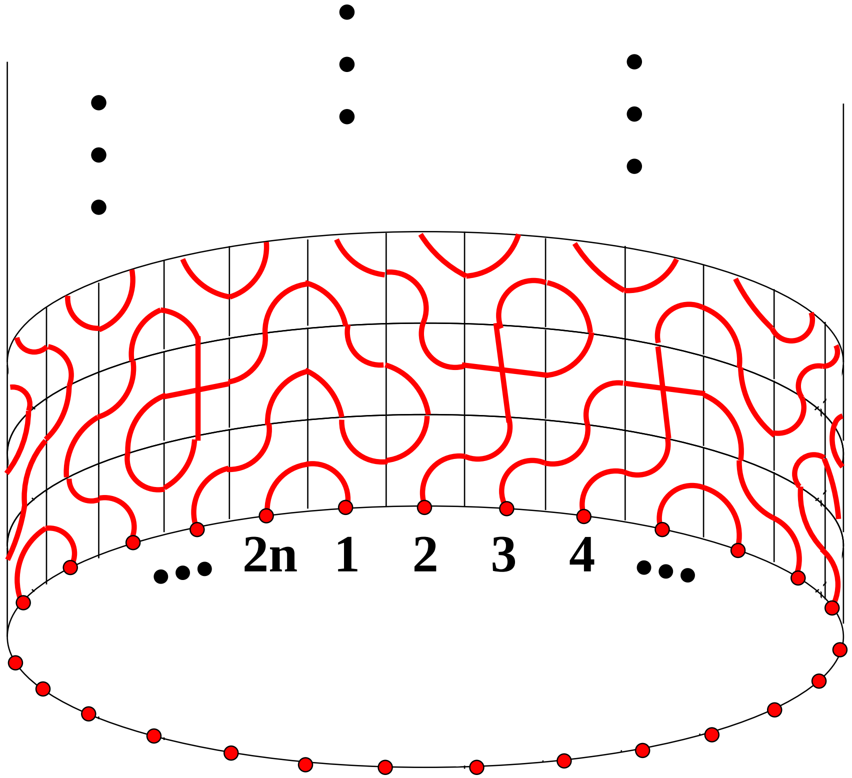
Plan of the talk

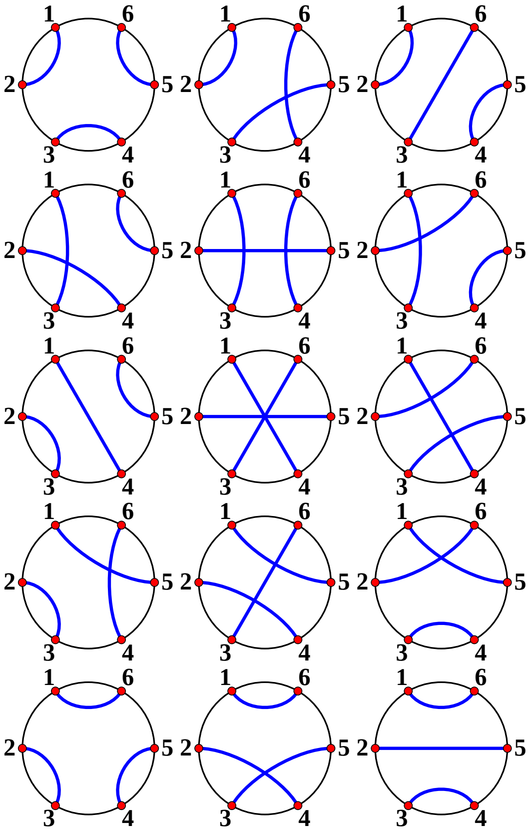
- The Brauer B(1) Loop model:
  - ◇ Definition
  - ◇ Transfer Matrix and Perron–Frobenius eigenvector
  - ◇ Multi-parameter generalization
  - ◇ qKZ equation
- The Brauer Loop scheme:
  - ◇ Definition of the scheme (infinite periodic triangular matrices)
  - ◇ Torus action and Equivariant Cohomology
  - ◇ Geometric action of Brauer
  - ◇ Application: degree of the commuting variety
- Relation to Orbital Varieties:
  - ◇ Nilpotent orbits of order 2, Orbital Varieties and B-orbits
  - ◇ From the Brauer loop scheme to B-orbits
  - ◇ Temperley–Lieb action and Hotta construction
  - ◇ Relation to Schubert varieties

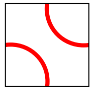
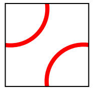
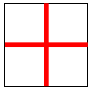
References

P. Di Francesco, P. Zinn-Justin, *Inhomogeneous model of crossing loops...*, `math-ph/0412031`.
A. Knutson, P. Zinn-Justin, *A scheme related to the Brauer loop model*, `math.AG/0503224`.
A. Knutson, P. Zinn-Justin, *The Brauer loop scheme and orbital varieties*, `1001.3335`.

Brauer model of loops





Probability that external vertex i is connected to vertex j? (proba:  $=$  $= 4/9$,  $= 1/9$)

→ vector Ψ_n, whose components are indexed by **crossing link patterns**, satisfying

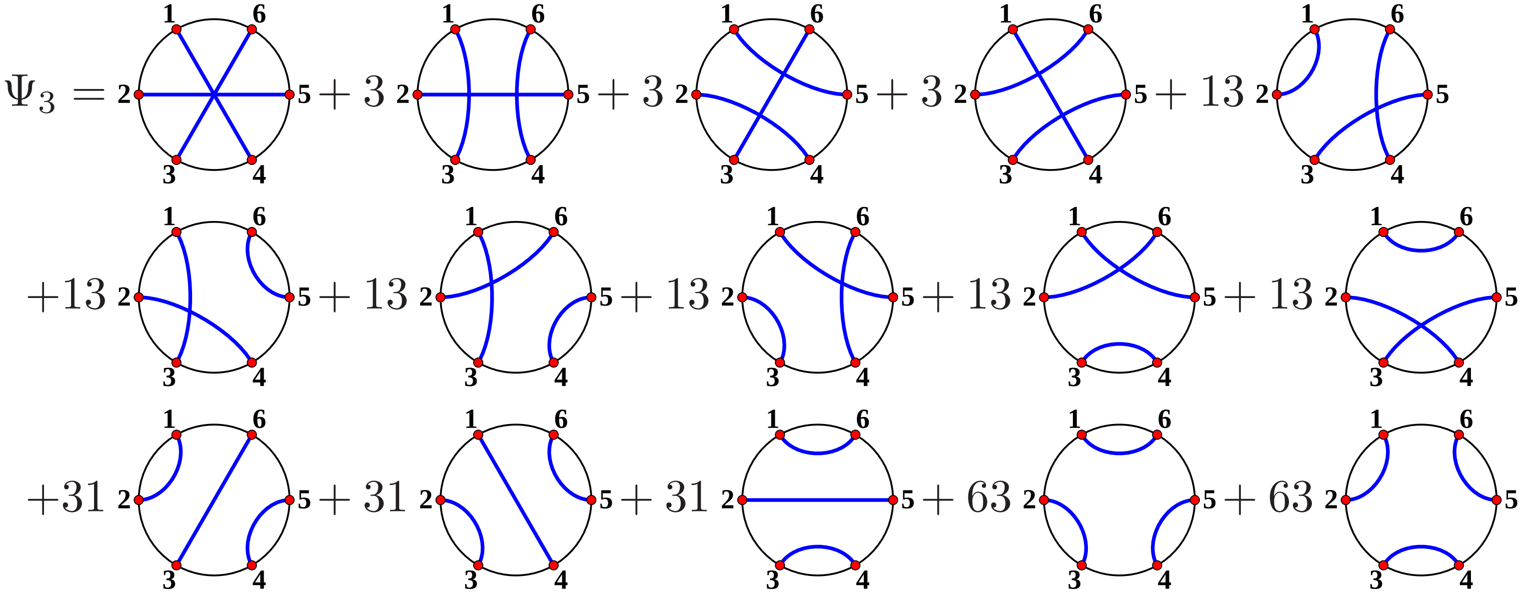
$$T_n \Psi_n = \Psi_n$$

where T_n is the **transfer matrix** that adds a row to the semi-infinite cylinder.

Brauer model of loops cont'd

NB: $\pi =$ crossing link pattern, or chord diagram, or Brauer diagram, or fixed-point free involution.

Example: for $n = 3$ ($N = 2n = 6$), up to normalization, Ψ_3 reads

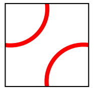


Conjecture [PZJ '04] (now theorem [AK, PZJ '05]): these numbers are degrees of the irreducible components of the Brauer loop scheme.

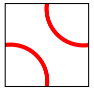
Inhomogeneous Brauer model of loops [PDF, PZJ '04]

Introduce local probabilities dependent on the column i via a parameter z_i respecting **integrability** of the model (i.e. satisfying Yang–Baxter equation).

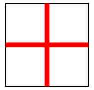
$$T_n(t|z_1,\dots,z_{2n}) = \prod_{i=1}^{2n} \Big(a(t-z_i)$$



$$+\, a(a-t+z_i)$$



$$+\, \frac{(t-z_i)(a-t+z_i)}{2}$$



$$\Big)$$

$$T_n(t; z_1 \dots, z_{2n})\Psi_n(z_1,\dots,z_{2n}) = \Psi_n(z_1,\dots,z_{2n})$$

⋆ *Polynomiality.*

The $\Psi_\pi(z_1,\dots,z_{2n})$ can be chosen to be coprime polynomials; they are then of total degree $2n(n-1)$ and of partial degree at most $2(n-1)$ in each z_i, with integer coefficients.

⋆ *Factorization, Recursion relations*. . . → entirely fixed (see next slides)

⋆ *Sum rule.*

$$\sum_\pi \Psi_\pi(z_1,\dots,z_{2n}) = \mathrm{Pf}\left(\frac{z_i-z_j}{a-(z_i-z_j)^2}\right)_{1\le i,j\le 2n} \times \prod_{1\le i<j\le 2n} \frac{a-(z_i-z_j)^2}{z_i-z_j}$$

Brauer algebra B(β)

- ◇ Generators e_i, f_i, $i = 1, \ldots, N-1$ and relations

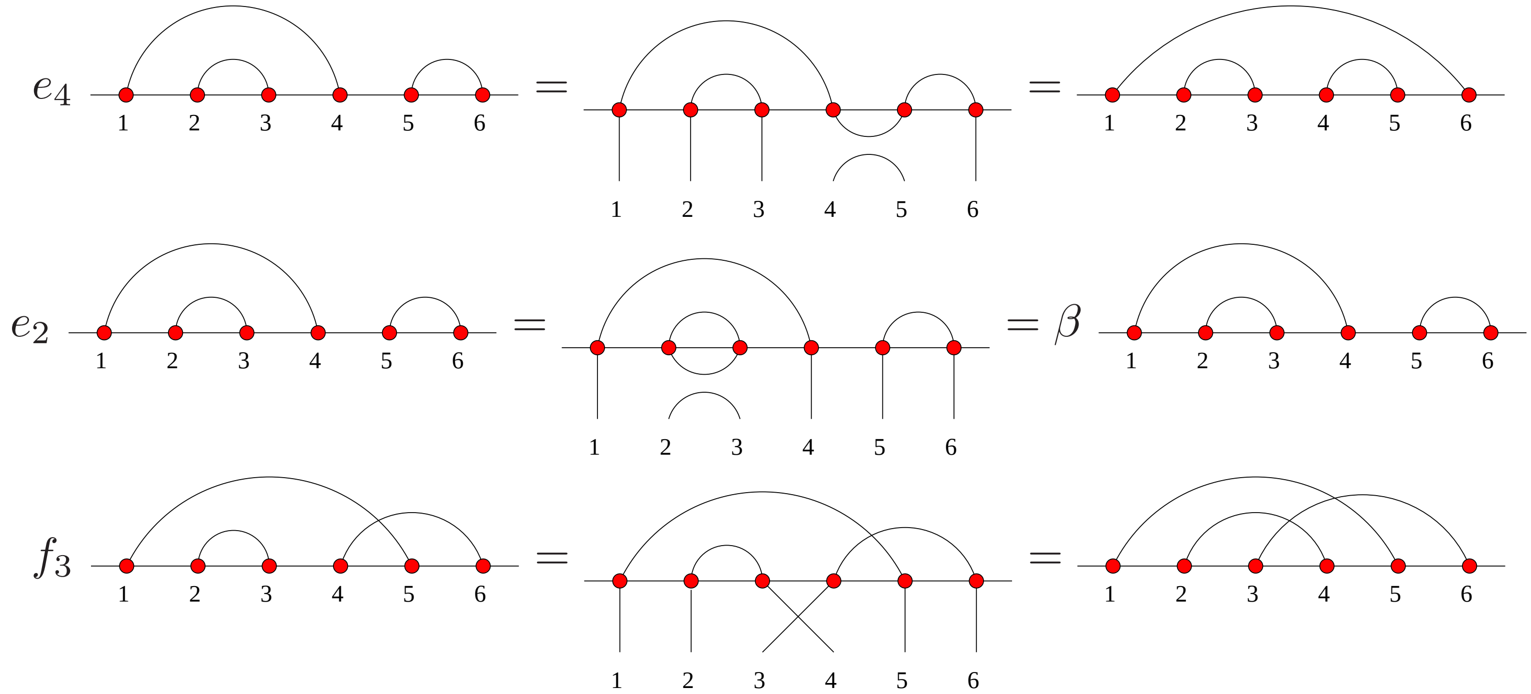
$$e_i^2 = \beta e_i \qquad e_i e_{i\pm1} e_i = e_i \qquad e_i e_j = e_j e_i \quad |i-j| > 1$$

$$f_i^2 = 1 \qquad (f_i f_{i+1})^3 = 1 \qquad f_i f_j = f_j f_i \quad |i-j| > 1$$

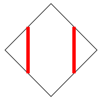
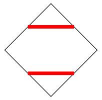
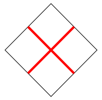
$$f_i e_i = e_i f_i = e_i \quad e_i f_i f_{i+1} = e_i e_{i+1} = f_{i+1} f_i e_{i+1} \quad e_{i+1} f_i f_{i+1} = e_{i+1} e_i = f_i f_{i+1} e_i \quad e_i f_j = f_j e_i$$

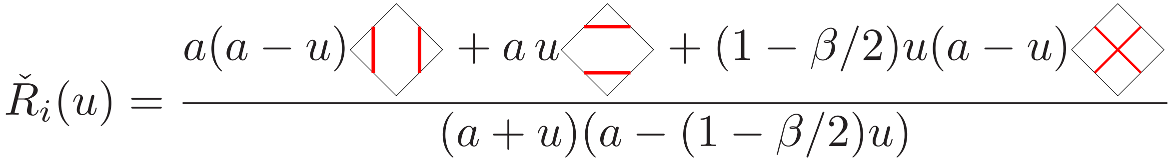
$$|i-j| > 1$$

- ◇ Action on link patterns: rewrite link patterns on a line



Rational qKZ equation

◇ R-matrix: $1 =$ , $e_i =$ , $f_i =$ 



$$\check{R}_i(u) = \frac{a(a-u)\,1 + a\,u\,e_i + (1-\beta/2)u(a-u)\,f_i}{(a+u)(a-(1-\beta/2)u)}$$

Satisfies Yang–Baxter equation: $\check{R}_i(u)\check{R}_{i+1}(u+v)\check{R}_i(v) = \check{R}_{i+1}(v)\check{R}_i(u+v)\check{R}_{i+1}(u)$ and unitarity equation: $\check{R}_i(u)\check{R}_i(-u) = 1$.

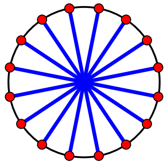
Fix ϵ and consider the following system of equations:

$$\begin{cases} \check{R}_i(z_i - z_{i+1})\Psi_n^{(\epsilon)}(z_1, \ldots, z_N) = \Psi_n^{(\epsilon)}(z_1, \ldots, z_{i+1}, z_i, \ldots, z_N) & i = 1, \ldots, N-1 \\ \rho\Psi_n^{(\epsilon)}(z_1, \ldots, z_N) = \Psi_n^{(\epsilon)}(z_2, \ldots, z_N, z_1 + \epsilon) & \end{cases}$$

where ρ is the rotation of link patterns.

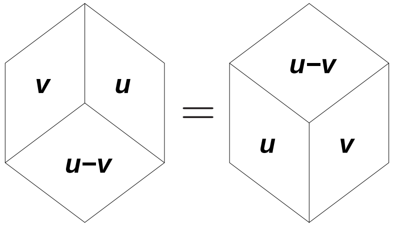
In general, no polynomial solutions. But if $\beta = \frac{2(a-\epsilon)}{2a-\epsilon}$, there is a solution uniquely fixed by

$$\Psi_{\pi_0}^{(\epsilon)} = \prod_{\substack{1\le i<j\le 2n\\ j-i<n}} (a + z_i - z_j) \prod_{\substack{1\le i<j\le 2n\\ j-i>n}} (a + z_j - z_i - \epsilon) \qquad \pi_0 =$$

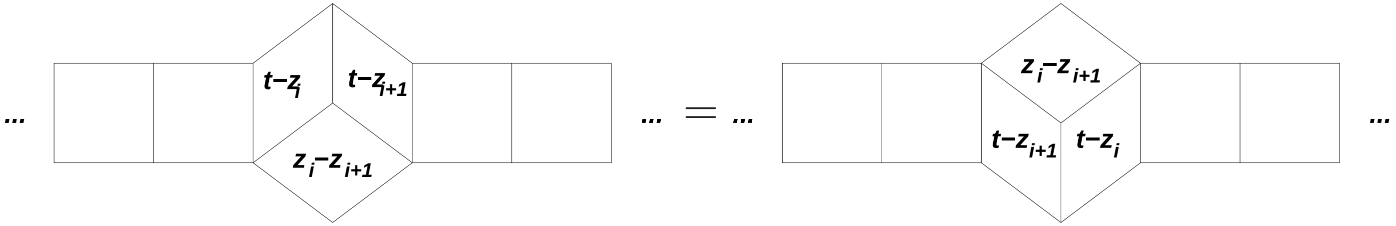


Claim: when $\epsilon = 0$ we recover our eigenvector Ψ_n.

From qKZ equation back to the Brauer loop model



Applied to the transfer matrix:



or more explicitly

$$\check{R}_i(z_i - z_{i+1}) T_n(t|z_1, \ldots, z_i, z_{i+1}, \ldots, z_{2n}) = T_n(t|z_1, \ldots, z_{i+1}, z_i, \ldots, z_{2n}) \check{R}_i(z_i - z_{i+1})$$

The intertwining relation implies (NB: fixing the normalization is non-trivial!)

$$\Psi_n(\ldots, z_{i+1}, z_i, \ldots) = \check{R}_i(z_i - z_{i+1}) \Psi_n(\ldots, z_i, z_{i+1}, \ldots)$$

Infinite periodic upper triangular matrices

$R_{\mathbb{Z}}$ = algebra of upper triangular complex matrices with rows and columns indexed by $\mathbb{Z}$.

$R_{\mathbb{Z} \bmod N}$ = subalgebra of matrices with the periodicity

$$A_{ij} = A_{i+N,j+N} \quad \forall i, j \in \mathbb{Z}$$

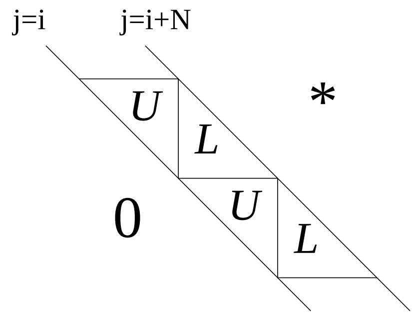
This subalgebra contains the "shift" matrix S carrying 1s just above the main diagonal, $S_{ij} = \delta_{i,j-1}$.

$\mathcal{M}$ is the quotient algebra

$$\mathcal{M} := R_{\mathbb{Z} \bmod N} / \left\langle S^N \right\rangle$$

An element $M \in \mathcal{M}$ is determined by the entries $M_{i,j}$ with $1 \leq i \leq N$, $i \leq j < i + N$.

Sometimes it is convenient to separate these entries into an upper triangular matrix U and a strict lower triangular matrix L:



The affine scheme E

Define in the space $\mathcal{M}^0$ of matrices with zero diagonals:

$$E := \{\, M \in \mathcal{M}^0 : M^2 = 0 \,\}$$

Explicitly, the equations defining the scheme E read:

$$\sum_{j:i<j<k} M_{i,j} M_{j,k} = 0 \qquad \forall i, k < i + N$$

What are the components of E? what is their dimension?

Experimental answer: to simplify, in what follows we assume N even ($N = 2n$). Then

1) E is equidimensional:

$$E = \bigcup_{\pi} E_\pi$$

with $\dim E_\pi = N^2/2$.

2) The E_π are normal.

Example 1: $N = 4$. Three components:

⋆ One cyclic-invariant component:

$$E_1 = \left\{ M = \begin{pmatrix} 0 & 0 & m_{13} & m_{14} & & & \\ & 0 & 0 & m_{24} & m_{25} & & \\ & & 0 & 0 & m_{35} & m_{36} & \\ & & & 0 & 0 & m_{46} & m_{47} \end{pmatrix} \right\}$$

⋆ Two components:

$$E_2 = \left\{ M = \begin{pmatrix} 0 & m_{12} & m_{13} & m_{14} & & & \\ & 0 & 0 & m_{24} & m_{25} & & \\ & & 0 & m_{34} & m_{35} & m_{36} & \\ & & & 0 & 0 & m_{46} & m_{47} \end{pmatrix} \quad \begin{matrix} m_{12}m_{24} + m_{13}m_{34} = 0 \\ m_{35}m_{56} + m_{34}m_{46} = 0 \\ m_{13}m_{35} - m_{24}m_{46} = 0 \end{matrix} \right\}$$

$$E_3 = \rho(E_2)$$

related to each other by the cycling automorphism $\rho : M_{i,j} \mapsto M_{i+1,j+1}$.

NB: E_2, E_3 are not complete intersections. . .

Torus action and equivariant cohomology

Action of $T = (\mathbb{C}^\star)^{N+1}$ on $\mathcal{M}$:

$$M_{i,j} \mapsto \mathrm{e}^{a + z_i - z_j} M_{i,j}$$

where $z_{i+N} = z_i + \epsilon$.

Remark: the action is simply conjugation by $\mathrm{diag}(\mathrm{e}^{z_i})$ ($\notin \mathcal{M}$ if $\epsilon \neq 0$) and scaling by e^a.

→ Equivariant cohomology $H_T^*(M_N(\mathbb{C})) \subset \mathbb{C}[a, \epsilon, z_1, \ldots, z_N]$ generated by $[M_{i,j}]_T = a + z_i - z_j$.

This action preserves E and its components E_π.

→ Each E_π is pushed forward by inclusion to some cohomology class in $H_T^*(\mathcal{M}^0)$.

Multidegrees

Algebraic formulation: Purely algebraic framework of equivariant cohomology for invariant subschemes of a (complex) vector space W with a linear action:

multidegree $\mathrm{mdeg}_W X$ of a T-invariant scheme $X \subset W$ defined by

(1) If $X = W = \{0\}$ then $\mathrm{mdeg}_W X = 1$.

(2) If X has top-dimensional components X_i with multiplicity m_i, $\mathrm{mdeg}_W X = \sum_i m_i \,\mathrm{mdeg}_W X_i$.

(3) If X is a variety and H is a T-invariant hyperplane in W,

- (a) If $X \not\subset H$, then $\mathrm{mdeg}_W X = \mathrm{mdeg}_H (X \cap H)$.
- (b) If $X \subset H$, then $\mathrm{mdeg}_W X = \mathrm{mdeg}_H X \,\cdot$ (weight of T on W/H).

Remark 1: $\mathrm{mdeg}_W X$ is a homogeneous polynomial, of degree the codimension of X in W.

Remark 2: Integral formula:

$$\mathrm{mdeg}\, X \propto \int_X \mathrm{d}\mu(x) \exp\left(-\pi \sum_i wt(x_i)\, |x_i|^2 \right)$$

Remark 3: here, $\mathrm{mdeg}\, X|_{a=1, z_i=0} = \deg X$.

Multidegree of E_π

What is $\operatorname{mdeg} E_\pi$? ($\deg E_\pi$?)

Example 1: $N = 4$.

⋆ One component of degree 1:

$$E_1 = \left\{ M = \begin{pmatrix} 0 & 0 & m_{13} & m_{14} & & & \\ & 0 & 0 & m_{24} & m_{25} & & \\ & & 0 & 0 & m_{35} & m_{36} & \\ & & & 0 & 0 & m_{46} & m_{47} \end{pmatrix} \right\}$$

$$\operatorname{mdeg} E_1 = (a + z_1 - z_2)(a + z_2 - z_3)(a + z_3 - z_4)(a - \epsilon + z_4 - z_1)$$

⋆ Two components of degree 3: $(b := a - \epsilon)$

$$E_2 = \left\{ M = \begin{pmatrix} 0 & m_{12} & m_{13} & m_{14} & & & \\ & 0 & 0 & m_{24} & m_{25} & & \\ & & 0 & m_{34} & m_{35} & m_{36} & \\ & & & 0 & 0 & m_{46} & m_{47} \end{pmatrix} \quad \begin{matrix} m_{12}m_{24} + m_{13}m_{34} = 0 \\ m_{35}m_{56} + m_{34}m_{46} = 0 \\ m_{13}m_{35} - m_{24}m_{46} = 0 \end{matrix} \right\}$$

$$E_3 = \rho(E_2)$$

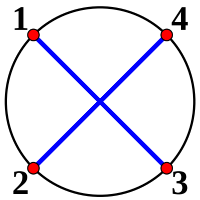
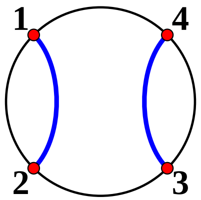
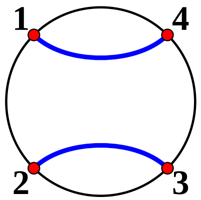
$$\operatorname{mdeg} E_2 = (a + z_2 - z_3)(b + z_4 - z_1)(a^2 + 2ab + bz_1 - az_2 - z_1 z_2 + az_3 + z_2 z_3 - bz_4 + z_1 z_4 - z_3 z_4)$$

Example 2: $N = 6$: $(\deg E_\pi) = (1, 3, 3, 3, 13, 13, 13, 13, 13, 13, 31, 31, 31, 63, 63)$. $\deg E = 307$.

General relation scheme ↔ statistical model

Conjecture [PZJ]: There is a natural way to index irreducible components E_π of E with crossing link patterns π of size $N = 2n$, in such a way that their multidegrees are solutions of rational qKZ equation associated to the Brauer algebra

$$\operatorname{mdeg} E_\pi = \Psi_\pi^{(\epsilon)}(z_1, \ldots, z_{2n})$$

Example: $E_1 \leftrightarrow$  , $E_2 \leftrightarrow$  , $E_3 \leftrightarrow$  .

In particular, at $\epsilon = 0$ and all $z_i = 0$, the degrees of the E_π are the unnormalized probabilities in the (homogeneous) Brauer loop model.

Proof for $\epsilon = 0$ in [AK, ZJ '05]; full proof in [AK,ZJ '10].

Corollary: the sum $\sum_\pi \Psi_\pi^{(\epsilon)}(z_1, \ldots, z_{2n})$ is the multidegree of E itself.

Definition of the E_π

Note that $s_i(M) := \sum_{j:i<j<i+N} M_{i,j} M_{j,i+N}$ is well-defined for $M \in E = \{M^2 = 0\}$.

Two simple lemmas:

(1) E (and therefore each E_π) is stable by conjugation.

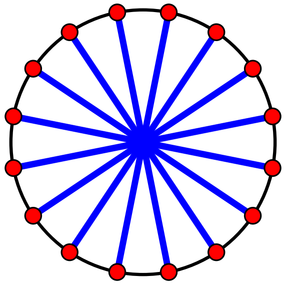
(2) $s_i(M) = s_i(PMP^{-1})$ for all i, $M \in E$, P invertible.

Motivates the following two equivalent definitions:

$$E_\pi = \overline{\left\{\, M \in E : s_i(M) = s_j(M) \text{ if and only if } j \in \{i, \pi(i)\} \,\right\}}$$

$$= \overline{\bigcup_{t\ \text{diag}} Orb(\pi t)} = \overline{\left\{\, P\pi t P^{-1}, t \text{ diag}, P \text{ inv} \,\right\}} \qquad \pi_{ij} = 1 \text{ iff } j = \pi(i), i < j < i + N$$

Special case: "trivial" component. $\pi_0 =$



$$, \quad E_{\pi_0} = \begin{pmatrix} 0 & \cdots & 0 & \star & \cdots & \star & \\ & 0 & \cdots & 0 & \star & \cdots & \star \\ & & \ddots & & \ddots & \ddots & \end{pmatrix}$$

$$\operatorname{mdeg} E_{\pi_0} = \prod_{\substack{1 \le i < j \le 2n \\ j - i < n}} (a + z_i - z_j) \prod_{\substack{1 \le i < j \le 2n \\ j - i > n}} (a + z_j - z_i - \epsilon)$$

Geometric action of Brauer algebra

⋆ "Sweeping": Define $L_i = \{\text{invertible matrices with off-diagonal elements at } (i,i+1),\ (i+1,i)\}$, $B_i = \{\text{invertible matrices with off-diagonal elements at } (i,i+1)\}$ and

$$S_i : L_i \times_{B_i} \mathcal{M} \to \mathcal{M}_i$$

$$(P,M) \to PMP^{-1}$$

If $S_{i|L_i \times_{B_i} X}$ generically one-to-one, then

$$\mathrm{mdeg}_{\mathcal{M}_i}\, S_i(L_i \times_{B_i} X) = -\partial_i \mathrm{mdeg}_{\mathcal{M}_i}\, X$$

where $\partial_i = \frac{1}{z_{i+1}-z_i}(\tau_i - 1)$ and $\tau_i F(z_i, z_{i+1}) = F(z_{i+1}, z_i)$.

Remark: $\mathrm{mdeg}_{\mathcal{M}_i}\, X = (a + z_{i+1} - z_i)\mathrm{mdeg}_{\mathcal{M}}\, X$.

⋆ "Cutting": Imposing an additional equation that decreases dimension by 1 amounts to multiplying by the weight of the equation.

Geometric action of Brauer algebra cont'd

Now consider a component E_π. Sweeping with L_i stays within upper triangular matrices only if $M_{i,i+1} = 0$. Therefore we must distinguish two cases:

⋆ Assume π has no arch between i and $i+1$. Then $E_\pi \subset \{M : M_{i,i+1} = 0\}$. Thus, sweep first. The result is upper triangular but not in $E \Rightarrow$ impose $(M^2)_{i+1,i} = 0$.

One can show that the result is $E_\pi \cup E_{f_i \pi}$.

$$-(a+b+z_{i+1}-z_i)(a+z_i-z_{i+1})\partial_i\left(\frac{\mathrm{mdeg}\,E_\pi}{a+z_i-z_{i+1}}\right) = \mathrm{mdeg}\,E_{f_i\cdot\pi} + \mathrm{mdeg}\,E_\pi$$

⋆ Assume π has an arch between i and $i+1$. Then cut with $M_{i,i+1} = 0$, [throw away the L_i-invariant pieces and] sweep, then cut with $(M^2)_{i+1,i} = 0$. One can show (after some hard work!) that the result is $\bigcup_{\pi' \neq \pi : e_i \pi' = \pi} E_{\pi'} \cap \{M \in E : s_i(M) = s_{\pi(i)}(M)\ \forall i\}$.

$$-(a+b+z_{i+1}-z_i)(a+z_i-z_{i+1})\partial_i \mathrm{mdeg}\,E_\pi = (a+b) \sum_{\pi' \neq \pi : e_i\pi' = \pi} \mathrm{mdeg}\,E_{\pi'}$$

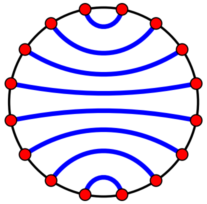
Application: (multi)degree of the commuting variety

Define the **commuting variety** to be the scheme

$$C = \{ (X, Y) \in M_n(\mathbb{C})^2 : XY = YX \}$$

It is a classical difficult problem to compute the degree of C. (previously known up to $n = 4$ only)

Observation [A. Knutson '03]: there is a Gröbner degeneration from $C \times V$ to E_π where $N = 2n$

and $\pi =$  .

In particular, $\deg C = \deg E_\pi = 1,\ 3,\ 31,\ 1145,$

[dG, N] $154881,\ 77899563,\ 147226330175,\ 1053765855157617,$

[PZJ] $28736455088578690945,\ 3000127124463666294963283,\ 1203831304687539089648950490463,$

$\ldots$

$$\log \deg C \sim n^2 \times \log 2 \qquad n \to \infty$$

Orbital varieties

We work with $G = GL(N)$, $\mathfrak{g} = \mathfrak{gl}(N)$. $B = \{\text{invertible upper triangular matrices}\}$,

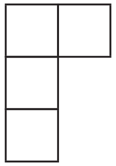
$\mathfrak{b} = \{\text{upper triangular matrices}\}$.

We are interested in nilpotent orbits:

$$\mathcal{O} = \{gMg^{-1}, g \in G\} \qquad M^N = 0$$

Nilpotent orbits are entirely characterized by the sizes of blocks of the Jordan decomposition of M:

$$M = \begin{pmatrix} \boxed{0} & \\ & \boxed{\begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix}} \end{pmatrix} \longrightarrow \quad \text{Young diagram :}$$



$\lambda_1 = 2$
$\lambda_2 = 1$
$\lambda_3 = 1$

Nilpotent orbit closures $\overline{\mathcal{O}}$ are (irreducible) algebraic varieties:

$$\overline{\mathcal{O}} = \{\, M : \operatorname{rank} M^i \leq \sum_{j>i} \lambda_j \quad i = 1, \ldots, k\}$$

To $\overline{\mathcal{O}}$ one associates its **orbital varieties** $\{X_\gamma\}$ which are the irreducible components of $\overline{\mathcal{O}} \cap \mathfrak{b}$.

Orbital varieties cont'd

Orbital varieties are indexed by standard Young tableaux. [Spaltenstein, 1976]

Indeed, to an $M \in \mathcal{O} \cap \mathfrak{b}$, one can associate a tableau as the sequence of Young diagrams of successive restrictions of M to the first n basis vectors. Components are closures of M with a given SYT.

$$M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \ydiagram{1} \quad \ydiagram{2} \quad \ydiagram{3} \quad \equiv \quad \ytableaushort{123}$$

$$M = \begin{pmatrix} 0 & 0 & \star \\ 0 & 0 & \star \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \ydiagram{1} \quad \ydiagram{2} \quad \ydiagram{2,1} \quad \equiv \quad \ytableaushort{12,3}$$

$$M = \begin{pmatrix} 0 & \star & \star \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \ydiagram{1} \quad \ydiagram{1,1} \quad \ydiagram{2,1} \quad \equiv \quad \ytableaushort{13,2}$$

$$M = \begin{pmatrix} 0 & \star & \star \\ 0 & 0 & \star \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \ydiagram{1} \quad \ydiagram{1,1} \quad \ydiagram{1,1,1} \quad \equiv \quad \ytableaushort{1,2,3}$$

In particular, the number of components of $\overline{\mathcal{O}} \cap \mathfrak{b}$ is the dimension of the corresponding irrep of $\mathcal{S}_N$.

(extended) Joseph polynomials

There is a natural torus action on $\overline{\mathcal{O}} \cap \mathfrak{b}$ and each of its components: conjugation by diagonal matrices.

$$M \to DMD^{-1}, \quad D \in (\mathbb{C}^\star)^N \qquad \Rightarrow \ [M_{ij}] = z_i - z_j$$

Joseph polynomials = multidegrees of orbital varieties.

Form a basis of an irreducible representation of the symmetric group [Joseph]. Identical to the Springer representation. (also same as KL basis in many cases)

Additional $\mathbb{C}^\star$ action by scaling: $[M_{ij}] = a + z_i - z_j$, $i < j$.

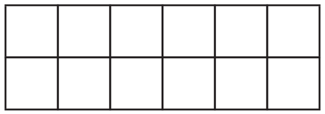
→ (extended) Joseph polynomials

$$J_\gamma(a, z_1, \ldots, z_N) = \mathrm{mdeg}_{\mathfrak{b}}\, X_\gamma$$

The usual Joseph polynomials are $J_\gamma(0, z_1, \ldots, z_N)$.

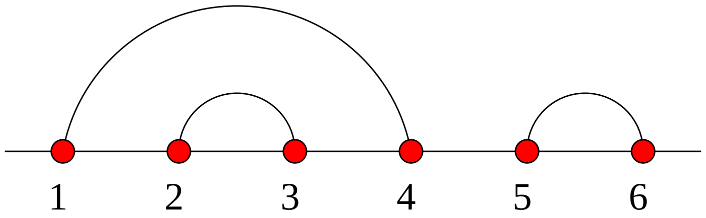
Orbital varieties of order 2

We now specialize to orbits of matrices of maximal rank that square to zero:



Standard Young tableaux can be more conveniently described as non-crossing **link patterns**:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & 6 \\ \hline \end{array} \quad \longrightarrow$$

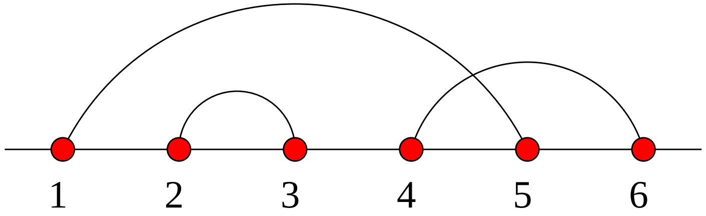


Orbital varieties of order 2 can then be described more explicitly as closures of B-orbits of upper triangles of involutions corresponding to the link pattern:

$$\pi_< = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ \vdots & \ddots & 1 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ \vdots & & & & \ddots & 1 \\ 0 & \cdots & & & \cdots & 0 \end{pmatrix} \qquad X_\pi = \overline{\{g\pi_< g^{-1}, g \text{ upper triangular}\}}$$

More upper triangular orbits

In fact, there are more B-orbits than the orbital varieties. To **any** fixed-point-free involution, i.e. to any (not necessarily non-crossing) link pattern, is associated a B-orbit.



$$\pi_< = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ \vdots & \ddots & 1 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 1 \\ \vdots & & & & \ddots & 0 \\ 0 & \cdots & & & \cdots & 0 \end{pmatrix} \qquad X_\pi = \overline{\{g\pi_< g^{-1}, g \text{ upper triangular}\}}$$

What can we say about the multidegrees $J_\pi(a, z_1, \ldots, z_N) = \mathrm{mdeg}_{\mathfrak{b}}\, X_\pi$?

The Brauer loop scheme as a normal cone

There is a better way to "break into pieces" an orbit closure $\overline{\mathcal{O}}$: take its "flat limit" as one scales to zero the strict lower triangular part of the matrix. The result is the **normal cone** of the orbital scheme inside $\overline{\mathcal{O}}$.

In the order 2 case we obtain the Brauer loop scheme of E. Indeed, an alternate definition of E ("interpolation" between usual and deformed product) is:

if $R_N(\mathbb{C})$ is the subspace of upper triangular matrices and

$$R_N(\mathbb{C}[t]) = R_N(\mathbb{C}) \oplus t M_N(\mathbb{C}) \oplus t^2 M_N(\mathbb{C}) \oplus \cdots$$

then our algebra is isomorphic to $R_N(\mathbb{C}[t])/tR_N(\mathbb{C}[t])$: $M \mapsto U + tL$.

In this language, it is more convenient to rewrite the weights in the following (non-cyclic invariant) way:

$$[M_{ij}] = \begin{cases} [U_{ij}] = a + z_i - z_j & i \leq j \\ [L_{ij}] = b + z_i - z_j & i > j \end{cases}$$

with $b = a - \epsilon$.

From the Brauer loop scheme to Orbital Varieties

Consider the operation: $E_\pi \mapsto E_\pi \cap \mathfrak{b}$. We find easily: $E_\pi \cap \mathfrak{b} = X_\pi$ i.e. components of the Brauer scheme are in one-to-one correspondence with B-orbits.

In the multidegree language this corresponds to $b \to \infty$:

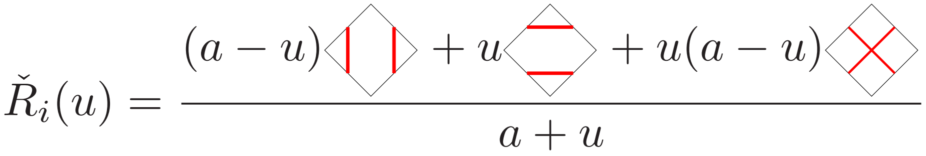
$$\Psi_\pi(a, b, z_1, \ldots, z_N) \overset{b\to\infty}{\sim} b^{\#} J_\pi(a, z_1, \ldots, z_N)$$

Now, take $b \to \infty$ limit in the Brauer B(β) qKZ equation. Recall that $\beta = \frac{2b}{a+b} \Rightarrow$ limit of the degenerate Brauer algebra B(2).

$$e_i^2 = 2e_i \qquad e_i e_{i\pm1} e_i = e_i \qquad e_i e_j = e_j e_i \quad |i-j| > 1$$

$$f_i^2 = 0 \qquad f_i f_{i+1} f_i = f_{i+1} f_i f_{i+1} \qquad f_i f_j = f_j f_i \quad |i-j| > 1$$

$$f_i e_i = e_i f_i = 0 \quad f_{i+1} f_i e_{i+1} = f_i f_{i+1} e_i = 0 \qquad e_i f_j = f_j e_i \quad |i-j| > 1$$



qKZ equation for Orbital Varieties/B-orbits

$$\check{R}_i(z_i - z_{i+1})J(z_1,\ldots,z_N) = J(z_1,\ldots,z_{i+1},z_i,\ldots,z_N)$$

◇ e_i equation:

$$-(a+z_i-z_{i+1})\partial_i J_\pi = \sum_{\pi'\neq\pi : e_i\pi'=\pi} J_{\pi'}$$

Related to Hotta's construction of the Joseph polynomials: cut with $M_{i\,i+1}=0$ then sweep.

Indeed TL(2) is a quotient of the symmetric group! Equivalently the usual generators of the symmetric group $s_i = 1 - e_i$ are given by $s_i = -\tau_i + a\partial_i$.

◇ f_i equation: if i and $i+1$ are unconnected and the arches starting from i, $i+1$ do not cross,

$$-(a+z_i-z_{i+1})\partial_i \frac{J_\pi}{a+z_i-z_{i+1}} = J_{f_i\pi}$$

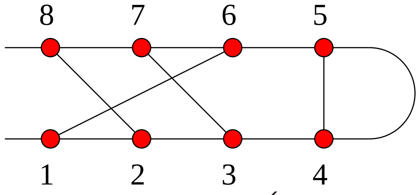
NB: $f_i\pi$ has one more crossing than π.

Looks very similar to relations between Schubert polynomials. Indeed. . .

Matrix Schubert varieties and (double) Schubert polynomials

Consider the crossing link patterns π for which $\pi(i) > n$ for $i \leq n$. $(N = 2n)$

Such patterns are in one-to-one correspondence with $\sigma \in \mathcal{S}_n$:



The corresponding matrices are contained in the upper right square: $M = \begin{pmatrix} 0 & p(M) \\ 0 & 0 \end{pmatrix}$.

Also, recall that the matrix Schubert varieties are defined by

$$\tilde{X}_\sigma = \{\, M \in M(n,\mathbb{C}) : \operatorname{rank} M_{i\times j} \leq \operatorname{rank} \sigma_{i\times j} \quad i,j = 1,\ldots,n\} = \overline{B_-\sigma B_+}$$

Proposition: $p(X_\pi)$ is the mirror image of matrix Schubert variety $\tilde{X}_\sigma$; thus,

$$J_\pi(a, z_1, \ldots, z_N) = \prod_{1\leq i<j\leq n} (a + z_i - z_j) \prod_{n+1\leq i<j\leq N} (a + z_i - z_j)$$
$$S_\sigma(a + z_n, \ldots, a + z_1; z_{n+1}, \ldots, z_N)$$

where the S_σ are the double Schubert polynomials.

Remark: relation to the flag variety G/B: ($G = GL(n)$, $T = \mathbb{C}^n$)

$$H^*(G/B) \simeq H^*_B(G) \simeq H^*_T(G) \overset{i^*}{\twoheadleftarrow} H^*_T(\mathfrak{g}) = \mathbb{C}[z_1, \ldots, z_n]$$

$i^*(S_\sigma(z_1, \ldots, z_n; 0, \ldots, 0))$ linear basis of $H^*(G/B)$.

Open problems

⋆ Conjectured equations of E_π:

(1) $M^2 = 0$.

(2) $s_i(M) = s_{\pi(i)}(M)$ for all i.

(3) For any matrix entry (i, j), $i < j < i + N$, we have $r_{ij}(M) \leq r_{ij}(\pi)$, where r_{ij} denotes the rank of the submatrix south-west of entry (i, j). In polynomial terms, this asserts the vanishing of all minors of size $r_{ij}(\pi) + 1$ in the submatrix southwest of entry (i, j).

⋆ Structure of orbits by conjugation.

inside $\mathcal{M}$? inside E?

problems: additional coincidences in (2). structure of the poset of rank conditions generalizing (3).