Refined enumeration of Alternating Sign Matrices and Descending Plane Partitions

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Introduction

- Plane Partitions were introduced by Mac Mahon about a century ago. However Descending Plane Partitions (DPPs), as well as other variations on plane partitions (symmetry classes), were considered in the 80s. [Andrews]
- Alternating Sign Matrices (ASMs) also appeared in the 80s, but in a completely different context, namely in Mills, Robbins and Rumsey's study Dodgson's condensation algorithm for the evaluation of determinants.
- One of the possible formulations of the Alternating Sign Matrix conjecture is that these objects are in bijection (for every size n). (proved by Zeilberger in '96 in a slightly different form)

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1. Kuperberg's alternative proof of the Alternating Sign Matrix conjecture using the connection to the six-vertex model. ('96)
2. The Razumov–Stroganov correspondence and related conjectures. ('01)

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T. Fonseca and P. Zinn-Justin: proof of the doubly refined Alternating Sign Matrix conjecture ('08).

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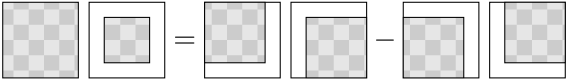
Today's talk is about the proof of another generalization of the ASM conjecture formulated in '83 by Mills, Robbins and Rumsey.

Iterative use of the Desnanot–Jacobi identity:



allows to compute the determinant of a $n \times n$ matrix by computing the determinants of the connected minors of size $1, \ldots, n$.

What happens when we replace the minus sign with an arbitrary parameter?

Laurent phenomenon: the result is still a Laurent polynomial in the entries of the matrix.

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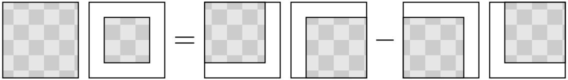

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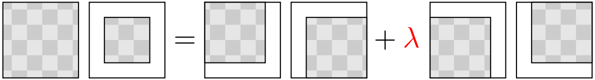

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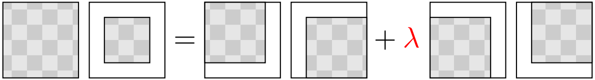

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Theorem (Robbins, Rumsey, '86)

If M is an $n \times n$ matrix, then

$$\det_\lambda M = \sum_{A \in \mathrm{ASM}(n)} \lambda^{\nu(A)} (1+\lambda)^{\mu(A)} \prod_{i,j=1}^{n} M_{ij}^{A_{ij}}$$

Here $\mathrm{ASM}(n)$ is the set of $n \times n$ Alternating Sign Matrices, that is matrices such that in each row and column, the non-zero entries form an alternation of $+1$s and -1s starting and ending with $+1$.

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Example

For $n = 3$, there are 7 ASMs:

$$\mathrm{ASM}(3) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\}$$

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$\mu(A)$ is the number of -1s in A.

$\nu(A)$ is a generalization of the inversion number of A:

$$\nu(A) = \sum_{\substack{1 \le i \le i' \le n \\ 1 \le j' < j \le n}} A_{ij} A_{i'j'}$$

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For the constant matrix 1_n, we have the recurrence $\det_\lambda 1_{n+1} \det_\lambda 1_{n-1} = (1+\lambda) \det_\lambda^2 1_n$ and therefore:

$$\det_\lambda 1_n = (1+\lambda)^{n(n-1)/2} = \sum_{A \in \mathrm{ASM}(n)} \lambda^{\nu(A)} (1+\lambda)^{\mu(A)}$$

In what follows, we shall be interested in more general weighted enumerations of ASMs, of the type $\sum_{A\in\mathrm{ASM}(n)} x^{\nu(A)}\ y^{\mu(A)}$ and refinements.

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A Descending Plane Partition is an array of positive integers ("parts") of the form

$$\begin{array}{lllll} D_{11} & D_{12} \ldots\ldots\ldots\ldots\ldots\ldots D_{1,\lambda_1} & & & \\ & D_{22} \ldots\ldots\ldots\ldots D_{2,\lambda_2+1} & & & \\ & \quad \ddots \quad \ldots & & & \\ & \qquad\quad D_{tt} \ldots D_{t,\lambda_t+t-1} & & & \end{array}$$

such that

- The parts decrease weakly along rows, i.e., $D_{ij} \geq D_{i,j+1}$.
- The parts decrease strictly down columns, i.e., $D_{ij} > D_{i+1,j}$.
- The first parts of each row and the row lengths satisfy

$$D_{11} > \lambda_1 \geq D_{22} > \lambda_2 \geq \ldots \geq D_{t-1,t-1} > \lambda_{t-1} \geq D_{tt} > \lambda_t$$

Let $\mathrm{DPP}(n)$ be the set of DPPs in which each part is at most n, i.e., such that $D_{ij} \in \{1, \ldots, n\}$.

Example

For $n = 3$, there are 7 DPPs:

$$\mathrm{DPP}(3) = \left\{ \emptyset,\ \begin{matrix} 3 & 3 \\ & 2 \end{matrix},\ 2,\ 3\,3,\ 3,\ 3\,2,\ 3\,1 \right\}$$

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Define statistics for each $D \in \mathrm{DPP}(n)$ as:

$$\nu(D) = \text{number of parts of } D \text{ for which } D_{ij} > j - i,$$

$$\mu(D) = \text{number of parts of } D \text{ for which } D_{ij} \leq j - i.$$

DPP enumeration

Theorem (Andrews, 79)

The number of DPPs with parts at most n is:

$$|\mathrm{DPP}(n)| = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!} = 1, 2, 7, 42, 429 \ldots$$

The Alternating Sign Matrix conjecture

The following result was first conjectured by Mills, Robbins and Rumsey in '82:

Theorem (Zeilberger, '96; Kuperberg, '96)

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NB: a third family is also known to have the same enumeration as ASMs and DPPs: TSSCPPs. In fact, Zeilberger's proof consists of a (non-bijective) proof of equienumeration of ASMs and TSSCPPs.

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Theorem (Behrend, Di Francesco, Zinn-Justin, '11)

The sizes of $\{A \in \mathrm{ASM}(n) \mid \nu(A) = p,\ \mu(A) = m\}$ *and* $\{D \in \mathrm{DPP}(n) \mid \nu(D) = p,\ \mu(D) = m\}$ *are equal for any* n, p, m.

(in fact, an even more general result that was conjectured by Mills, Robbins and Rumsey in '83 is proved)
Equivalently, if one defines generating series:

$$Z_{\mathrm{ASM}}(n, x, y) = \sum_{A \in \mathrm{ASM}(n)} x^{\nu(A)}\, y^{\mu(A)}$$

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$$Z_{\mathrm{ASM/DPP}}(3, x, y) = 1 + x^3 + x + x^2 + x + x^2 + xy$$

Strategy: write the two generating series as determinants:

modified Izergin

$$Z_{\mathrm{ASM}}(n,x,y) = \det M_{\mathrm{ASM}}(n,x,y)$$
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Let $6\mathrm{VDW}(n)$ be the set of all configurations of the six-vertex model on the $n \times n$ grid with DWBC, i.e., decorations of the grid's edges with arrows such that:

- The arrows on the external edges are fixed, with the horizontal ones all incoming and the vertical ones all outgoing.
- At each internal vertex, there are as many incoming as outgoing arrows.

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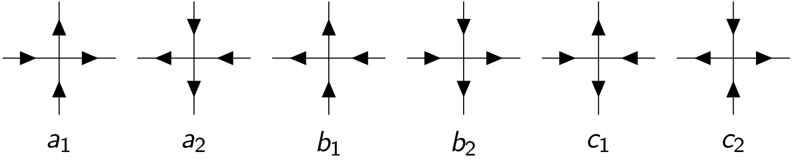

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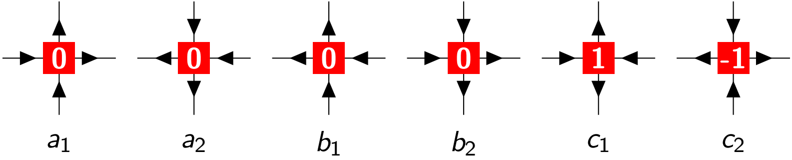
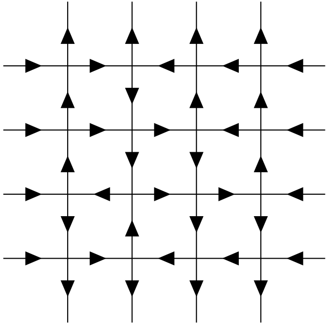


The bijection from $6\mathrm{VDW}(n)$ to $\mathrm{ASM}(n)$



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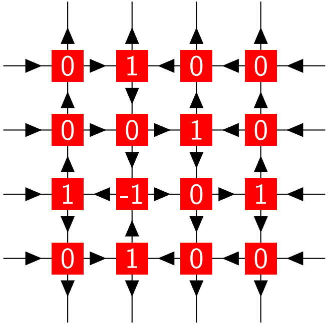

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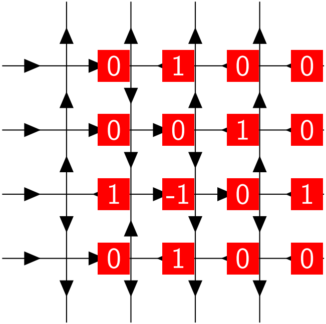

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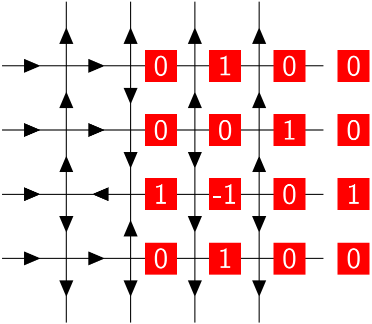

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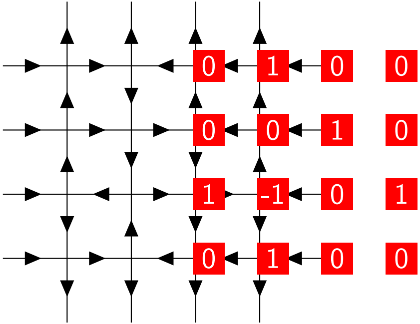

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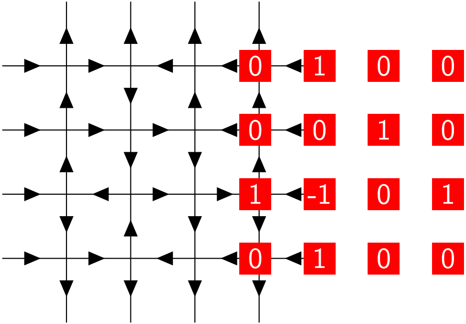

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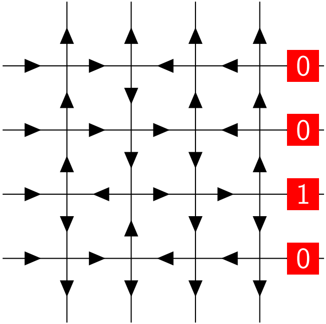

0	1	0	0
0	0	1	0
1	-1	0	1
0	1	0	0

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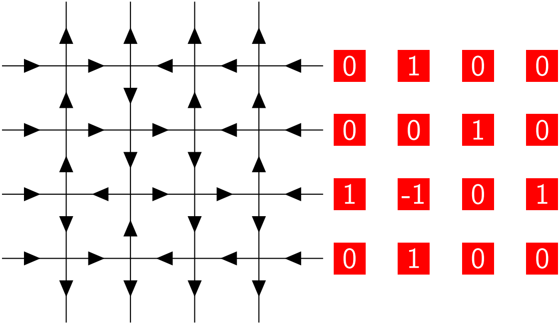

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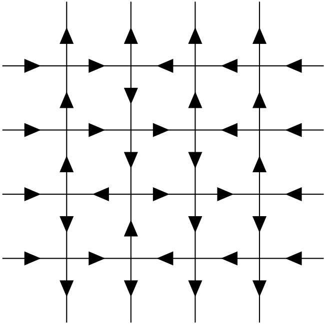

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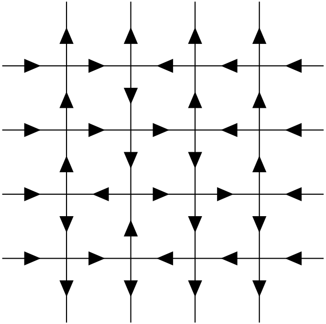

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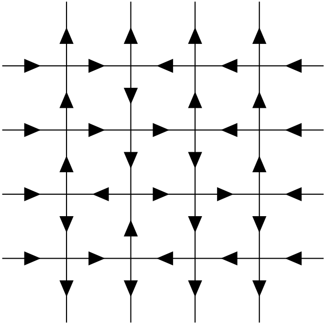

$$\begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{matrix}$$

Statistics

Statistics also have a nice interpretation in terms of the six-vertex model: if $A \in \mathrm{ASM}(n) \mapsto C \in 6\mathrm{VDW}(n)$,

$$\mu(A) = \frac{1}{2}\left((\text{number of vertices of type } c \text{ in } C) - n\right)$$
$$\nu(A) = \frac{1}{2}(\text{number of vertices of type } a \text{ in } C)$$

Define the six-vertex partition function of the six-vertex model with DWBC to be:

$$Z_{6\mathrm{VDW}}(u_1,\ldots,u_n;v_1,\ldots,v_n) = \sum_{C\in 6\mathrm{VDW}(n)} \prod_{i,j=1}^{n} C_{ij}(u_i,v_j)$$

where the u_i (resp. the v_j) are parameters attached to each row (resp. a column), and C_{ij} is the type of configuration at vertex (i,j).

$$a(u,v) = uq - \frac{1}{vq}, \qquad b(u,v) = \frac{u}{q} - \frac{q}{v}, \qquad c(u,v) = \left(q^2 - \frac{1}{q^2}\right)\sqrt{\frac{u}{v}}$$

Based on Korepin's recurrence relations for Z_{6VDW}, Izergin found the following determinant formula:

Theorem (Izergin, '87)

$$Z_{\mathrm{6VDW}}(u_1,\ldots,u_n;v_1,\ldots,v_n) \propto \frac{\det\limits_{1\le i,j\le n}\left(\dfrac{c(u_i,v_j)}{a(u_i,v_j)b(u_i,v_j)}\right)}{\prod\limits_{1\le i<j\le n}(u_j-u_i)(v_j-v_i)}$$

Problem: what happens in the homogeneous limit $u_1,\ldots,u_n,v_1,\ldots,v_n \longrightarrow r$?

Based on Korepin's recurrence relations for Z_{6VDW}, Izergin found the following determinant formula:

Theorem (Izergin, '87)

$$Z_{\mathrm{6VDW}}(u_1,\ldots,u_n;v_1,\ldots,v_n) \propto \frac{\det\limits_{1\le i,j\le n}\left(\dfrac{c(u_i,v_j)}{a(u_i,v_j)b(u_i,v_j)}\right)}{\prod\limits_{1\le i<j\le n}(u_j-u_i)(v_j-v_i)}$$

Problem: what happens in the homogeneous limit $u_1,\ldots,u_n,v_1,\ldots,v_n \to r$?

The “naive” homogeneous limit:

$$Z_{\mathrm{6VDW}}(r,\ldots,r;r,\ldots,r) \propto \det_{0\le i,j\le n-1} \frac{\partial^{i+j}}{\partial u^i \partial v^j}\left(\frac{c(u,v)}{a(u,v)b(u,v)}\right)_{|u,v=r}$$

$$\propto \det_{0\le i,j\le n-1} \frac{\partial^{i+j}}{\partial u^i \partial v^j}\left(\frac{1}{uv-q^2}-\frac{1}{uv-q^{-2}}\right)_{|u,v=r}$$

Define L_{ij} to be the $n \times n$ lower-triangular matrix with entries $\binom{i}{j}$, and D to be the diagonal matrix with entries $\left(\frac{qr-q^{-1}r^{-1}}{q^{-1}r-qr^{-1}}\right)^i$, $i = 0, \ldots, n-1$.

Proposition (Behrend, Di Francesco, Zinn-Justin, '11)

$$Z_{6\mathrm{VDW}}(r, \ldots, r; r, \ldots, r) \propto \det\left(I - \frac{r^2 - q^{-2}}{r^2 - q^2} DLDL^T\right)$$

Proof: write the determinant as $\det(A_+ - A_-)$, note that $A_\pm$ is up to a diagonal conjugation $\frac{1}{r^2-q^{\pm 2}} D_\pm L D_\pm L^T$, pull out $\det A_+$ and conjugate $I - A_- A_+^{-1} \ldots$

Define L_{ij} to be the $n \times n$ lower-triangular matrix with entries $\binom{i}{j}$, and D to be the diagonal matrix with entries $\left(\frac{qr-q^{-1}r^{-1}}{q^{-1}r-qr^{-1}}\right)^i$, $i = 0, \ldots, n-1$.

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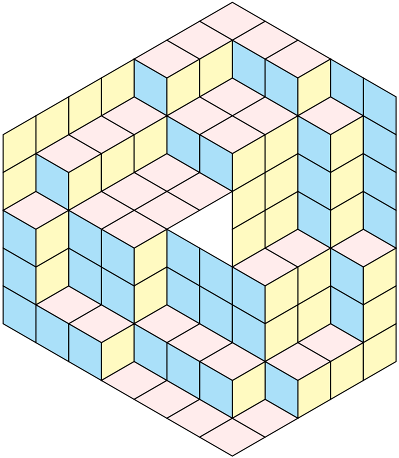
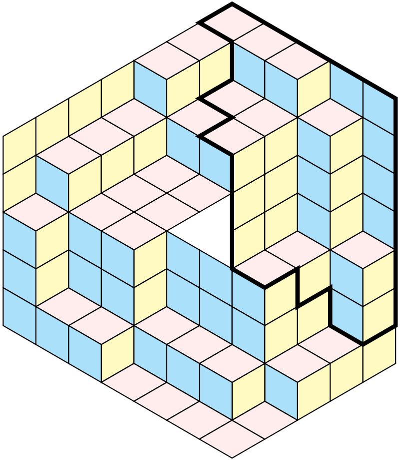
Rewriting the previous proposition in terms of Boltzmann weights a, b, c, and then switching to $x = (a/b)^2$, $y = (c/b)^2$, we finally find $Z_{\mathrm{ASM}}(n, x, y) = \det M_{\mathrm{ASM}}(n, x, y)$ with

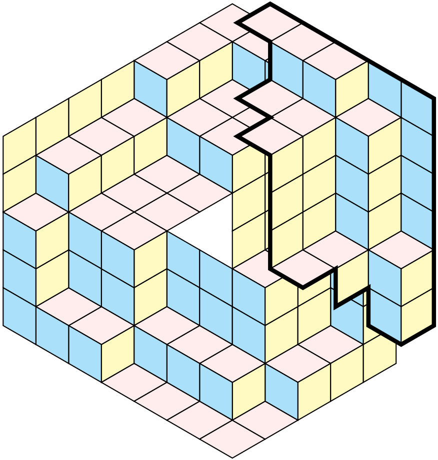
$$M_{\mathrm{ASM}}(n, x, y)_{ij} = (1 - \omega)\delta_{ij} + \omega \sum_{k=0}^{\min(i,j)} \binom{i}{k}\binom{j}{k} x^k\, y^{i-k}$$

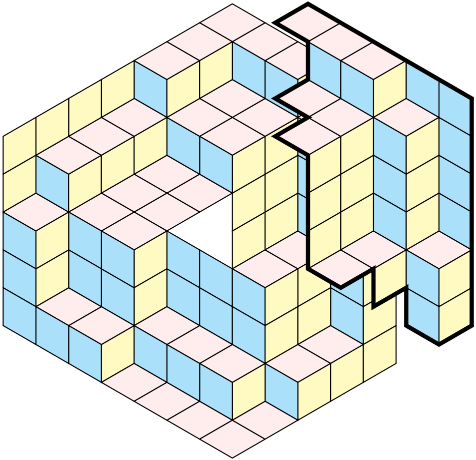
with $i, j = 0, \ldots, n-1$ and ω a solution of

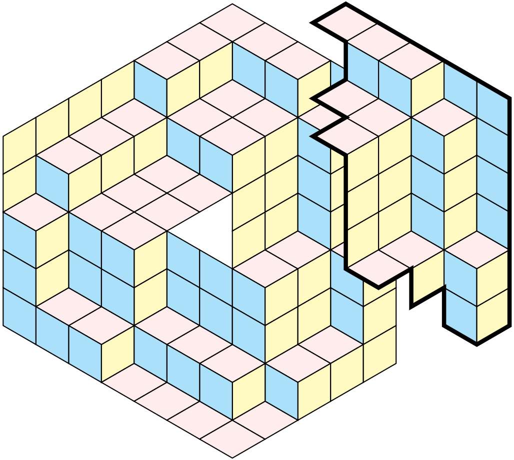
$$y\omega^2 + (1 - x - y)\omega + x = 0$$

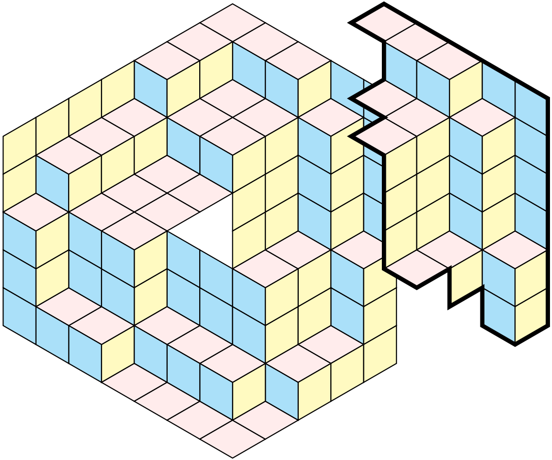


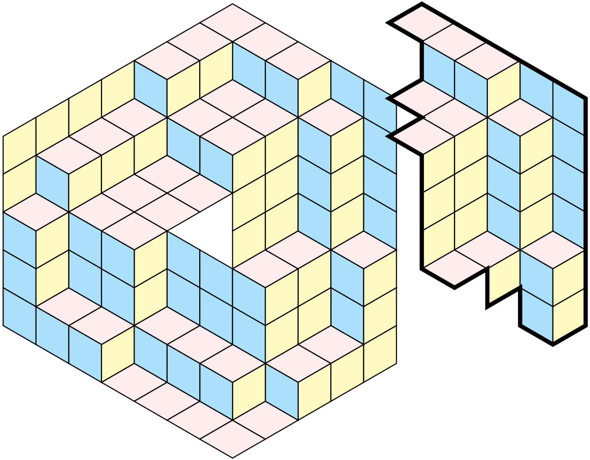


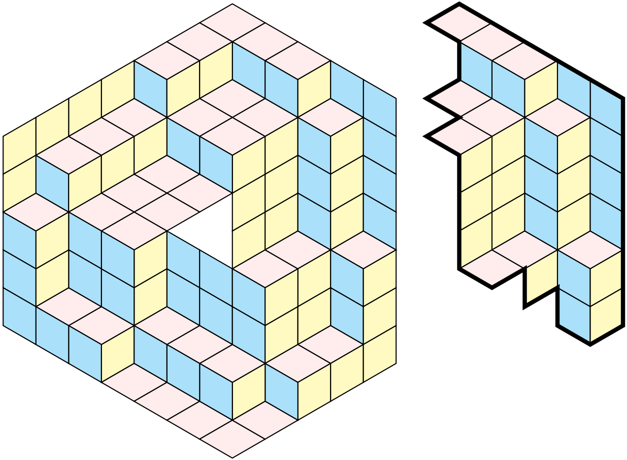


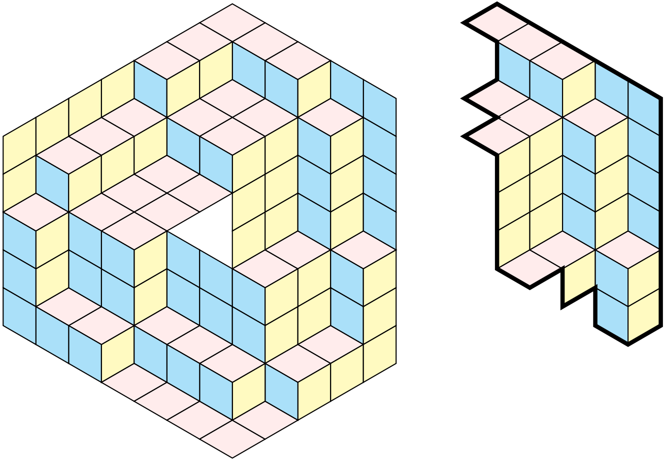


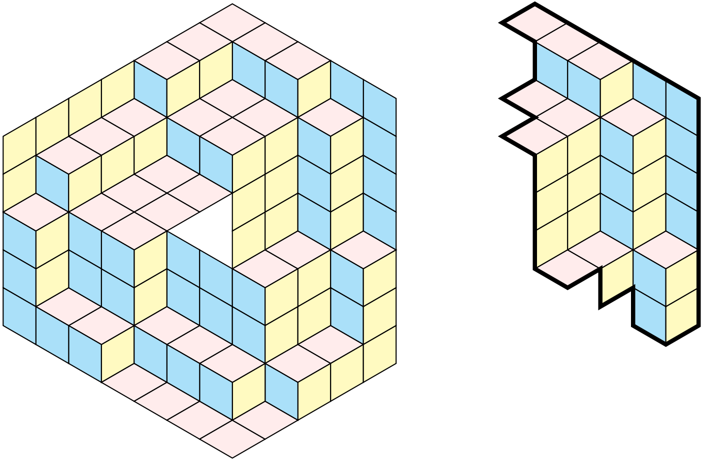


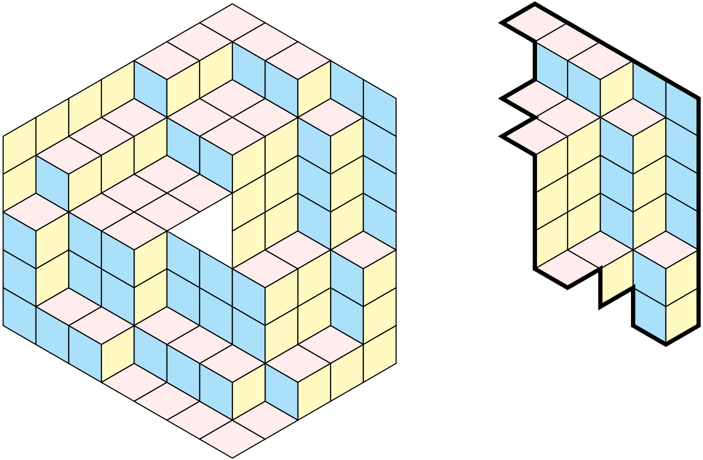


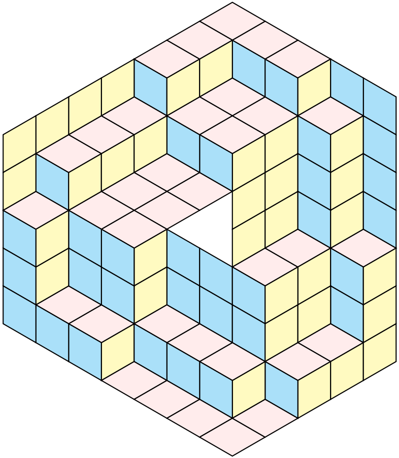


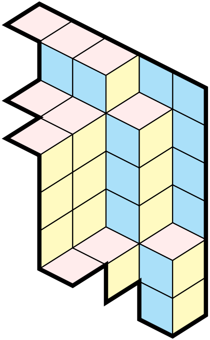


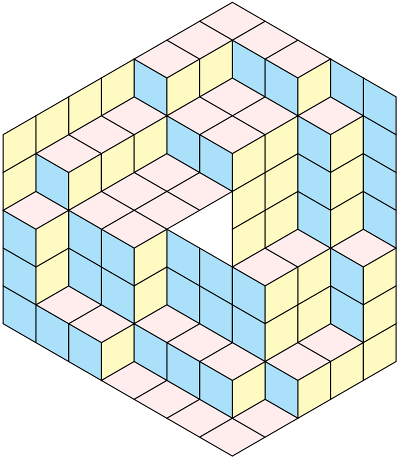


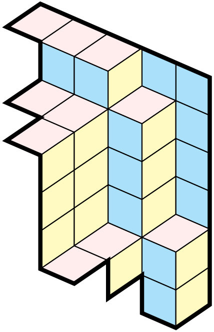


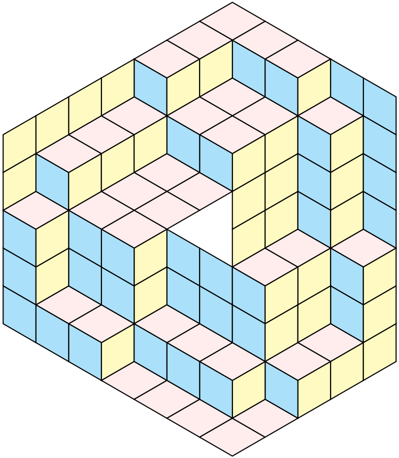


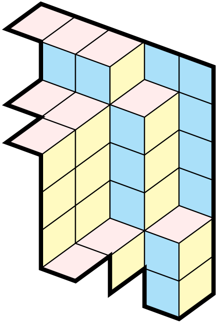


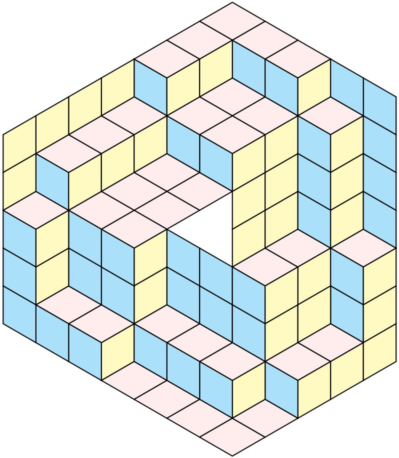


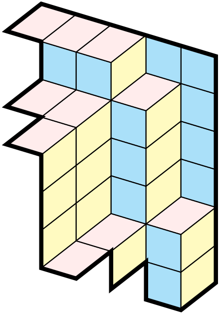


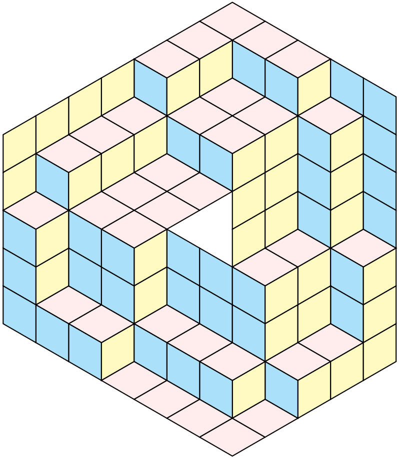


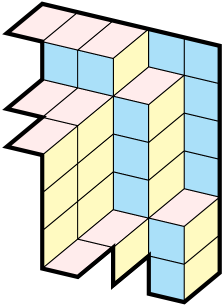


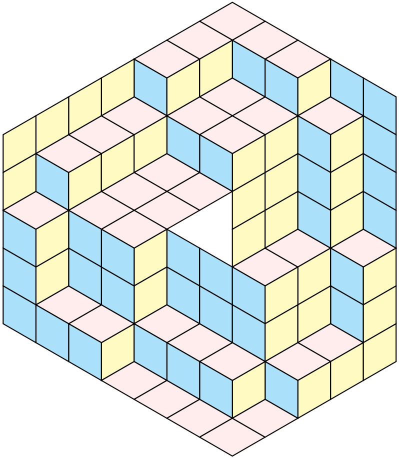


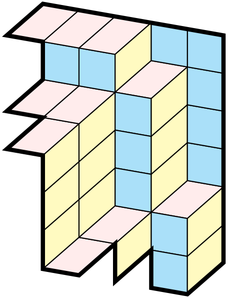


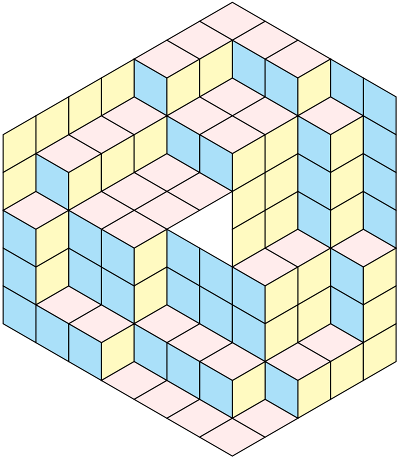


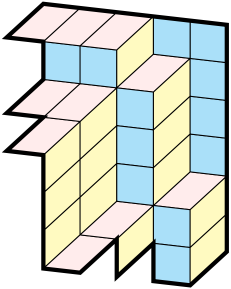


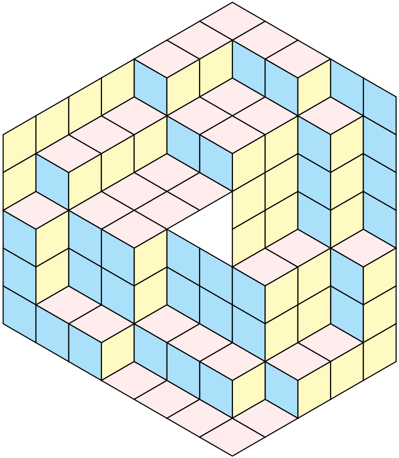


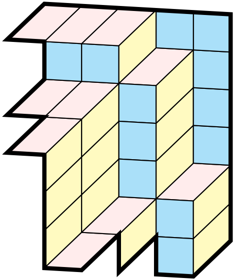


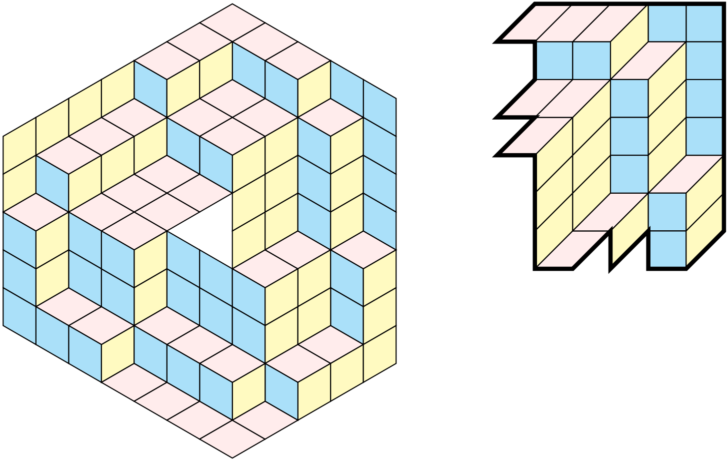


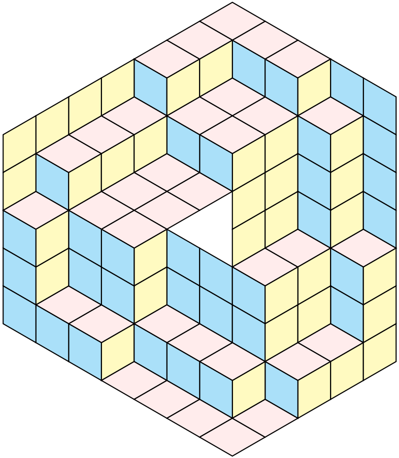


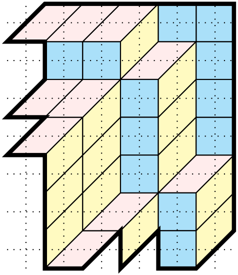


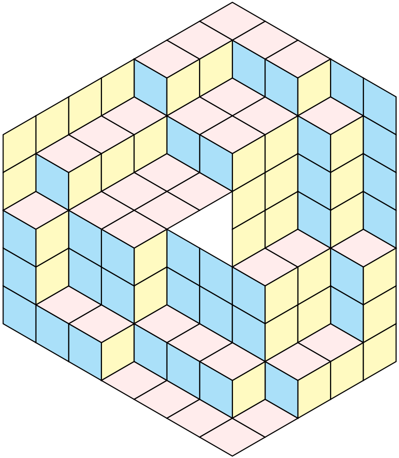


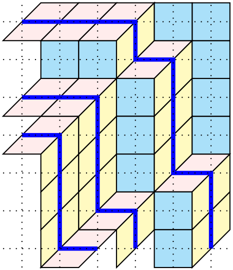


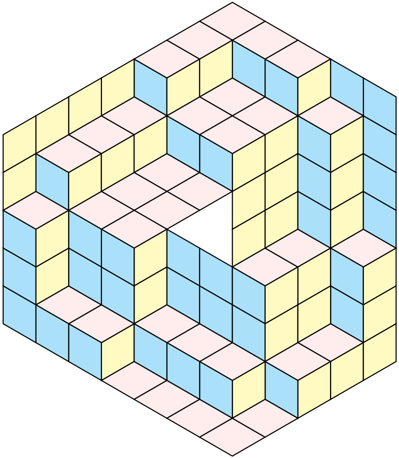


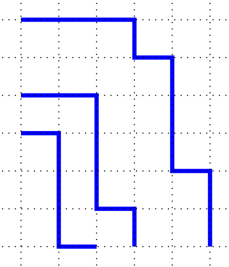


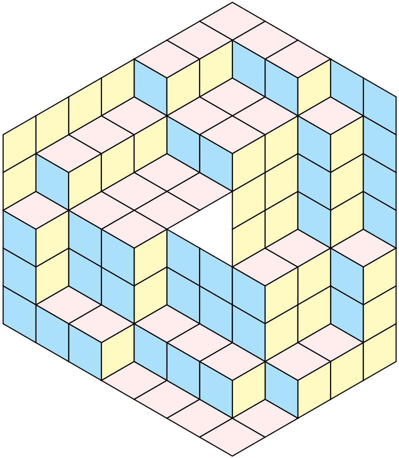


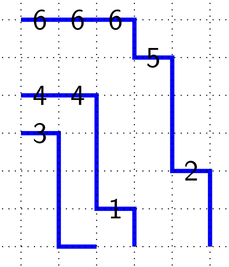








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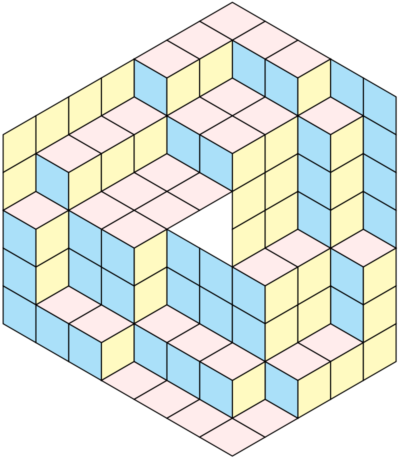

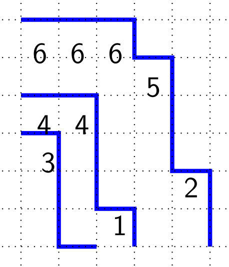
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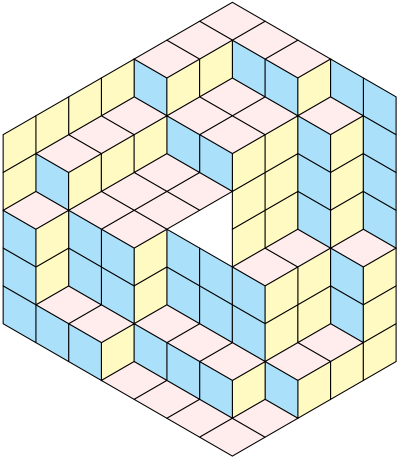

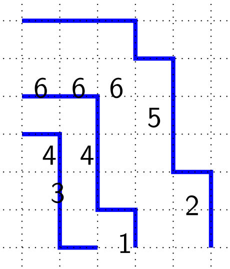
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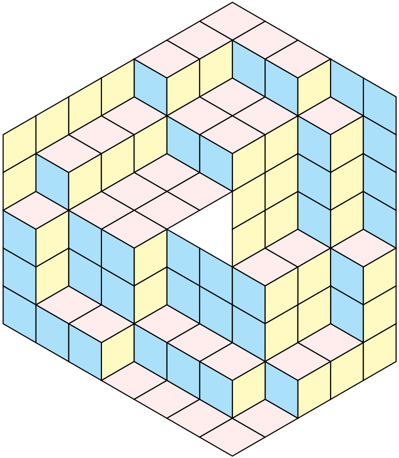

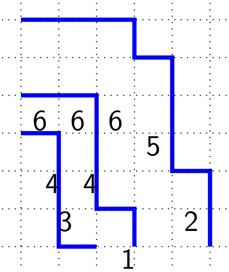
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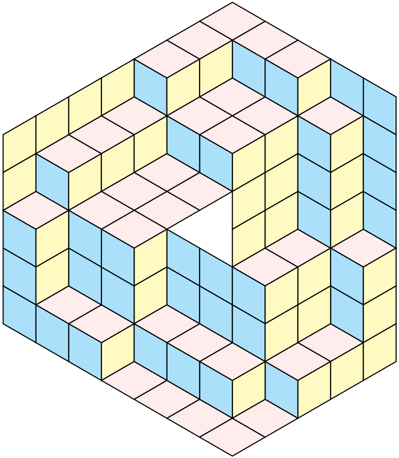

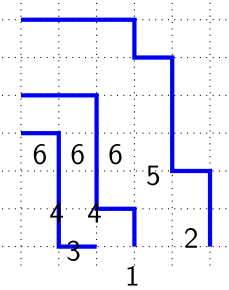
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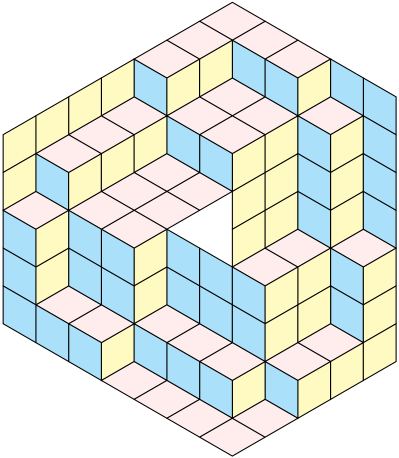

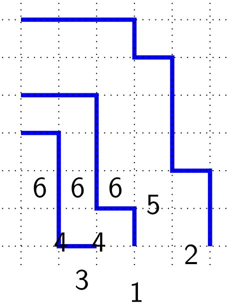
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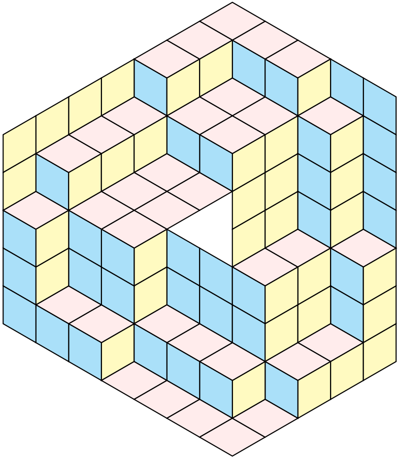

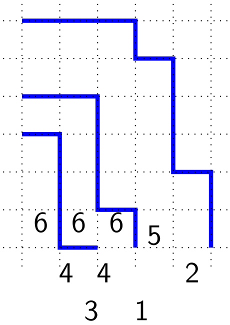
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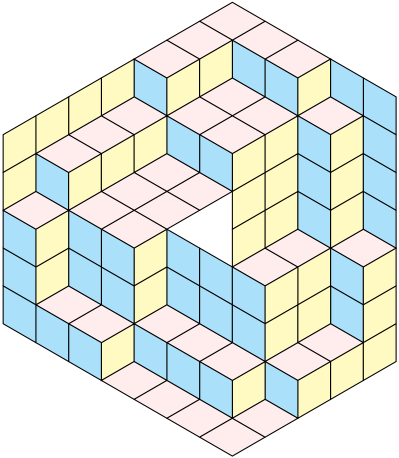

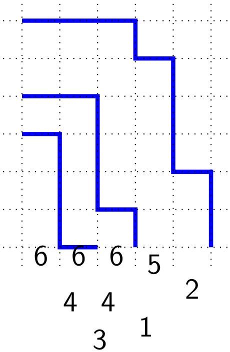
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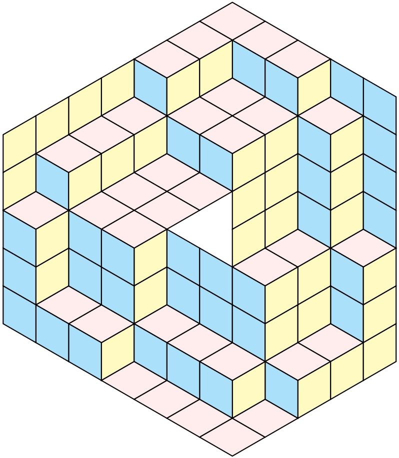

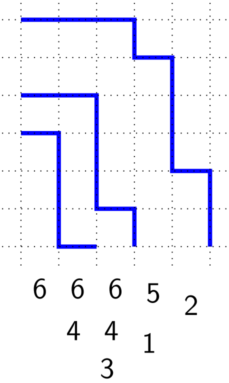
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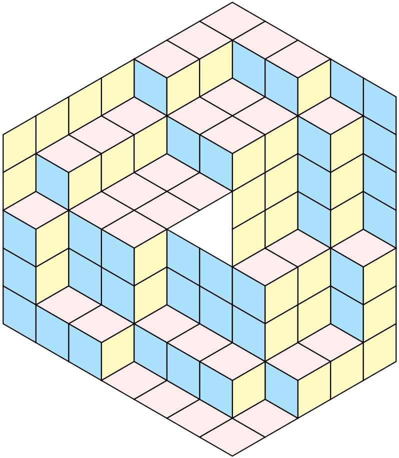

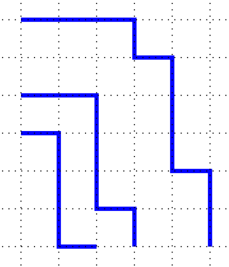

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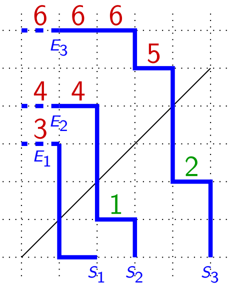
Statistics

Statistics also have a nice interpretation in terms of NonIntersecting lattice Paths (NILPs):

$D =$ 

$\nu(D) = 7$

$\mu(D) = 2$

LGV formula / free fermions

NILPs are (lattice) free fermions:

Number of NILPs from S_i to E_i, $i = 1, \ldots, n$

$$= \det_{i,j=1,\ldots,n} (\text{Number of (single) paths from } S_i \text{ to } E_j)$$

and similarly with weighted sums.

LGV formula / free fermions

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Number of NILPs from S_i to E_i, $i = 1, \ldots, n$

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and similarly with weighted sums.

Here we are also summing over endpoints and the number of paths ("grand canonical partition function"):
$Z_{\mathrm{DPP}}(n,x,y) = \det M_{\mathrm{DPP}}(n,x,y)$ with

$$M_{\mathrm{DPP}}(n,x,y) = \delta_{ij} + \sum_{k=0}^{i-1} \sum_{\ell=0}^{\min(j,k)} \binom{j}{\ell} \binom{k}{\ell} x^{\ell+1} y^{k-\ell}$$

Note that the second term is a product of three discrete transfer matrices...

We have

$$(I - S)M_{\mathrm{DPP}}(n, x, y)(I + (\omega - 1)S^T) = (I + (x - \omega y - 1)S)M_{\mathrm{ASM}}(n, x, y)(I - S^T)$$

where $I_{ij} = \delta_{i,j}$ and $S_{ij} = \delta_{i,j+1}$.

Therefore,

$$Z_{\mathrm{DPP}}(n, x, y) = Z_{\mathrm{ASM}}(n, x, y)$$

We have

$$(I - S)M_{\mathrm{DPP}}(n, x, y)(I + (\omega - 1)S^T) = (I + (x - \omega y - 1)S)M_{\mathrm{ASM}}(n, x, y)(I - S^T)$$

where $I_{ij} = \delta_{i,j}$ and $S_{ij} = \delta_{i,j+1}$.

Therefore,

$$Z_{\mathrm{DPP}}(n, x, y) = Z_{\mathrm{ASM}}(n, x, y)$$

Define refined enumeration by introduction "boundary" statistics:

- For ASMs:

$\rho_1(A)$ = number of 0's to the left of the 1 in the first row of A,

$\rho_2(A)$ = number of 0's to the right of the 1 in the last row of A.

- For DPPs:

$\rho_1(D)$ = number of n's in D,

$\rho_2(D)$ = (number of $(n-1)$'s in D)
+ (number of rows of D of length $n-1$).

$$Z_{\text{ASM/DPP}}(n, x, y, z_1, z_2) = \sum_X x^{\nu(X)} y^{\mu(X)} z_1^{\rho_1(X)} z_2^{\rho_2(X)}$$

Define refined enumeration by introduction "boundary" statistics:

- For ASMs:

$$\rho_1(A) = \text{number of 0's to the left of the 1 in the first row of } A,$$
$$\rho_2(A) = \text{number of 0's to the right of the 1 in the last row of } A.$$

- For DPPs:

$$\rho_1(D) = \text{number of } n\text{'s in } D,$$
$$\rho_2(D) = (\text{number of } (n-1)\text{'s in } D) + (\text{number of rows of } D \text{ of length } n-1).$$

$$Z_{\mathrm{ASM/DPP}}(n, x, y, z_1, z_2) = \sum_X x^{\nu(X)} y^{\mu(X)} z_1^{\rho_1(X)} z_2^{\rho_2(X)}$$

Define refined enumeration by introduction "boundary" statistics:

- For ASMs:

$$\rho_1(A) = \text{number of 0's to the left of the 1 in the first row of } A,$$
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$$\rho_1(D) = \text{number of } n\text{'s in } D,$$
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Example ($n = 3$)

$$\mathrm{ASM}(3) = \left\{ \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}, \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}, \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}, \begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix} \right\}$$

$$\mathrm{DPP}(3) = \left\{ \emptyset,\ \begin{smallmatrix}3&3\\&2\end{smallmatrix},\ 2,\ 3\,3,\ 3,\ 3\,2,\ 3\,1 \right\}$$

$$Z_{\mathrm{ASM/DPP}}(3, x, y, z_1, z_2) = 1 + x^3 z_1^2 z_2^2 + x z_2 + x^2 z_1^2 z_2 + x z_1 + x^2 z_1 z_2^2 + x y z_1 z_2$$

Strategy of proof

1. Generalize the unrefined proof to a single refinement.
   Involves modifying one row of the matrices...
2. Show the bilinear identity for both ASMs and DPPs:

$$(z_1 - z_2)(z_3 - z_4) Z_n(x,y,z_1,z_2) Z_n(x,y,z_3,z_4) - (z_1 - z_3)(z_2 - z_4) Z_n(x,y,z_1,z_3) Z_n(x,y,z_2,z_4) + (z_1 - z_4)(z_2 - z_3) Z_n(x,y,z_1,z_4) Z_n(x,y,z_2,z_3) = 0.$$

This allows to express the double refinement in terms of the single refinement: $(z_3 = 1, z_4 = 0)$

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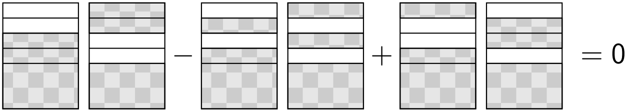
An equivalent form of Desnanot–Jacobi



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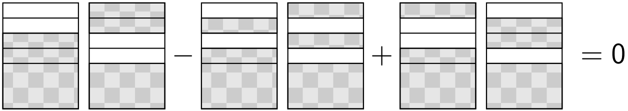

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The double refinement of ASMs simply corresponds to keeping two spectral parameters free and letting the others tend to r.

$\longrightarrow$ Apply directly Desnanot–Jacobi to the Izergin matrix

$$\left(\frac{c(u_i, v_j)}{a(u_i, v_j)b(u_i, v_j)}\right)$$

A similar formula appears in [Colomo, Pronko, '05].

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Direct application of the LGV formula leads to:

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