Limiting shapes in the six-vertex model

P. Zinn-Justin

LPTHE, Université Paris 6

May 2, 2010

based on work by F. Colomo, A. Pronko, P. Zinn-Justin

- This work originated in the observation [Korepin, PZJ; PZJ, '00] that the bulk free energy of the six-vertex model depends on the boundary conditions.
- At the same time, a lot of work on dimer models [Jockusch, Propp, Shor, '98; …; Kenyon, Okounkov, '04] showed the existence of spatial phase separation and limiting shapes in these models.
- For a long time, only qualitative [PZJ, '02] or numerical results [Syljuasen, Zvonarev, '04; Allison, Reshetikhin, '05] in the case of the six-vertex model.
- Colomo and Pronko recently managed to compute a certain correlation function for the six-vertex model with Domain Wall Boundary Conditions and to describe the arctic curve in the disordered regime ['07–'09].
- Their work can be extended to describe the arctic curve in all regimes.

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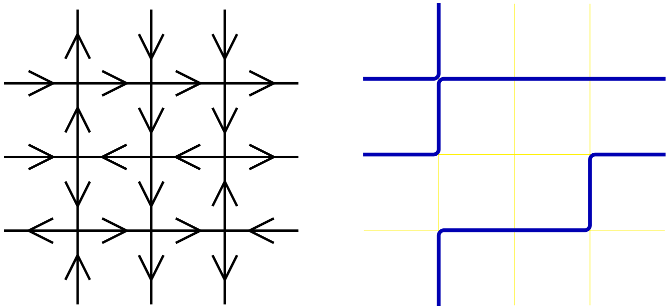
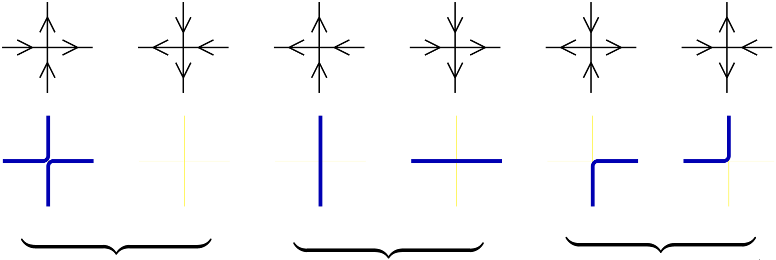
Configurations



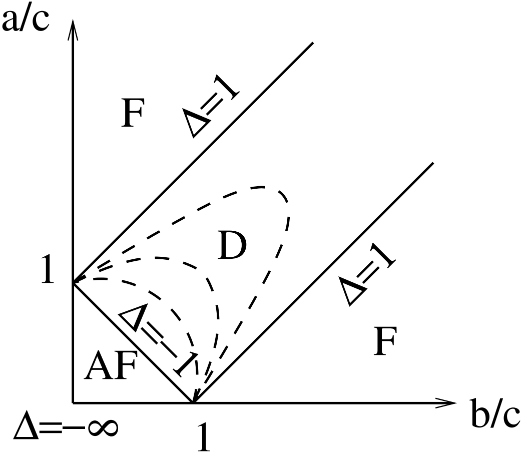
Boltzmann weights



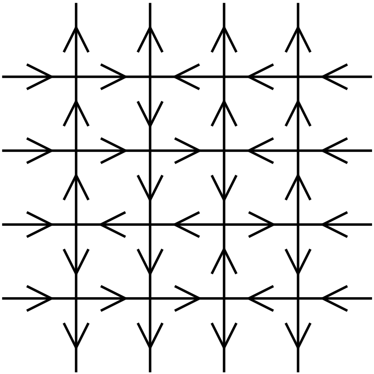
$$a = q\,u - q^{-1} \qquad b = u - 1 \qquad c = (q - q^{-1})u^{1/2}$$

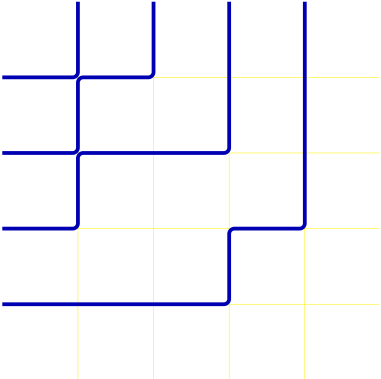
Phase diagram

Set $\Delta = \frac{1}{2}(q + q^{-1}) = (a^2 + b^2 - c^2)/(2ab)$.

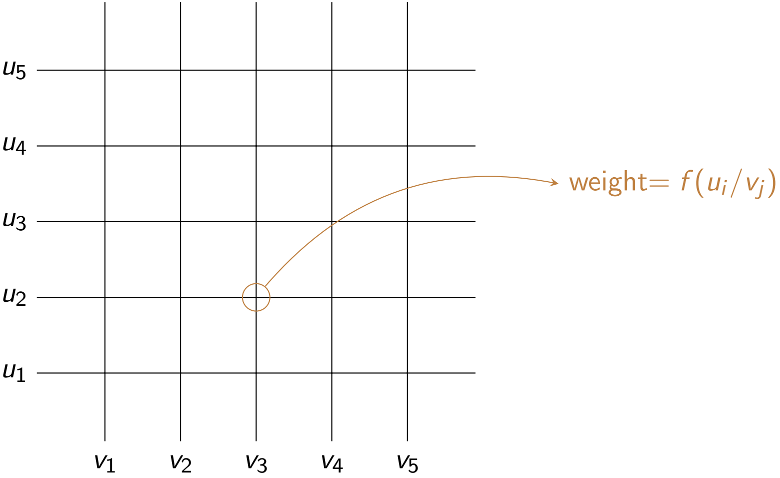


Domain Wall Boundary Conditions





Inhomogeneous weights



Izergin's determinant formula ['87]

The partition function Z_N of the six-vertex model with DWBC can be written as:

$$Z_N(u_1, \ldots, u_N; v_1, \ldots, v_N) \propto \frac{\det \phi(u_i/v_j)}{\Delta(u_i)\Delta(v_i)},$$

$$\phi(u) = \frac{q - q^{-1}}{(u-1)(qu - q^{-1})}$$

Matrix model-like formulation

Write

$$\phi(u) = \int_{-\infty}^{+\infty} d\rho_0(\lambda) u^{\lambda/2}$$

where $d\rho_0(\lambda)$ is some measure to rewrite

$$Z_N \propto \int d\rho_0(\lambda_1) \cdots d\rho_0(\lambda_N) \frac{\det(u_i^{\lambda_j/2})}{\Delta(u_i)} \frac{\det(v_i^{-\lambda_j/2})}{\Delta(v_i)}$$

$\rightarrow$ one-matrix model with "double external field".

Bulk free energy

Take the homogeneous limit $u_i \to 1$, $v_i \to v$:

$$Z_N \propto \int d\rho(\lambda_1) \cdots d\rho(\lambda_N) \Delta(\lambda_i)^2$$

where $d\rho(\lambda) = v^{-\lambda/2} d\rho_0(\lambda)$. $\to$ usual one-matrix model.
One can compute the large N limit by standard matrix model techniques. . . [PZJ, '00]

- The ferroelectric regime $\Delta > 1$ is trivial.
- The disordered regime $|\Delta| < 1$ corresponds to a one-cut matrix model with non-polynomial potential $V(\lambda) = a\lambda + b|\lambda|$.
- The antiferroelectric regime $\Delta < -1$ produces a two-cut solution separated by a Douglas–Kazakov type saturated region.

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One point boundary correlation function

In the limit where all $u_i \to 1$ but one, which is sent to u, we can first expand the determinant, and then replace it with a contour integral:

$$
\begin{aligned}
Z_N &\propto \int d\rho(\lambda_1)\cdots d\rho(\lambda_N)\Delta(\lambda_j)\sum_{i=1}^{N} u^{\lambda_i/2}\Delta(\lambda_j)_{j\neq i} \\
&\propto \int d\rho(\lambda_1)\cdots d\rho(\lambda_N)\Delta(\lambda_j)^2\sum_{i=1}^{N} \frac{(-1)^{i-1}u^{\lambda_i/2}}{\prod_{\substack{1\leq j\leq N\\ j\neq i}}(\lambda_i-\lambda_j)} \\
&\propto \int d\rho(\lambda_1)\cdots d\rho(\lambda_N)\Delta(\lambda_j)^2 \oint d\lambda\, \frac{u^{\lambda/2}}{\prod_{i=1}^{N}(\lambda-\lambda_i)}
\end{aligned}
$$

One point boundary correlation function cont'd

As $N \to \infty$, the integral over $\mu = \lambda/(2N)$ is dominated by a saddle point; the saddle point equation is

$$\log u = \omega(\mu_\star)$$

where $\omega(\mu)$ is the resolvent of the $\mu_i = \lambda_i/(2N)$:

$$\omega(\mu) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\mu - \mu_i}$$

Therefore, the one point boundary correlation function

$$\lim_{N\to\infty} \frac{1}{N} u \frac{d}{du} \log Z_N(u, 1, \ldots, 1; v, \ldots, v) = analytic + \omega^{-1}(\log u)$$

is given by the functional inverse of the resolvent.

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From general principles, one expects that with fixed boundary conditions, the six-vertex model should undergo spatial phase separation in the thermodynamic limit.

Based on its phase diagram in electric field, one should have:

- Ferroelectric and disordered regions for $\Delta > -1$.
- Ferroelectric, disordered and antiferroeletric regions for $\Delta < -1$.

These different regions can in principle be determined by applying a variational principle, but it is too complicated to solve in practice.

Here we shall be concerned with the arctic curve, that is the curve separating the ferroeletric and disordered regions. In the south-east corner it coincides with the trajectory of the lowest path.

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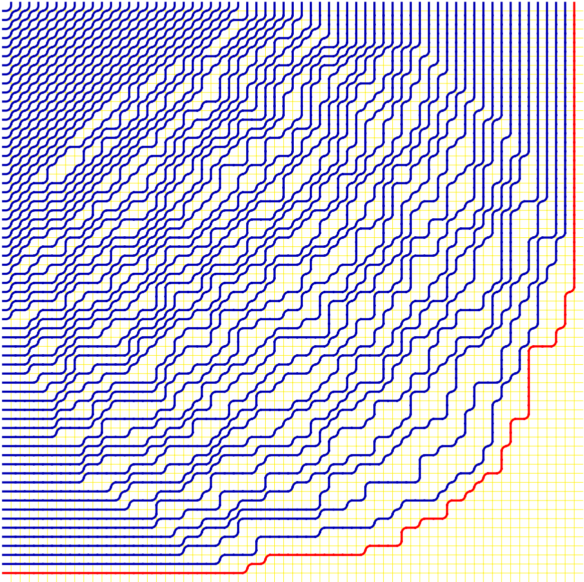
$a = 1$
$b = 1$
$c = 1$
$\Delta = 1/2$

Coupling from the past [Propp and Wilson '96], implementation by Blum and Woolever '97, Wieland '07, PZJ '09.



$a = 1$
$b = 1$
$c = \sqrt{2}$
$\Delta = 0$

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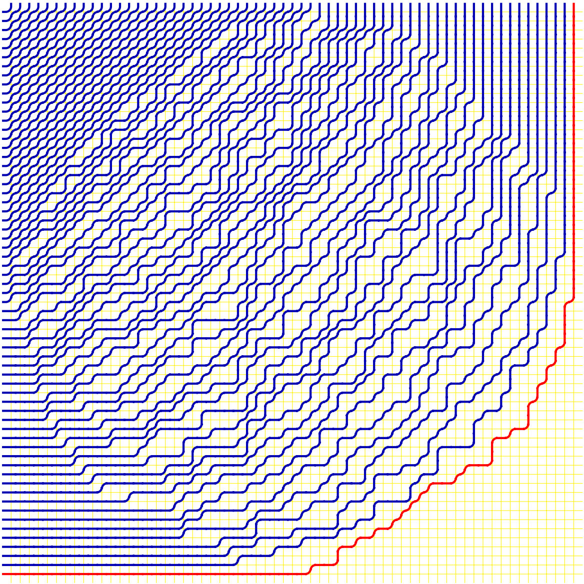
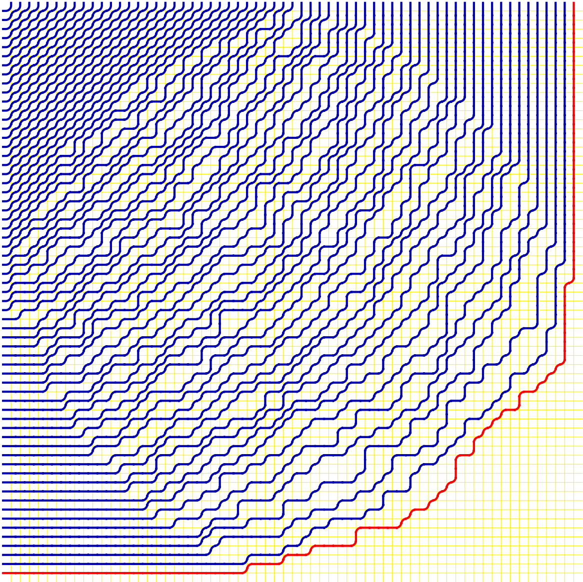


$a = 1$
$b = 1$
$c = \sqrt{3}$
$\Delta = -1/2$

Coupling from the past [Propp and Wilson '96], implementation by Blum and Woolever '97, Wieland '07, PZJ '09.



$a = 1$
$b = 1$
$c = 2$
$\Delta = -1$

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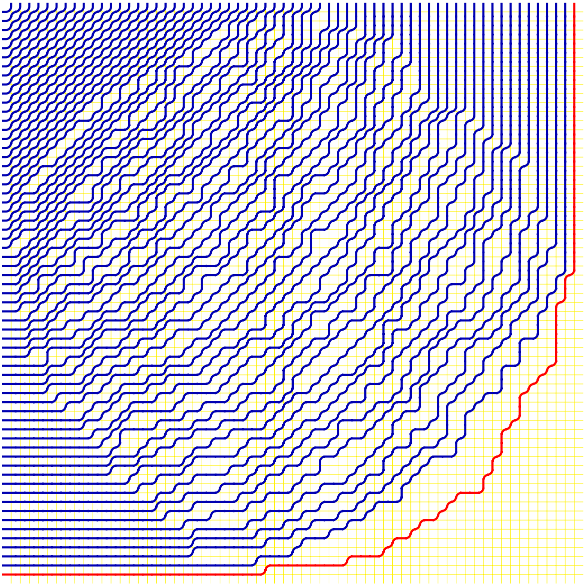

$a = 1$
$b = 1$
$c = 3$
$\Delta = -7/2$

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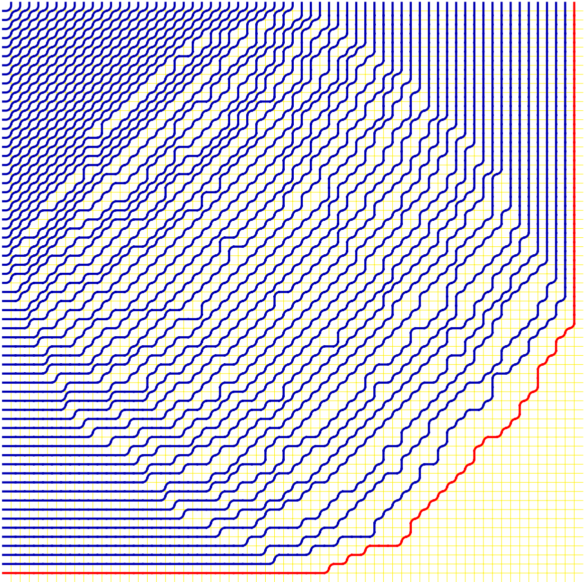

$a = 1$
$b = 1$
$c = 4$
$\Delta = -7$

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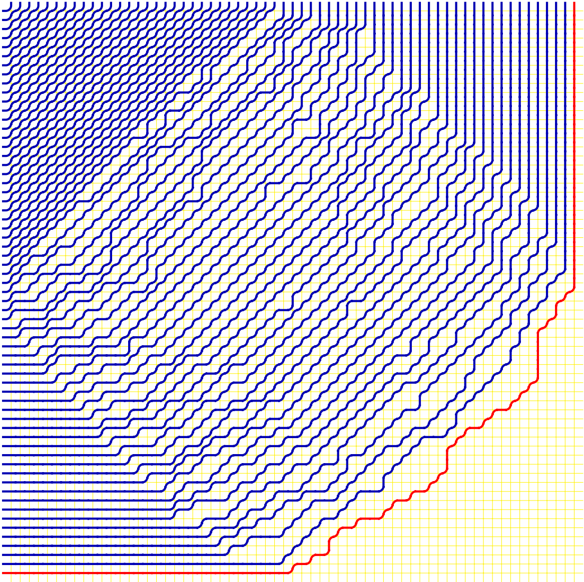

$a = 3$
$b = 4$
$c = 5$
$\Delta = 0$

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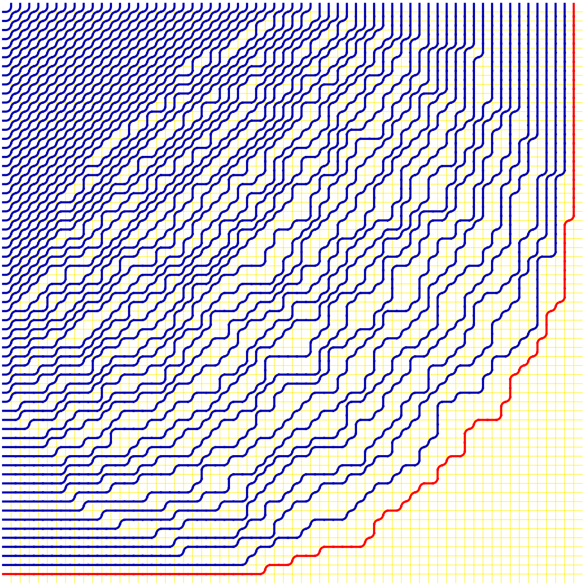

$a = 1$
$b = 3$
$c = 5$
$\Delta = -15/8$

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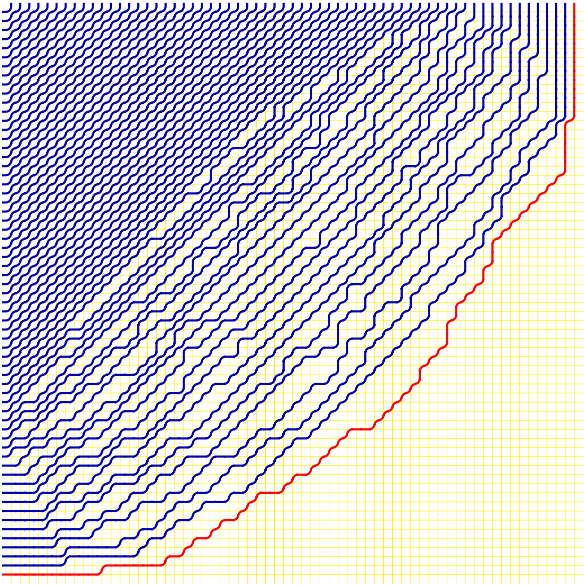

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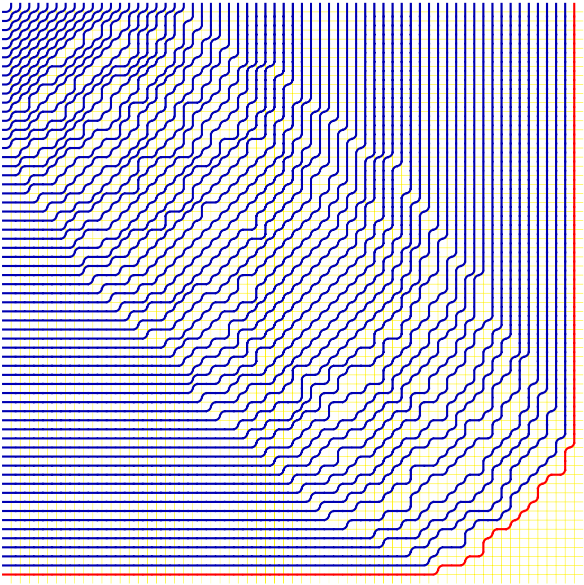

P. Colomo and A. Pronko managed to find an integral expression for the Emptiness Formation Probability in the rectangle south-east of a given point (r, s).

$$F_N^{(r,s)} \propto \oint \cdots \oint \prod_{i=1}^{s} z_i^{-s} \hat{z}_i^{-r} dz_i \, \Delta(z_i)^2 \det(h_{N-j}(z_i)) \det(h_{s-j}(\hat{z}_i))$$

$$\hat{z}_i = -\frac{z_i - 1}{(t^2 - 2\Delta t) z_i + 1}$$

$$h_N(z) \propto \lim_{N \to \infty} \frac{1}{N} \log Z_N(u(z), 1, \dots, 1; v, \dots, v)$$

$$t = b/a$$

By carefully analyzing the critical behavior as $N \to \infty$ of this matrix model-like expression, Colomo and Pronko found the following parametric form of [one quarter of] the arctic curve in terms of rescaled coordinates $X = r/N$ and $Y = s/N$:

$$X(u) = x_1(u) + x_2(u)\omega^{-1}(u) + x_3(u)\frac{d}{du}\omega^{-1}(u)$$
$$Y(u) = y_1(u) + y_2(u)\omega^{-1}(u) + y_3(u)\frac{d}{du}\omega^{-1}(u)$$

where $x_i(u)$, $y_i(u)$ are rational functions. $|u| = 1$ in the disordered phase, $u > 1$ in the antiferroelectric phase.

Set $q = -\exp(-i\pi/\alpha)$.

- In the disordered phase, $\omega^{-1}(u)$ is a rational function of u^α:

$$\omega^{-1}(u) = \frac{\alpha}{2}\left(\frac{u^\alpha + 1}{u^\alpha - 1} - \frac{u^\alpha + v^\alpha}{u^\alpha - v^\alpha}\right)$$

  If α is rational, the curve (X, Y) is algebraic.

- In the antiferroelectric phase, $\omega^{-1}(\log u)$ is an elliptic function:

$$\omega^{-1}(u) = \frac{\alpha}{2}\left(\frac{u^\alpha + 1}{u^\alpha - 1} - \frac{u^\alpha + v^\alpha}{u^\alpha - v^\alpha} + 2\sum_{n=1}^{\infty} \frac{\tilde{q}^{2n}}{1 - \tilde{q}^{2n}}\left(u^{n\alpha} - u^{-n\alpha} - (uv^{-1})^{n\alpha} + (u^{-1}v)^{n\alpha}\right)\right)$$

  where $\tilde{q} = \exp(i\pi\alpha)$, $|\tilde{q}| < 1$.

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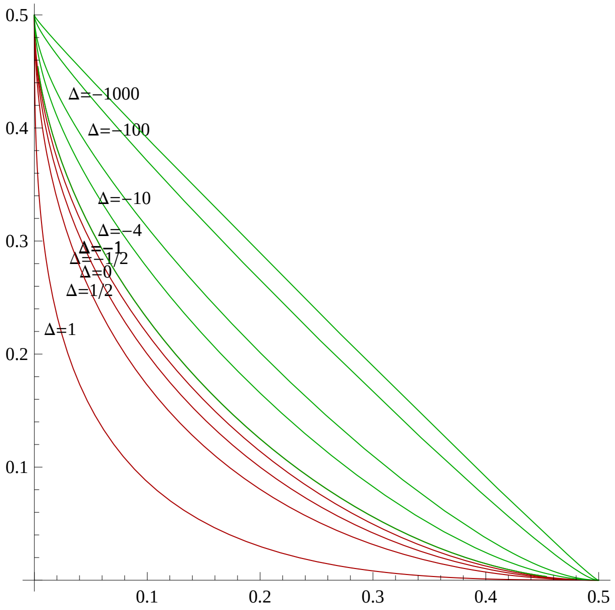
0.5
0.4
0.3
0.2
0.1
0.1
0.2
0.3
0.4
0.5
Δ=−1000
Δ=−100
Δ=−10
Δ=−4
Δ=−1
Δ=−1/2
Δ=0
Δ=1/2
Δ=1

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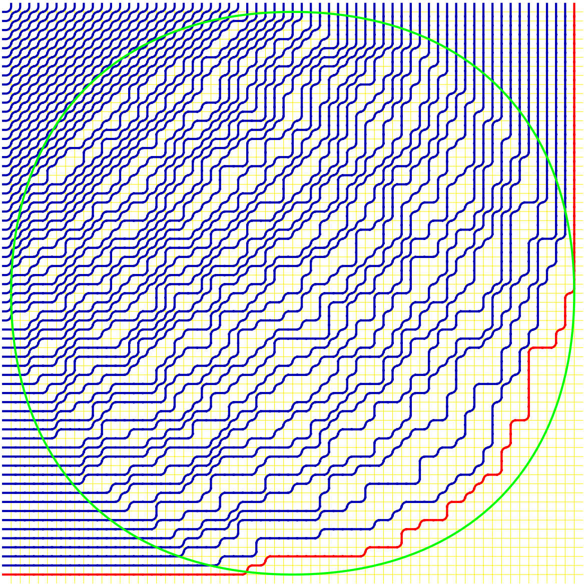

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