Fluctuations of Real Random Matrices and Second-Order Freeness

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March 8, 2012

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Families of matrices are *asymptotically free* if

$$\lim_{N \to \infty} \mathbb{E}\left(\mathrm{tr}\left(\mathring{A}_{1,N} \cdots \mathring{A}_{p,N}\right)\right) = 0$$

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- $\varphi_2(1_A, a) = \varphi_2(a, 1_A) = 0$.

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Subalgebras $A_1, \ldots, A_n$ of a second-order noncommutative probability space $(A, \varphi_1, \varphi_2)$ are *complex second-order free* if they are free and for $a_1, \ldots, a_p$ and $b_1, \ldots, b_q$ centred and either cyclically alternating or consisting of a single term, we have

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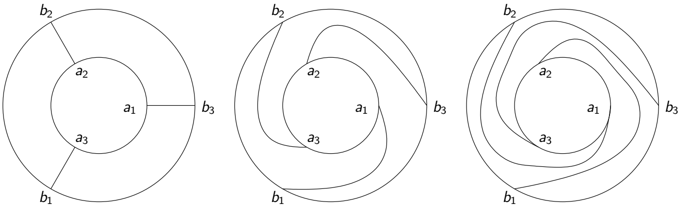
- and when $p = q$:

$$\varphi_2 \left(a_1 \cdots a_p, b_1 \cdots b_p \right) = \sum_{k=0}^{p-1} \prod_{i=1}^{p} \varphi_1 \left(a_i b_{k-i} \right).$$

Spoke diagrams:



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$$=\sum_{k=0}^{p-1}\prod_{i=1}^{p}\left(\lim_{N\to\infty}\left(\mathbb{E}\left(\mathrm{tr}\left(A_iB_{k-i}\right)\right)-\mathbb{E}\left(\mathrm{tr}\left(A_i\right)\right)\mathbb{E}\left(\mathrm{tr}\left(B_{k-i}\right)\right)\right)\right).$$

Let $X : \Omega \to M_{M\times N}(\mathbb{R})$ be a random matrix with $X_{ij} = \frac{1}{\sqrt{N}} f_{ij}$, where the f_{ij} are independent $N(0,1)$ random variables.

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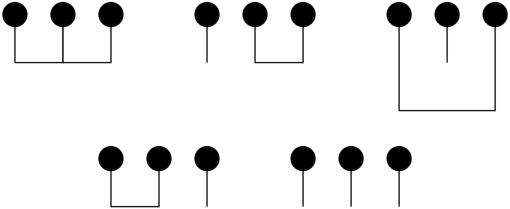
Real Wishart matrices are matrices $W := X^T D_k X$ for some deterministic matrix D_k.

There are 5 partitions of 3 elements:



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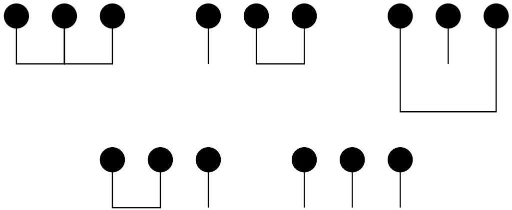

We define cumulants k_1, k_2, k_3 to satisfy:

$$\mathbb{E}(XYZ) = k_3(X, Y, Z) + k_1(X)\, k_2(Y, Z) + k_2(X, Z)\, k_1(Y) + k_2(X, Y)\, k_1(Z) + k_1(X)\, k_1(Y)\, k_1(Z).$$

Definition

The *nth mixed moment* of (classical) random variables $X_1, \ldots, X_n$ is an n-linear function defined to be the expectation of their product:

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Definition

We define the *cumulants* k_i to satisfy the *moment-cumulant formula*:

$$a_n(X_1, \ldots, X_n) = \sum_{\pi \in \mathcal{P}(n)} \prod_{V = \{i_1, \ldots, i_r\} \in \pi} k_r(X_{i_1}, \ldots, X_{i_r}).$$

The first four cumulants are:

$$k_1(X) = \mathbb{E}(X)$$

$$k_2(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

$$k_3(X, Y, Z) = \mathbb{E}(XYZ) - \mathbb{E}(X)\mathbb{E}(YZ) - \mathbb{E}(XY)\mathbb{E}(Y) - \mathbb{E}(XY)\mathbb{E}(Z) + 2\mathbb{E}(X)\mathbb{E}(Y)\mathbb{E}(Z)$$

$$\begin{aligned} k_4(X, Y, Z, W) = {} & \mathbb{E}(XYZW) - \mathbb{E}(X)\mathbb{E}(YZW) \\ & - \mathbb{E}(XZW)\mathbb{E}(Y) - \mathbb{E}(XYW)\mathbb{E}(Z) - \mathbb{E}(XYZ)\mathbb{E}(W) \\ & - \mathbb{E}(XY)\mathbb{E}(ZW) - \mathbb{E}(XZ)\mathbb{E}(YW) - \mathbb{E}(XW)\mathbb{E}(YZ) \\ & + 2\mathbb{E}(XY)\mathbb{E}(Z)\mathbb{E}(W) + 2\mathbb{E}(XZ)\mathbb{E}(Y)\mathbb{E}(W) \\ & + 2\mathbb{E}(XW)\mathbb{E}(Y)\mathbb{E}(Z) + 2\mathbb{E}(X)\mathbb{E}(YZ)\mathbb{E}(W) \\ & + 2\mathbb{E}(X)\mathbb{E}(YW)\mathbb{E}(Z) + 2\mathbb{E}(X)\mathbb{E}(Y)\mathbb{E}(ZW) \\ & - 6\mathbb{E}(X)\mathbb{E}(Y)\mathbb{E}(Z)\mathbb{E}(W). \end{aligned}$$

Say we wish to calculate

$$\mathbb{E}\left(\mathrm{tr}\left(XY_1XY_2X^TY_3XY_4X^TY_5\right)\mathrm{tr}\left(X^TY_6XY_7XY_8\right)\right).$$

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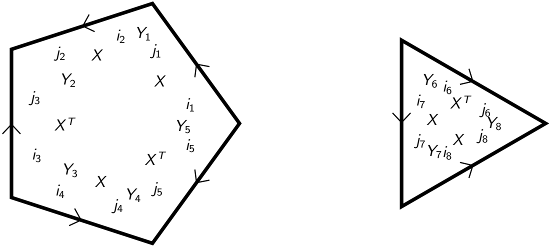
The traces of products are a sum over

$$X_{i_1j_1}Y^{(1)}_{j_1i_2}X_{i_2j_2}Y^{(2)}_{j_2j_3}X^T_{j_3i_3}Y^{(3)}_{i_3i_4}X_{i_4j_4}Y^{(4)}_{j_4j_5}X^T_{j_5i_5}Y^{(5)}_{i_5i_1}X^T_{j_6i_6}Y^{(6)}_{i_6i_7}X_{i_7j_7}Y^{(7)}_{j_7i_8}X_{i_8j_8}Y^{(8)}_{j_8j_6}.$$

We construct the faces:



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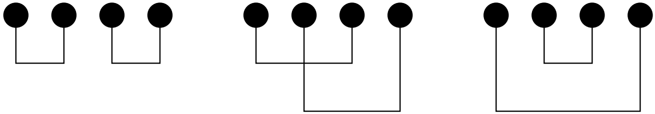
There are three pairings on 4 elements:



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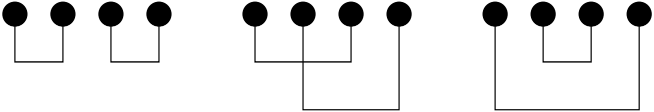

If X_1, X_2, X_3, X_4 are components of a multivariate Gaussian random variable, then

$$\mathbb{E}\left(X_1X_2X_3X_4\right) = \mathbb{E}\left(X_1X_2\right)\mathbb{E}\left(X_3X_4\right) + \mathbb{E}\left(X_1X_3\right)\mathbb{E}\left(X_2X_4\right) + \mathbb{E}\left(X_1X_4\right)\mathbb{E}\left(X_2X_3\right).$$

Let $\mathcal{P}_2(n)$ be the set of pairings on n elements.

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Theorem

Let $\{f_\lambda : \lambda \in \Lambda\}$, for some index set Λ, be a centred Gaussian family of random variables. Then for $i_1, \ldots, i_n \in \Lambda$,

$$\mathbb{E}\left(f_{i_1} \cdots f_{i_n}\right) = \sum_{\mathcal{P}_2(n)} \prod_{\{k,l\} \in \mathcal{P}_2(n)} \mathbb{E}\left(f_{i_k} f_{i_l}\right).$$

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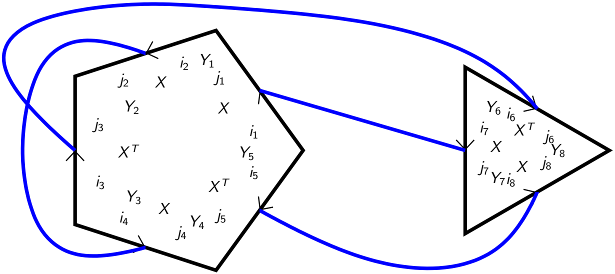
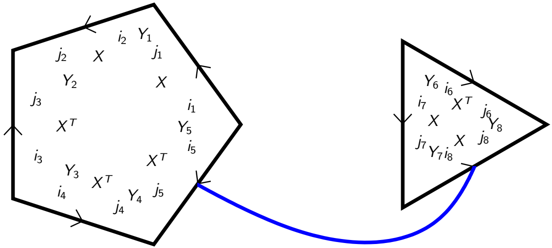
Here, for a pairing $\pi \in \mathcal{P}_2(n)$:

$$\prod_{\{k,l\}} \mathbb{E}\left(f_{i_k j_k} f_{i_l j_l}\right) = \begin{cases} 1, & \text{if } i_k = i_l \text{ and } j_k = j_l \text{ for all } \{k,l\} \in \pi \\ 0, & \text{otherwise} \end{cases}.$$

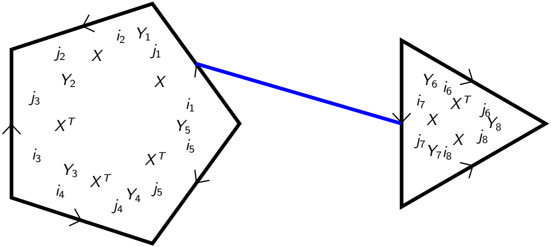
Putting indices which must be equal next to each other, we get a surface gluing:



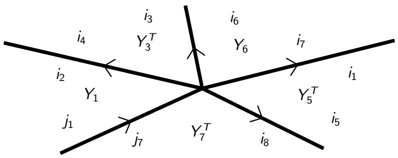
We note that if one term is from X and the other from X^T, the edge identification is untwisted:



If both terms are from X or from X^T, the edge identification is twisted:



The following vertex appears on the surface:



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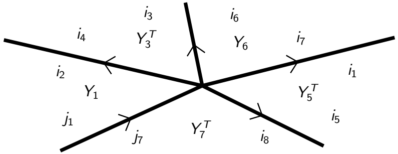

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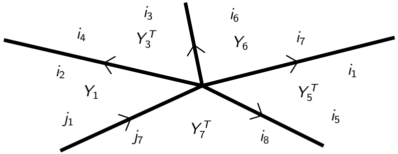

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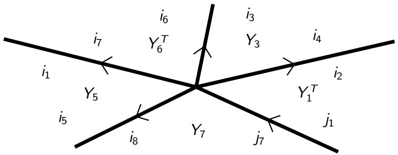
It contributes

$$\mathrm{Tr}\left(Y_1 Y_3^T Y_6 Y_5^T Y_7^T\right).$$

The same vertex viewed from the opposite side contributes the same value:



$$\mathrm{Tr}\left(Y_7 Y_5 Y_6^T Y_3 Y_1^T\right) = \mathrm{Tr}\left(Y_1 Y_3^T Y_6 Y_5^T Y_7^T\right).$$

Each vertex gives us a trace, and hence a factor of N when normalized.

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Crossings require handles, so highest order terms typically correspond to noncrossing diagrams with untwisted identifications.

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Highest order terms must have a relative orientation of the faces in which none of the edge-identifications are twisted.

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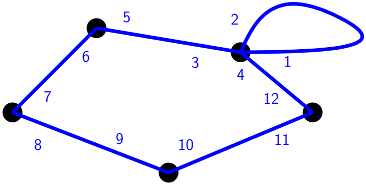
The permutation $\pi^{-1}\gamma^{-1}$ encodes vertex information.

Consider the map:



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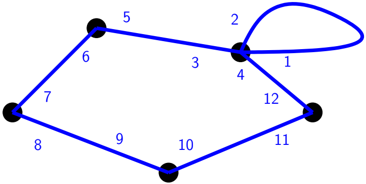

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$$\sigma = (1, 2, 3, 4)\,(5, 6)\,(7, 8)\,(9, 10)\,(11, 12)\,.$$

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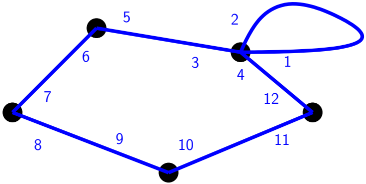

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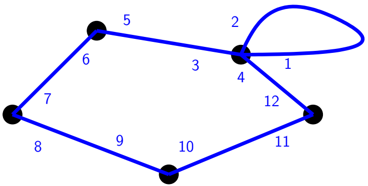
$$\sigma = (1,2,3,4)\,(5,6)\,(7,8)\,(9,10)\,(11,12)\,.$$

The edge information can be encoded in another permutation

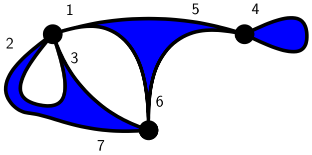
$$\alpha = (1,2)\,(3,5)\,(4,12)\,(6,7)\,(8,9)\,(10,11)\,.$$



The face information is encoded in

$$\varphi := \sigma^{-1}\alpha^{-1} = (1)\,(2, 4, 11, 9, 7, 5)\,(3, 6, 8, 10, 12)\,.$$

This construction works equally well with oriented hypermaps:



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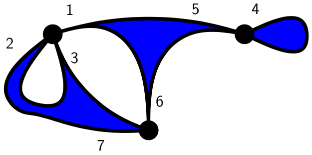

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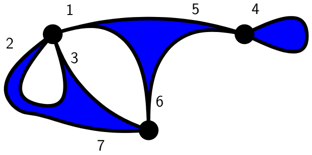

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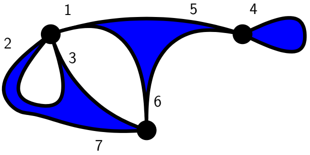

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$$\varphi = \sigma^{-1}\alpha^{-1} = (1,4,5,7)\,(2)\,(3,6)$$

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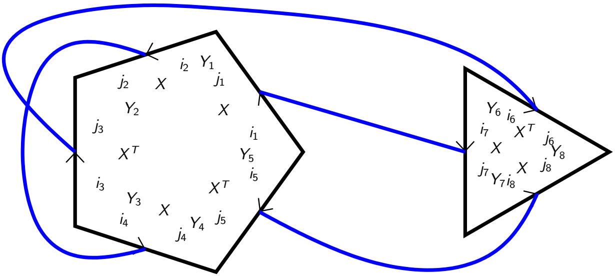
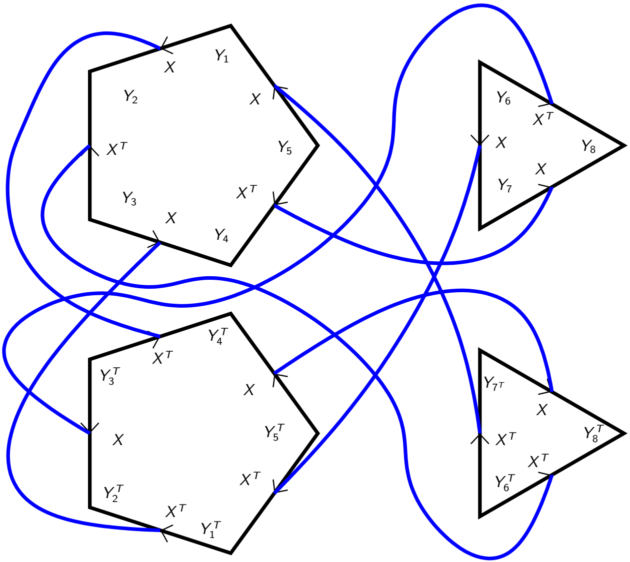
An untwisted edge-identification connects front to front and back to back, while a twisted edge-identification connects front to back and back to front.


i_2 Y_1
j_2 X j_1
Y_2 X
j_3 i_1
X^T Y_5
i_3 i_5
Y_3 X^T
X
i_4 Y_4 j_5
j_4
Y_6 i_6
i_7 X^T
X j_6 Y_8
j_7 Y_7 i_8 X j_8


Y_1
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X^T
Y_5
Y_3
X^T
X
Y_4
Y_6
X^T
X
Y_8
X
Y_7
Y_4^T
X^T
Y_3^T
X
X
Y_5^T
Y_2^T
X^T
X^T
Y_1^T
Y_{7^T}
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Y_8^T
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We let $\gamma_+ = \gamma$, and $\gamma_- = \delta\gamma\delta$.

Vertex information is given by $\gamma_+^{-1}\pi^{-1}\gamma_-$.

In the example,

$$\pi = (1, -7)\,(7, -1)\,(2, -4)\,(4, -2)\,(3, -6)\,(6, -3)\,(5, 8)\,(-8, -5)\,.$$

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This diagram contributes the term:

$$N^{-2}\mathbb{E}\left(\mathrm{tr}\left(Y_1 Y_3^T Y_6 Y_5^T Y_7^T\right)\mathrm{tr}\left(Y_2 Y_8^T Y_4^T\right)\right)$$

Let:

- $\mathrm{tr} := \frac{1}{N}\mathrm{Tr}$,
- $n_1, \ldots, n_r$ positive integers, $n := n_1 + \cdots + n_r$,
- $A^{(1)} = A$, $A^{(-1)} = A^T$,
- $[n] = \{1, \ldots, n\}$,
- $\varepsilon : [n] \to \{1, -1\}$,
- $\delta_\varepsilon : k \mapsto \varepsilon(k)\, k$.

For $\gamma = (c_1, \ldots, c_{n_1}) \cdots (c_{n_1+\cdots+n_{r-1}}, \ldots, c_n) \in S_n$, we define:

$$\mathrm{Tr}_\gamma (A_1, \ldots, A_n) := \mathrm{Tr}\left(A_{c_1} \cdots A_{c_{n_1}}\right) \cdots \mathrm{Tr}\left(A_{c_{n_1+\cdots+n_{r-1}}} \cdots A_{c_n}\right).$$

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Then

$$\mathrm{Tr}_\gamma (A_1, \ldots, A_n) = \sum_{1 \leq i_1, \ldots, i_n \leq N} A_{i_1 i_{\gamma(1)}} \cdots A_{i_n i_{\gamma(n)}}.$$

For example:

$$
\begin{aligned}
& \mathrm{Tr}_{(1,2,3,4,5,6)(7,8,9,10)}\left(A_1,\ldots,A_{10}\right) \\
= & \mathrm{Tr}\left(A_1A_2A_3A_4A_5A_6\right)\mathrm{Tr}\left(A_7A_8A_9A_{10}\right) \\
= & \sum_{i_1,\ldots,i_6=1}^{N} A^{(1)}_{i_1,i_2}A^{(2)}_{i_2,i_3}A^{(3)}_{i_3,i_4}A^{(4)}_{i_4,i_5}A^{(5)}_{i_5,i_6}A^{(6)}_{i_6,i_1}A^{(7)}_{i_7,i_8}A^{(8)}_{i_8,i_9}A^{(9)}_{i_9,i_{10}}A^{(10)}_{i_{10},i_1}
\end{aligned}
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Reversing the order of summation,

$$\sum_{\pi \in \mathcal{P}_2(n)} \sum_{\substack{1 \leq \iota_1^+, \ldots \iota_n^+ \leq M \\ 1 \leq \iota_1^-, \ldots, \iota_n^- \leq N \\ \iota_k^\pm = \iota_l^\pm : \{k,l\} \in \pi}} N^{-\#(\gamma)-n} \mathbb{E}\left(Y^{(1)}_{\iota_1^{-\varepsilon(1)} \iota_{\gamma(1)}^{\varepsilon(\gamma(1))}} \cdots Y^{(n)}_{\iota_n^{-\varepsilon(n)} \iota_{\gamma(n)}^{\varepsilon(\gamma(n))}} \right)$$

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$$= \sum_{\pi \in \mathcal{P}_2(n)} N^{\#\left(\gamma_-^{-1}\delta_\varepsilon \pi \delta \pi \delta_\varepsilon \gamma_+\right)/2 - \#(\gamma) - n} \mathbb{E}\left(\mathrm{tr}_{\gamma_-^{-1}\delta_\varepsilon \pi \delta \pi \delta_\varepsilon \gamma_+ / 2}\left(Y_1, \ldots, Y_n \right) \right).$$

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This is a sum over all gluings compatible with the edge directions given by the transposes.

If we expand out the GOE matrix $T := \frac{1}{\sqrt{2}}\left(X + X^{T}\right)$, we get

$$\begin{aligned}
&\mathbb{E}\left(\mathrm{tr}_{\gamma}\left(TY_1, \ldots, TY_n\right)\right) \\
&\quad = \sum_{\varepsilon:\{1,\ldots,n\}\to\{1,-1\}} \frac{1}{2^{n/2}} \mathbb{E}\left(\mathrm{tr}_{\gamma}\left(X^{(\varepsilon(1))} Y_1 \cdots X^{(\varepsilon(n))} Y_n\right)\right).
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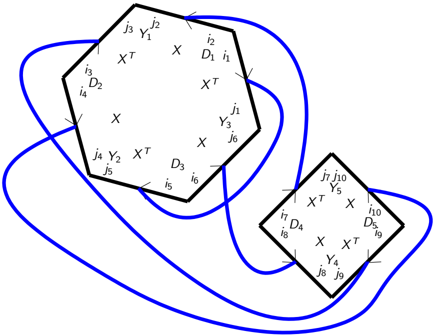
With Wishart matrices $W := X^T D_k X$, we can collapse the edges corresponding to each matrix to a single edge. We can think of the connecting blocks as (possibly twisted) hyperedges.



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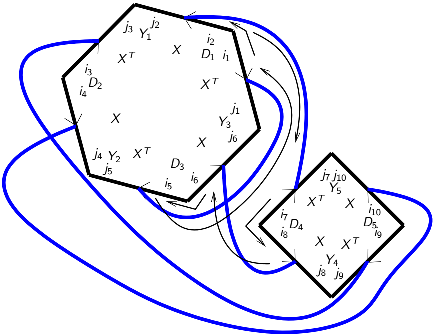

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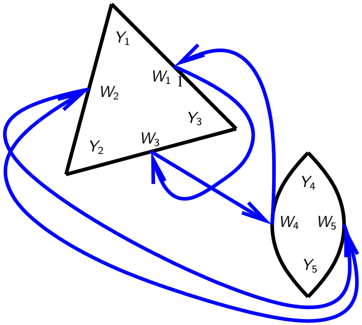

Thus:

$$\mathbb{E}\left(\mathrm{tr}_{\gamma}\left(W_1 Y_1, \cdots, W_n Y_n\right)\right) = \sum_{\pi \in PM([n])} N^{\chi(\gamma,\pi)-\#(\gamma)} \mathrm{tr}_{\pi^{-1}/2}\left(D_1, \ldots, D_n\right) \mathbb{E}\left(\mathrm{tr}_{\gamma_-^{-1}\pi\gamma_+/2}\left(Y_1, \ldots, Y_n\right)\right).$$

We note that all of the matrix ensembles satisfy

$$\mathbb{E}\left(\mathrm{tr}_{\gamma}\left(X_{\lambda_1}^{(\varepsilon(1))}Y_1,\cdots,X_{\lambda_n}^{(\varepsilon(n))}Y_n\right)\right)$$
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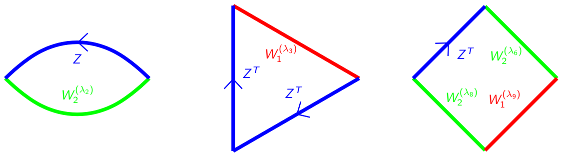

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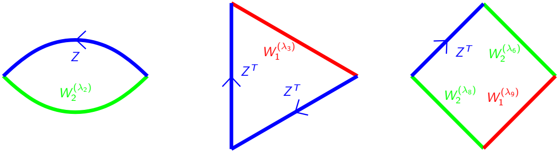

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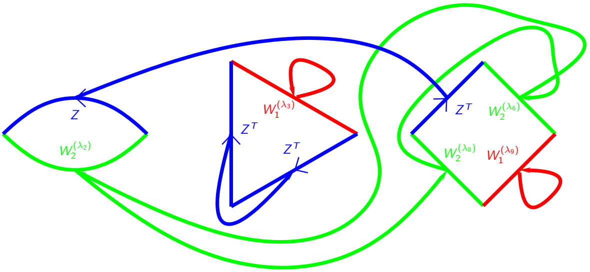
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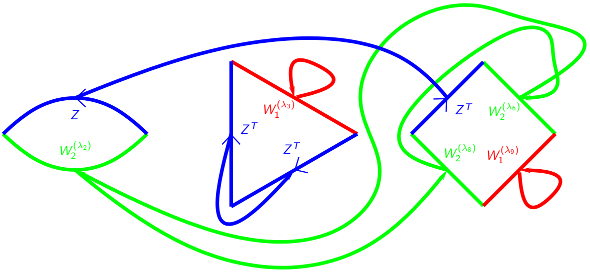
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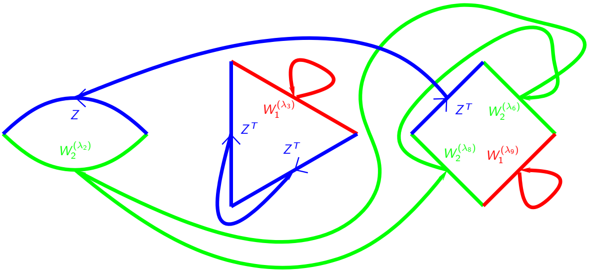

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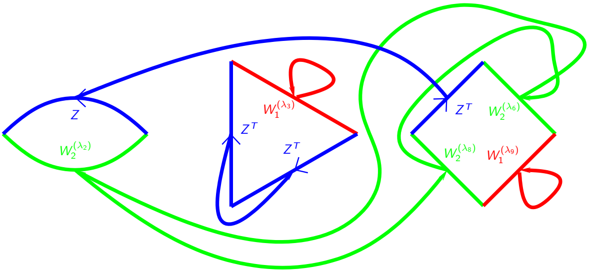

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$$\mathrm{tr}\left(A_{\lambda_3}\right)\mathrm{tr}\left(A_{\lambda_9}\right)\mathrm{tr}\left(B_{\lambda_2}B_{\lambda_6}^T B_{\lambda_8}\right) N^{-5}$$

The nth cumulant is the sum over connected surfaces constructed out of the n faces.

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For any cumulant, we have an Euler characteristic expansion:

$$\text{(sphere terms)}\, N^{-2r+2} + \text{(projective plane terms)}\, N^{-2r+1} + \\ \text{(torus and Klein bottle terms)}\, N^{-2r} + \\ \text{(connected sum of 3 projective planes terms)}\, N^{-2r-2} + \cdots .$$

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If we expand out an expression of the form

$$\mathbb{E}\left(\operatorname{tr}\left(\left(A_1 - \mathbb{E}\left(\operatorname{tr}\left(A_1\right)\right)\right) \cdots \left(A_r - \mathbb{E}\left(\operatorname{tr}\left(A_r\right)\right)\right)\right)\right)$$

we get

$$\sum_{I \subseteq [r]} (-1)^{|I|} \prod_{i \in I} \mathbb{E}\left(\operatorname{tr}\left(A_i\right)\right) \mathbb{E}\left(\operatorname{tr}\left(\prod_{i \notin I} A_i\right)\right).$$

Expressions like this one can be interpreted in terms of the Principle of Inclusion and Exclusion.

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Since diagrams with connected A_i require crossings, these vanish asymptotically.

In order to find an appropriate definition of second-order freeness, we want to consider values of

$$\lim_{N\to\infty} k_2\left(\mathrm{Tr}\left(\left(A_1 - \mathbb{E}\left(\mathrm{tr}\left(A_1\right)\right)\right)\cdots\left(A_p - \mathbb{E}\left(\mathrm{tr}\left(A_p\right)\right)\right)\right),\right.$$
$$\left.\mathrm{Tr}\left(\left(B_1 - \mathbb{E}\left(\mathrm{tr}\left(B_1\right)\right)\right)\cdots\left(B_q - \mathbb{E}\left(\mathrm{tr}\left(B_q\right)\right)\right)\right)\right).$$

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We can apply the Principle of Inclusion and Exclusion to this expression as well, with the same interpretation.

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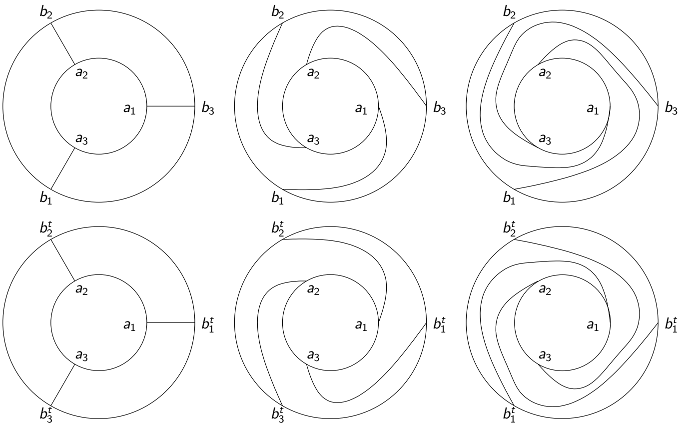
In the real case, unlike the complex case, we need to consider spoke diagrams with both relative orientations.

Spoke diagrams for the real case:



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The contribution of such a spoke is

$$\mathbb{E}\left(\operatorname{tr}\left(A_i B_j^{(\pm 1)}\right)\right) - \mathbb{E}\left(\operatorname{tr}\left(A_i\right)\right)\mathbb{E}\left(\operatorname{tr}\left(B_j^{(\pm 1)}\right)\right).$$

Definition

Families of matrices are *asymptotically real second-order free* if they are asymptotically free, have a second-order limit distribution, and for A_i and B_i in algebras generated by cyclically alternating families

$$\lim_{N\to\infty} k_2\left(\mathrm{Tr}\left(\mathring{A}_1\cdots\mathring{A}_p\right),\mathrm{Tr}\left(\mathring{B}_1\cdots\mathring{B}_q\right)\right)$$

vanishes when $p \neq q$, and when $p = q$, is equal to

$$\lim_{N\to\infty} k_2\left(\mathrm{Tr}\left(\mathring{A}_1\cdots\mathring{A}_p\right),\mathrm{Tr}\left(\mathring{B}_1\cdots\mathring{B}_p\right)\right)$$
$$=\sum_{k=0}^{p-1}\prod_{i=1}^{p}\left(\lim_{N\to\infty}\left(\mathbb{E}\left(\mathrm{tr}\left(A_iB_{k-i}\right)\right)-\mathbb{E}\left(\mathrm{tr}\left(A_i\right)\right)\mathbb{E}\left(\mathrm{tr}\left(B_{k-i}\right)\right)\right)\right)$$
$$+\sum_{k=0}^{p-1}\prod_{i=1}^{p}\left(\lim_{N\to\infty}\left(\mathbb{E}\left(\mathrm{tr}\left(A_iB_{k+i}^T\right)\right)-\mathbb{E}\left(\mathrm{tr}\left(A_i\right)\right)\mathbb{E}\left(\mathrm{tr}\left(B_{k+i}^T\right)\right)\right)\right).$$

Definition

Subalgebras $A_1, \ldots, A_n$ of a second-order noncommutative probability space $(A, \varphi_1, \varphi_2)$ are *real second-order free* if they are free and for $a_1, \ldots, a_p$ and $b_1, \ldots, b_q$ centred and either cyclically alternating or consisting of a single term

$$\varphi_2\left(a_1 \cdots a_p, b_1 \cdots b_q\right) = 0$$

when $p \neq q$ and

$$\varphi_2\left(a_1 \cdots a_p, b_1 \cdots b_p\right) = \sum_{k=0}^{p-1} \prod_{i=1}^{p} \varphi_1\left(a_i b_{k-i}\right) + \sum_{k=0}^{p-1} \prod_{i=1}^{p} \varphi_1\left(a_i b_{k+i}^t\right).$$