Complex eigenvalues of quadratised matrices

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based on joint work (J Phys A, 2012) with:

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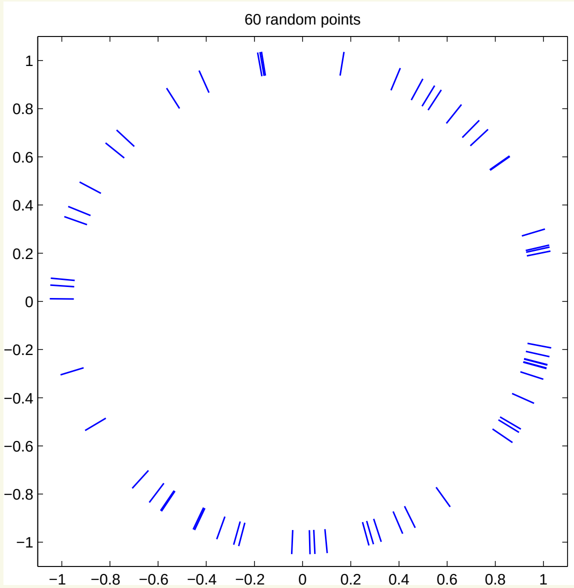
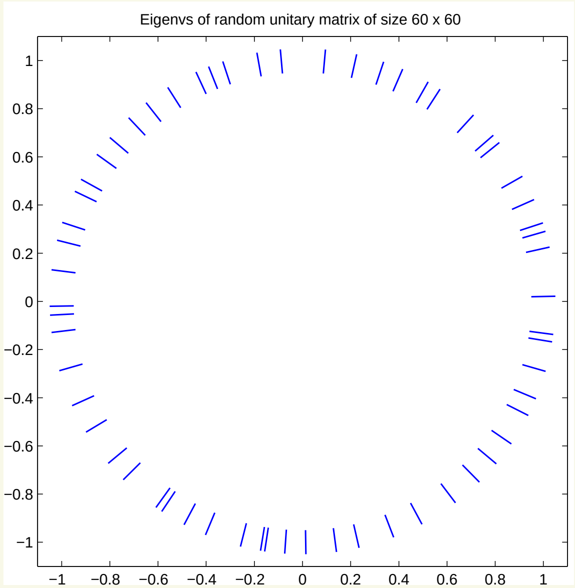
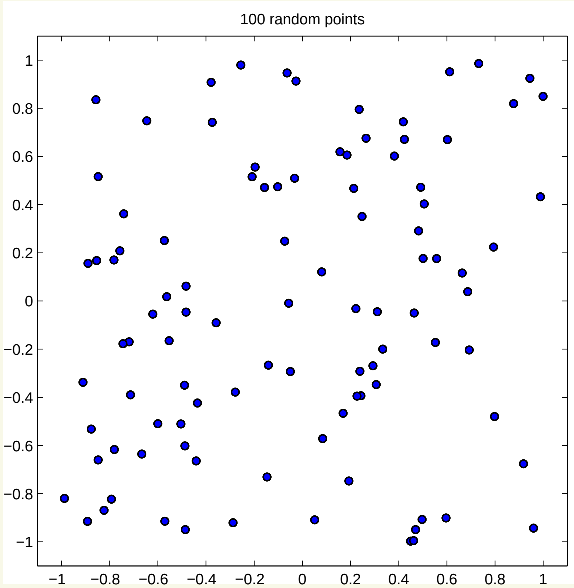
Outline of talk

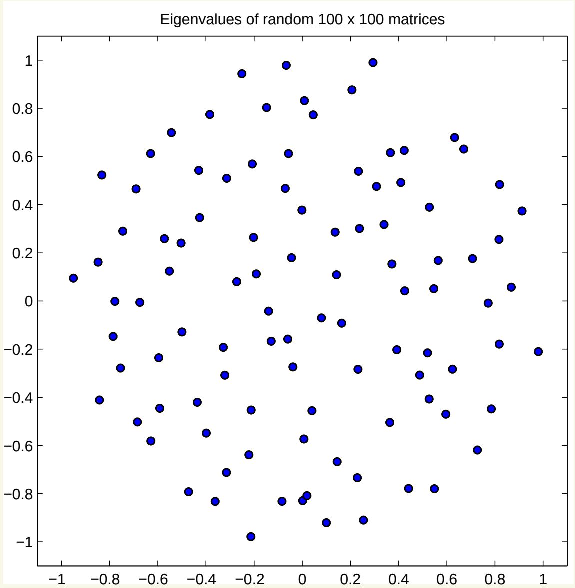
- EV distributions in the complex plane
- Quadratisation of rectangular matrices
- Induced complex Ginibre ensemble
- Induced real Ginibre ensemble
- Conclusions

Random matrices as a probability machine









Two matrix decompositions

Quite a lot of interest to EV stats in the complex plane in the last 20 yrs.

Two matrix decompositions (‘coordinate systems’):

- $G = \dfrac{G + G^\dagger}{2} + i\dfrac{G - G^\dagger}{2i}$.
  If G is complex then $G = H_1 + iH_2$ with $H_{1,2}$ Hermitian; and if G is real then $G = S + A$ with S symmetric and A asymmetric;
- $G = RU$ where $R = \sqrt{G^\dagger G}$ and U is unitary (orthogonal).

Correspondingly, one can describe matrix distributions in terms of the ‘real ’ and ‘imaginary’ part of G, or ‘radial’ and ‘angular’ part. Polar decomposition is natural for rotationally invariant matrix distributions like

$$p(G) \propto \exp\left[-\operatorname{Tr}\phi(GG^\dagger)\right]$$

Of course, there are many other matrix decompositions.

Cartesian picture: Complex Matrices

$G = H_1 + i\alpha H_2$, where $H_{1,2}$ are Hermitian, independent Gaussian (GUE)

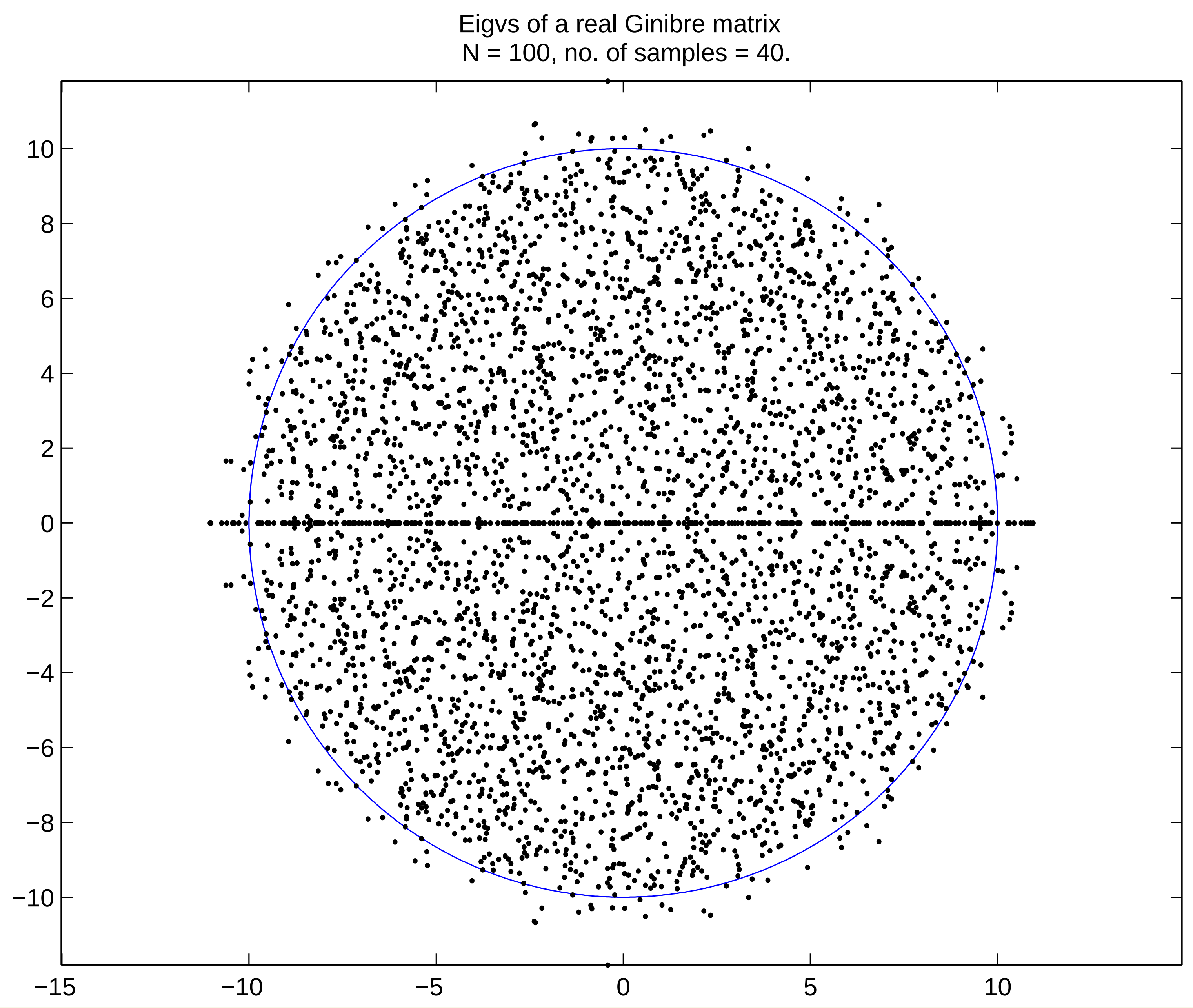
If $\alpha = 1$ then have Ginibre's ensemble $p(G) \propto \exp\left[-\operatorname{Tr} GG^\dagger\right]$. For large N have uniform distr of EV in a disk, and EV corr fncs are known in closed form (Ginibre 1965).

If $|\alpha| < 1$ then for large N have uniform distr of EV in a ellipse (Girko 1985) and the Ginibre EV correlations. If $\alpha \to 0$ the have weakly non-Hermitian matrices, crossover from Wigner-Dyson to Ginibre EV correlations (Fyodorov, Kh, Sommers 1997).

Beyond Gaussian distribution:

- G with i.i.d. entries: Circular Law (Girko 1984, Bai 1997, also Götze & Tikhomirov 2010, Tao & Vu 2008, 2010), if allow for correlated pairs symmetric about the main diagonal, $\langle G_{jk} G_{kj} \rangle = 1 - \alpha^2$, $|\alpha| < 1$, then have Elliptic Law (Girko 1985, Naumov 2012)). EV corr fncs are not known in this case.
- weak non-Hermiticity $H_1 \in$ GUE, H_2 is finite rank, fixed, EV corr fncs known in closed form (Fyodorov & Kh 1999).

Real Matrices



Real Matrices

Have the Elliptic Law of distribution of EVs for matrices with pairwise (G_{jk}, G_{kj}) correlations, and the Circular Law for i.i.d.

For Gaussian matrices finer details of EV distribution are available via EV jpdf (Lehmann & Sommers 1991, and Edelman 1993).

The expected no of real EV is propto $\sqrt{N}$ in the limit of large matrix dim N and real EV have uniform distribution (Edelman, Kostlan, & Shub 1994, Forrester & Nagao 2007)

Away from the real line have Ginibre correlations (Akemann & Kanzieper 2007, Forrester & Nagao 2007). New EV correlations on the real line (Forrester & Nagao 2007) and near the real line (Borodin & Sinclair 2009). Alternative derivation by Sommers 2007 and Sommers & Weiczorek 2008.

Weakly non-Hermitian limit $G = S + \alpha A$, with $\alpha \to 0$ by Efetov 1997 and Forrester & Nagao 2009.

Polar decomposition picture: EV density

Consider random matrices in the form $G = RU$, with R, U independent, $R \geq 0$ and U Haar unitary (orthogonal), equivalently $V\Lambda U$ (SVD)

Introduced by Fyodorov & Sommers 2003, with finite rank R, in the context of resonances in open chaotic sys.

Another example, the Feinberg-Zee ensemble $p_{FZ} \propto \exp[-N \operatorname{Tr} \phi(G^\dagger G)]$, with ϕ polynomial. Can be recast into RU by SVD, $G = V\Lambda U$.

For finite-N the mean EV density of RU is known in terms EV of R (Wei & Fyodorov 2008), the large-N limit performed by Bogomolny 2010.

Single Ring Theorem for $G = V\Lambda U$ by Guionnet, Krishnapur & Zeitouni 2011, proving Feinberg-Zee 1997 – an analogue of the Circular Law.

Finer details of EV distribution, e.g. corr fnc or distribution of real EVs for real matrices are difficult (if possible) for general R. Known in several exactly solvable cases beyond Gaussian G: spherical ensemble (Forrester & Krishnapur 2008, Forrester & Mays 2010) truncations of Haar unitaries/orthogonals (Życzkowski & Sommers 2000, Kh, Sommers & Życzkowski 2010)

Quadratisation of rectangular matrices

Consider X with M rows and N columns, $M > N$, matrices are 'standing'.

Y and Z are the upper $N \times N$ block and lower $(M-N) \times N$ of X. Want a unitary transformation $W \in U(M)$ s.t.

$$W^\dagger X = W^\dagger \begin{bmatrix} Y \\ Z \end{bmatrix} = \begin{bmatrix} G \\ 0 \end{bmatrix} .$$

W exists, def. up to right multiplication by $\mathrm{diag}[U, V]$, $U \in U(N), U(M-N)$

Thus $W \in U(M)/U(N) \times U(M-N)$ ...

Parameter count:

- X has $2MN$ real parameters.
- W has $M^2 - N^2 - (M-N)^2 = 2MN - N^2$ real parameters
- G has N^2 real parameters.

Call G quadratization of X

Quadratisation of rectangular matrices

Can parametrise cosets by $N \times (M-N)$ matrices C:

$$\tilde{W} = \begin{bmatrix} (1_N - CC^\dagger)^{1/2} & C \\ -C^\dagger & (1_{M-N} - C^\dagger C)^{1/2} \end{bmatrix},$$

Prop. 1 Let $M > N$. For any $X \in Mat(M,N)$ of full rank there exist unique $\tilde{W}$ as above and $G \in Mat(N,N)$ such that

$$\tilde{W}^\dagger X = \tilde{W}^\dagger \begin{bmatrix} Y \\ Z \end{bmatrix} = \begin{bmatrix} G \\ 0 \end{bmatrix}.$$

The square matrix G is given by $G = \left(1_N + \frac{1}{Y^\dagger} Z^\dagger Z \frac{1}{Y}\right)^{1/2} Y$.

Prop. 2 If $X \in Mat(M,N)$ is Gaussian with density $p(X) \propto e^{-\frac{\beta}{2} \operatorname{Tr} X^\dagger X}$ ($\beta = 1$ or $\beta = 2$ depending on whether X is real or complex) then it quadratisation $G \in Mat(N,N)$ has density

$$p_{\mathrm{IndG}}(G) \propto \left(\det G^\dagger G\right)^{\frac{\beta}{2}(M-N)} \exp\left(-\frac{\beta}{2} \operatorname{Tr} G^\dagger G\right)$$

Proof of Prop 2

Here is one based on SVD (works for non-Gaussian weights as well):

Ignoring a set of zero prob., $X = Q\,\Sigma^{1/2} P^\dagger$, where $\Sigma^{1/2}$ is diag mat of ordered SVs of X, and $Q \in U(M)/U(M-N)$ and $P \in U(N)/U(1)^N$. By computation, $d\nu(X) = d\mu(Q) d\mu(P) d\sigma(\Sigma)$, where $d\mu$ is Haar and

$$d\sigma(\Sigma) \propto (\det \Sigma)^{\frac{\beta}{2}(M-N+1-\frac{2}{\beta})}\, e^{-\frac{\beta}{2}\operatorname{Tr}\Sigma} \prod_{j<k} |s_k - s_j|^\beta \prod_{j=1}^{N} ds_j\ .$$

Introduce independent Haar unitary $U \in U(N)$ and rewrite SVD as

$$X = QUU^\dagger \Sigma^{1/2} P^\dagger = QUG; \quad G = U^\dagger \Sigma^{1/2} P^\dagger.$$

The pdf of G follows by rolling the argument back from $U^\dagger \Sigma^{1/2} P^\dagger$ to G.

Thus, with $\tilde{Q} := QU \in U(M)/U(M-N)$ and $W \in U(M)/U(M-N) \times U(N)$,

$$X = \tilde{Q} G = W \begin{bmatrix} G \\ 0 \end{bmatrix},$$

as required. □

Polar decomposition revisited

Relation $G = \left(1_N + \frac{1}{Y^\dagger} Z^\dagger Z \frac{1}{Y}\right)^{1/2} Y$ provides a recipe for sampling the distribution

$$p_{\mathrm{IndG}}(G) \propto \left(\det G^\dagger G\right)^{\frac{\beta}{2}(M-N)} \exp\left(-\frac{\beta}{2}\operatorname{Tr} G^\dagger G\right)$$

Interestingly, by rearranging $X = QUU^\dagger \Sigma^{1/2} P^\dagger = QUG$ one obtains another recipe (G is just Haar unitary times sq. root of Wishart).

Prop 3 Suppose that U is $N \times N$ Haar unitary (real orthogonal for $\beta = 1$) and X is $M \times N$ Gaussian, independent of U. Then the $N \times N$ matrix

$$G = U(X^\dagger X)^{1/2}$$

has the distribution $p_{\mathrm{IndG}}(G)$.

Proof. Since $(X^\dagger X)^{1/2} = P\Sigma^{1/2} P^\dagger$, then

$$G = U^\dagger Q^\dagger X = U^\dagger P^\dagger (X^\dagger X)^{1/2} = \tilde{U}^\dagger (X^\dagger X)^{1/2}.$$

$\tilde{U}^\dagger$ is Haar unitary and independent of X. $\square$

A few comments on quadratisation of random matrices

Quadratisation is well defined: as density of quadratised matrix doesn't depend on which rows W nullifies. Thus, can introduce *quasi spectrum* as the spectrum of the quadratised matrix.

Distributions other than Gaussian: Consider $X \in Mat(M,N)$ with density $p_{\rm FZ}(X) \propto \exp[-\operatorname{Tr}\phi(X^\dagger X)]$. On applying the procedure of quadratisation, one obtains the induced Feinberg-Zee ensemble

$$p_{\rm IndFZ}(G) \propto (\det G^\dagger G)^{\frac{\beta}{2}(M-N)} \exp[-\operatorname{Tr}\phi(G^\dagger G)] \, .$$

'Eigenvalue' map: Embed $Mat(M,N)$ into $Mat(M,M)$ by augmenting X with $M-N$ zero column-vectors and write the quadratisation rule in terms of square matrices albeit with zero blocks:

$$W^\dagger \begin{bmatrix} Y & 0 \\ Z & 0 \end{bmatrix} = \begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix}; \quad or \quad W^\dagger \tilde{X} = \tilde{G}$$

EV map: the zero EV of $\tilde{X}$ stays put, and its multiplicity is conserved, and, otherwise, the EV of Y are mapped onto those of G.

Quadratising complex Gaussian matrices: EV density

Complex matrices are straightforward, can apply the method of OP.

The EV-corr fnc are $R_n^{(N)}(\lambda_1, \ldots, \lambda_n) = \det(K_N(\lambda_k, \lambda_l))$,

$$K_N(\lambda_k, \lambda_l) = \frac{1}{\pi} e^{-\frac{1}{2}|\lambda_k|^2 - \frac{1}{2}|\lambda_l|^2} \sum_{j=M-N}^{M-1} \frac{(\lambda_k \bar{\lambda}_l)^j}{j!} .$$

For M, N large and $M - N = \alpha N$ the EV density $\rho_N(\lambda) = K_N(\lambda, \lambda)$ is uniform in the ring $r_{in} < |\lambda| < r_{out}$, $\quad r_{in} = \sqrt{M-N}, r_{out} = \sqrt{M}$, i.e.

$$\lim_{N\to\infty} \rho_N(\sqrt{N} z) = \frac{1}{\pi} \text{ for all } |z| \in (\sqrt{\alpha}, \sqrt{1+\alpha}).$$

Close to the circular edges of EV support,

$$\lim_{N\to\infty} \rho_N((r_{out} + \xi)e^{i\phi}) = \lim_{N\to\infty} \rho_N((r_{in} - \xi)e^{i\phi}) = \frac{1}{2\pi} \operatorname{erfc}(\sqrt{2}\xi) ,$$

where $\operatorname{erfc}(x) := \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$, hence the EV density falls from $1/\pi$ to zero very fast, at a Gaussian rate.

Quadratising complex Gaussian matrices: EV correlations

Not surprisingly in the regime of strong non-rectangularity ($M, N \to \infty$ and $M - N = \alpha N$) one recovers the Ginibre correlations both in the bulk and at the circular edges of the eigenvalue distribution.

In the bulk: Set $\lambda_k = \sqrt{N}u + z_k$. Then for any $|u| \in (\sqrt{\alpha}, \sqrt{\alpha + 1})$:

$$\lim_{N\to\infty} R_n^{(N)}(\lambda_1, \ldots, \lambda_n) = \det \left[\frac{1}{\pi} \exp\left(-\frac{|z_j|^2}{2} - \frac{|z_k|^2}{2} + z_j \bar{z}_k \right) \right]_{j,k=1}^{n}$$

In particular, $R_2(\lambda_1, \lambda_2) = 1 - \exp(-|z_1 - z_2|^2)$.

At the circular edges: Let $|u| = 1$ and set $\lambda_k = \sqrt{M}u + z_k$. Then:

$$\lim_{N\to\infty} R_n^{(N)}(\lambda_1, \ldots, \lambda_n) = \det \left[\frac{1}{2\pi} e^{-\frac{|z_j|^2}{2} - \frac{|z_k|^2}{2} + z_j \bar{z}_k} \operatorname{erfc}\left(\frac{z_j \bar{u} + \bar{z}_k u}{\sqrt{2}} \right) \right].$$

The same limiting expression is found around the inner edge $\sqrt{M - N}$ of the eigenvalue density by setting $\lambda_k = \sqrt{M - N}u - z_k$.

Almost square matrices: emergence of the hole in the spectrum

Explore a different regime when the rectangularity index $L = M - N \ll N$. This corresponds to the quadratisation of almost square matrices.

At the origin the EV density vanishes algebraically, $\rho_N(\lambda) \sim \frac{1}{\pi}\frac{|\lambda|^{2L}}{L!}$ as $\lambda \to 0$, uniformly in N. Away from the origin, the density reaches its asymptotic value $1/\pi$ very quickly. This plateau extends to a full circle of radius $\sqrt{N}$.

Also, $\lim_{N\to\infty} R_n^{(N)}(\lambda_1, \dots, \lambda_n) = \det\left(K_{origin}(\lambda_j, \lambda_k)\right)_{j,k=1}^{n}$, with

$$K_{origin}(\lambda_j, \lambda_k) = \frac{1}{\pi}\frac{1}{\Gamma(L)}\int_0^{\lambda_j\bar{\lambda}_k} t^{L-1}e^{-t}dt \,.$$

The hole prob $A(s)$ is the prob that no EV lies inside the disk $|\lambda| < s$:

$$A(s) = \prod_{j=1}^{N} \frac{\Gamma(j+L, s^2)}{\Gamma(j+L)} \text{ with } \Gamma(a,x) := \int_x^{\infty} e^{-t}t^{a-1}dt.$$

For almost square matrices $A(s) = 1 - \frac{s^{2(L+1)}}{(L+1)!} + O(\frac{s^{2(L+2)}}{(L+2)!})$.

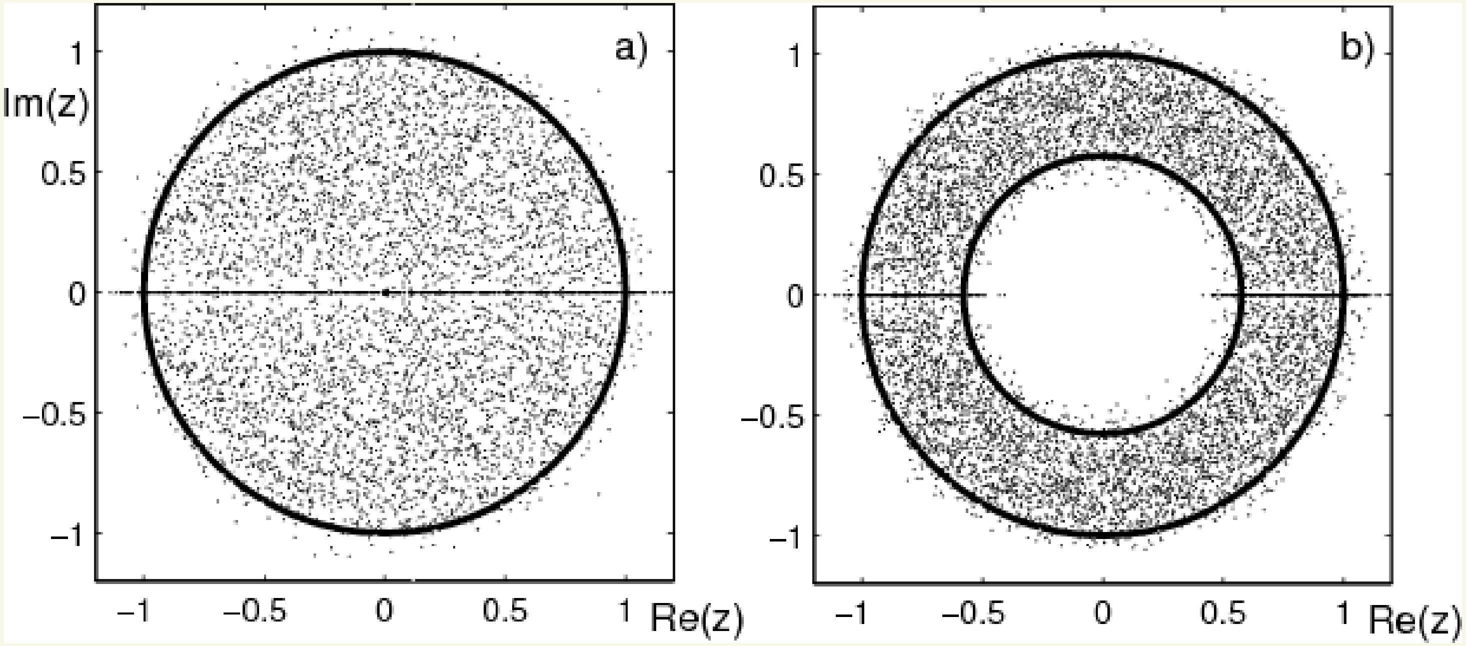
Quadratisation of real matrices

Consider real Gaussian matrices $X \in Mat(M, N)$, $X \mapsto G \in Mat(N, N)$

Gaussian measure on X induces a measure on G with density

$$p_{\mathrm{IndG}}(G) \propto \left(\det G^T G\right)^{\frac{1}{2}(M-N)} \exp\left(-\frac{1}{2}\,\mathrm{Tr}\, G^T G\right)$$

Spectra of G from the induced distribution for $N = 128$ and a) $M = N = 128$ (no hole) and b) $M = N + 32$. Each picture consists of 128 independent realisations and the spectra are rescaled by a factor $1/\sqrt{M}$



Real matrices

Building on the recent advances for the real Ginibre ensemble, the EV corr fnc are given in Pfaffian form, with the kernel expressed in terms of skew-OPs $\{q_j\}_{j=1,\dots}$, monic polynomials defined by

$$(q_{2j}, q_{2k}) = (q_{2j+1}, q_{2k+1}) = 0$$

$$(q_{2j}, q_{2k+1}) = -(q_{2j+1}, q_{2k}) = r_j \delta_{jk}$$

with the skew-symmetric inner product $(-,-)$:

$$(f, g) = (f, g)_{\mathbb{R}} + (f, g)_{\mathbb{C}}$$

$$(f, g)_{\mathbb{R}} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+y^2)} |xy|^L \mathrm{sgn}(y - x) f(x) g(y) dx dy$$

$$(f, g)_{\mathbb{C}} = 2i \int_{\mathbb{R}_+^2} e^{y^2-x^2} \operatorname{erfc}(\sqrt{2}y)(x^2 + y^2)^L \times \\ [f(x + iy)g(x - iy) - g(x + iy)f(x - iy)]\, dx dy.$$

Finding skew-OPs is a formidable task. However in the RM set up:

$$2 \sum_{j=0}^{\frac{N}{2}-1} \frac{1}{r_j} [q_{2j}(w)q_{2j+1}(w') - q_{2j+1}(w)q_{2j}(w')] = \\ \frac{1}{r_N} (w - w') \langle \det (G - w 1_{N-2}) \det (G - w' 1_{N-2}) \rangle_{N-2}$$

2nd moment of the characteristic polynomial

$\langle \det(G - w 1_{N-2}) \det(G - w' 1_{N-2}) \rangle_{N-2}$ can be evaluated in closed from.

Expand the product of determinants in Schur pols, decompose $G = O\sqrt{G^T G}$ and then integrate over the orthogonal group. The remaining integral is over the radial part of G:

$$\langle \det(G - wI) \det(G - w'I) \rangle_{N-2} = \sum_{j=0}^{N-2} \frac{\langle \epsilon_j(GG^T) \rangle_{N-2}}{\binom{n}{j}} (ww')^{N-2-j}$$

Here $\epsilon_j(GG^T)$ denotes the $j-$th elementary symm pol in EVs of GG^T and its average is given by the Selberg-Aomoto integral, leading to

$$2\sum_{j=0}^{\frac{N}{2}-1} \frac{1}{r_j} \left[q_{2j}(w) q_{2j+1}(w') - q_{2j+1}(w) q_{2j}(w') \right] =$$

$$\frac{1}{r_N}(w - w') \frac{(L+N-2)!}{\sqrt{2\pi}} \sum_{j=0}^{N-2} \frac{(ww')^j}{(L+j)!} \; ,$$

from which $q_0(w) = 1$, $q_1(w) = w$ and

$$q_{2j}(w) = w^{2j}, q_{2j+1}(w) = w^{2j+1} - (2j + L) w^{2j-1},$$

$$(q_{2j}, q_{2j+1}) = 2\sqrt{2\pi}\Gamma(L + 2j + 1) \; .$$

Large-N limit for real matrices

Density of complex EVs:

$$\rho_N^C(x+iy) = \sqrt{\frac{2}{\pi}}\, y\, \mathrm{erfc}(\sqrt{2}y) e^{y^2-x^2} \sum_{j=0}^{N-2} \frac{(x^2+y^2)^{j+L}}{(j+L)!}$$

vanishes on the real line. Same distribution in the bulk and at the edges as for complex matrices, though convergence is not uniform near $\mathbb{R}$.

Expected number of real EVs:

In the leading order the average no of real EV is $\sqrt{\frac{2}{\pi}}(\sqrt{N+L} - \sqrt{L})$

(Mean) Density of real EVs:

$$\lim_{N\to\infty} \rho_N^{(R)}(\sqrt{N}x) = \frac{1}{\sqrt{2\pi}} \text{ if } |x| \in (\sqrt{\alpha}, \sqrt{\alpha+1})$$

$$0 \quad \text{if } |x| \notin [\sqrt{\alpha}, \sqrt{\alpha+1}]$$

Tails of real density EVs:

$$\lim_{N\to\infty} \rho_N^R(\sqrt{L}-\xi) = \lim_{N\to\infty} \rho_N^R(\sqrt{L+N}+\xi) = \frac{1}{\sqrt{2\pi}}\left[\mathrm{erfc}(\sqrt{2}\xi) + \frac{1}{2\sqrt{2}} e^{-\xi^2}\,\mathrm{erfc}(-\xi)\right].$$

Same as in the real Ginibre, also the real Ginibre EV correlations in the bulk and at the edges!

Conclusions

A quadratisation procedure for rectangular matrices is introduced. When applied to Gaussian matrices, it leads to modified Ginibre ensemble with a pre-exponential term (the radial part squared is Wishart/Laguerre). This ensemble can be solved exactly for EV densities and corr fnc:

- for complex matrices the EV density is unifom in a ring, almost square matrices describe cross-over from disk to ring;
- for real matrices EV the density of complex EV is uniform in a ring, and the density of real EV is uniform on two intervals symmetric about the origin. The expected no of real EVs scales with $\sqrt{N}$;
- the EV corr functions are same as in the pure Ginibre ensembles

Work in progress (Fischmann's PHD thesis): applying quadratisation procedure to non-Gaussian distribution, e.g. rectangular truncations of Haar unitaries or orthogonals, EV densities are no longer uniform however recover the Ginibre corr fnc after unfolding the spectrum.

THANK YOU!