Vector Equilibrium Problems in Random Matrix Theory

Les Houches, 2012.

Adrien Hardy

Katholieke Universiteit Leuven / Institut de Mathématiques de Toulouse

Vector equilibrium problems ?

Vector equilibrium problem : Find the minimizer of

$$\sum_{1\leq i,j\leq d} c_{ij} \iint \log \frac{1}{|x-y|} d\mu_i(x) d\mu_j(y) + \sum_{i=1}^{d} \int V_i(x) d\mu_i(x)$$

when the vector of measures $(\mu_1, \ldots, \mu_d)$ on $\mathbb{C}$ lies in a prescribed set.

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when the vector of measures $(\mu_1, \ldots, \mu_d)$ on $\mathbb{C}$ lies in a prescribed set.

- $C = [c_{ij}]$: interaction matrix
- $V_1, \ldots, V_d$: potentials
  (with regularity and growth assumptions)

Plan

1. Motivations
2. Vector equilibrium problems
3. Large deviations

1) Motivations

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Unitary invariant Hermitian matrix models :

Consider the probability distribution on $\mathcal{H}_N(\mathbb{C})$

$$\frac{1}{Z_N} e^{-N\mathrm{Tr}V(M)} dM$$

with $V : \mathbb{R} \to \mathbb{R}$ continuous and satisfying

$$\lim_{|x|\to\infty} \big(V(x) - \log(1+x^2) \big) = +\infty.$$

1) Motivations

Eigenvalue density distribution :

$$\frac{1}{Z_N}\prod_{1\leq i<j\leq N}|x_j-x_i|^2\prod_{i=1}^{N}e^{-NV(x_i)}$$
$$=\frac{1}{Z_N}\exp\left\{-N^2\left(\frac{1}{N^2}\sum_{1\leq i\neq j\leq N}\log\frac{1}{|x_i-x_j|}+\frac{1}{N}\sum_{i=1}^{N}V(x_i)\right)\right\}$$

1) Motivations

$\Rightarrow$ Minimize

$$\iint \log \frac{1}{|x-y|} d\mu(x) d\mu(y) + \int V(x) d\mu(x)$$

for $\mu \in \mathcal{M}_1(\mathbb{R})$.

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for $\mu \in \mathcal{M}_1(\mathbb{R})$.

Theorem

1. *There exists a unique minimizer* μ^*.
2. $\mathrm{Supp}(\mu^*)$ *is compact.*
3. *As* $N \to \infty$, *the spectral measures converge to* μ^* *:*

$$\mu^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i} \longrightarrow \mu^*, \qquad a.s\ ,\ weakly.$$

Non-centered Wishart matrix :

Consider the probability distribution on $\mathcal{M}_{N+\alpha,N}(\mathbb{C})$ $(\alpha \geq 0)$

$$\frac{1}{Z_N} e^{-N(M-A)^*(M-A)} dM,$$

where, for $a > 0$,

$$A = \begin{bmatrix} \sqrt{a} & & \\ & \ddots & \\ & & \sqrt{a} \\ & \mathbf{0}_\alpha & \end{bmatrix} \in \mathcal{M}_{M,N}(\mathbb{C}).$$

1) Motivations

Singular value density distribution (the $x_1, \ldots, x_N \in \mathbb{R}_+$) :

$$\frac{1}{Z_N} \int_{\mathbb{R}_-^{N/2}} \frac{\prod_{i<j} |x_j - x_i|^2 \prod_{i<j} |y_j - y_i|^2}{\prod_{i,j} |x_i - y_j|} \prod_{i=1}^{N} e^{-NV_N(x_i)} \prod_{i=1}^{N/2} |y_i| d\sigma_N(y_i)$$

with $y_1, \ldots, y_{N/2} \in$ (a N-dependent discrete subset of) $\mathbb{R}_- \setminus \{0\}$.

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$\Rightarrow$ Minimize

$$\iint \log \frac{1}{|x-y|} d\mu_1(x) d\mu_1(y) - \iint \log \frac{1}{|x-y|} d\mu_1(x) d\mu_2(y) + \iint \log \frac{1}{|x-y|} d\mu_2(x) d\mu_2(y) + \int \left(x - 2\sqrt{ax}\right) d\mu_1(x)$$

with $\mu_1 \in \mathcal{M}_1(\mathbb{R}_+)$ and $\mu_2 \in \mathcal{M}_{1/2}(\mathbb{R}_-)$, $d\mu_2(x) \leq \frac{\sqrt{a}}{\pi} |x|^{-1/2} dx$

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Vector equilibrium problem : For $d \geq 1$, find the minimizer of the functional

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- $C = [c_{ij}]$: $d \times d$ real symmetric definite positive matrix
- $\Delta_1, \ldots, \Delta_d$: closed subsets of $\mathbb{C}$ with positive capacity.
- $m_1, \ldots, m_d > 0$.
- $V_1, \ldots, V_d$ with $V_i : \Delta_i \to \mathbb{R} \cup \{+\infty\}$, lower semi-continuous, and $\{x : V_i(x) < +\infty\}$ has positive capacity.

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Question : Existence and uniqueness of the minimizer ?

- Δ_i compacts, $\Delta_i \cap \Delta_j = \emptyset$ when $c_{ij} < 0$: **OK** (Gonchar, Rakhmanov)

- Δ_i compacts, or unbounded with strong growth assumptions

$$\lim_{|x|\to\infty} \frac{V_i(x)}{\log(1+|x|^2)} = +\infty \quad : \textbf{OK}$$

(Beckermann, Kalyagin, Matos, Wielonsky)

NB : Extra conditions for the measures

$$\iint \log \frac{1}{|x-y|} d\mu(x) d\mu(y) < +\infty, \quad \int \log(1+|x|) d\mu(x) < +\infty.$$

2) Vector equilibrium problems

- Δ_i compact, or unbounded with weak admissiblity condition

$$\liminf_{|x|\to\infty}\left(V_i(x)-\Big(\sum_{j=1}^{d}c_{ij}m_j\Big)\log(1+|x|^2)\right)>-\infty \qquad : \textbf{OK}$$

(H. , Kuijlaars)

NB : No extra conditions for the measures,

but *extension of the definition* for the functional.

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Key idea :

- $T : \mathbb{C} \to \mathcal{S}^2$, inverse stereographic projection.

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- $T_*\mu$: push-forward of a measure μ on $\mathbb{C}$ by T

$$\int_{\mathcal{S}^2} f(x) dT_*\mu(x) = \int_{\mathbb{C}} f \circ T(x) d\mu(x)$$

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Fact :

$$I\big(T_*\mu, T_*\nu\big) = I(\mu,\nu) + \frac{1}{2}\nu(\mathbb{C}) \int \log(1+|x|^2) d\mu(x) + \frac{1}{2}\mu(\mathbb{C}) \int \log(1+|x|^2) d\nu(x)$$

Consequence : If the μ_i's satisfy

$$I(\mu_i) < +\infty, \quad \int \log(1+|x|^2) d\mu(x) < +\infty$$

then

$$\sum_{1\leq i,j\leq d} c_{ij} I(\mu_i,\mu_j) + \sum_{i=1}^{d} \int_{\mathbb{C}} V_i(x) d\mu_i(x)$$

$$= \sum_{1\leq i,j\leq d} c_{ij} I\big(T_*\mu_i, T_*\mu_j\big) + \sum_{i=1}^{d} \int_{\mathcal{S}^2} \boldsymbol{\mathcal{V}}_i(x) dT_*\mu_i(x)$$

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where

$$\boldsymbol{\mathcal{V}}_i\big(T(x)\big) = V_i(x) - \Big(\sum_{j=1}^{d} c_{ij} m_j\Big) \log(1+|x|^2), \qquad x \in \Delta_i$$

$$\boldsymbol{\mathcal{V}}_i(\infty) = \liminf_{|x|\to\infty,\ x\in\Delta_i} \boldsymbol{\mathcal{V}}\big(T(x)\big) > -\infty \qquad \text{(by assumption !)}$$

2) Vector equilibrium problems

Definition (Extension of the energy functional)

For $(\mu_1, \dots, \mu_d) \in \mathcal{M}_{m_1}(\Delta_1) \times \cdots \times \mathcal{M}_{m_d}(\Delta_d)$,

$$J(\mu_1, \dots, \mu_d) :=$$

$$\sum_{1 \le i,j \le d} c_{ij} I\big(T_*\mu_i, T_*\mu_j\big) + \sum_{i=1}^{d} \int_{\mathcal{S}^2} \mathcal{V}_i(x) dT_*\mu_i(x)$$

if $I(T_*\mu_i) < +\infty$ for $i = 1, \dots, d$, and $J(\mu_1, \dots, \mu_d) = +\infty$ otherwise.

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if $I(T_*\mu_i) < +\infty$ for $i = 1, \dots, d$, and $J(\mu_1, \dots, \mu_d) = +\infty$ otherwise.

Theorem (H. , Kuijlaars)

1. *For all $\alpha \in \mathbb{R}$, the level set*

$$\Big\{(\mu_1, \dots, \mu_d) : \; J(\mu_1, \dots, \mu_d) \leq \alpha\Big\}$$

   is compact.

2. *J is strictly convex where it is finite.*

Simple example :

- Minimize

$$\iint \log \frac{1}{|x-y|} d\mu(x) d\mu(y) + \int \log(1+x^2) d\mu(x), \qquad \mu \in \mathcal{M}_1(\mathbb{R})$$

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- Then $\boldsymbol{\mathcal{V}} = 0$, which leads to minimize

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- Thus,

$$\mu^* = T_*^{-1}\big(\text{Uniform}(\mathcal{S}^1)\big) = \frac{dx}{\pi(1+x^2)}$$

3) Large deviations

Definition : The sequence of (random) measures $(\mu^N)_N$,

$$\mu^N = \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i},$$

satisfies a large deviation principle at speed N^2 and rate function Φ if for any Borel set $\mathcal{B}$ of probability measures

$$\mathbb{P}\Big(\mu^N \in \mathcal{B}\Big) \simeq \exp\Big(-N^2 \inf_{\mu\in\mathcal{B}} \Phi\Big)$$

as $N \to \infty$.

Precise statement :

- Φ is *non-negative* and *lower semi-continuous*.
- For any Borel set $\mathcal{B}$ of probability measures

$$\limsup_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}\left(\mu^N \in \mathcal{B}\right) \leq - \inf_{\mu \in clo(\mathcal{B})} \Phi$$

and

$$\liminf_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}\left(\mu^N \in \mathcal{B}\right) \geq - \inf_{\mu \in int(\mathcal{B})} \Phi.$$

3) Large deviations

Unitary invariant matrix models :
Consider the probability distribution on $\mathcal{H}_N(\mathbb{C})$

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with $V : \mathbb{R} \to \mathbb{R}$ continuous and satisfying

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Consider

$$I_V(\mu) \text{ " } = \text{ " } \iint \log \frac{1}{|x-y|} d\mu(x) d\mu(y) + \int V(x) d\mu(x)$$

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Theorem (Ben Arous, Guionnet, ...)

The spectral measures $(\mu^N)_N$ satisfy a large deviation principle on $\mathcal{M}_1(\mathbb{R})$ at speed N^2 and good rate function

$$\Phi(\mu) = I_V(\mu) - I_V(\mu^*)$$

- With

$$\mathcal{V}\big(T(x)\big) = V(x) - \log(1+x^2), \quad x \in \mathbb{R},$$
$$\mathcal{V}(\infty) = \liminf_{|x|\to\infty} \big(V(x) - \log(1+x^2)\big),$$

-

$$I_V(\mu) = \iint \log\frac{1}{|x-y|} dT_*\mu(x)dT_*\mu(y) + \int \mathcal{V}(x)dT_*\mu(x)$$

$$= \iint \left[\log\frac{1}{|x-y|} + \frac{\mathcal{V}(x)+\mathcal{V}(y)}{2}\right] dT_*\mu(x)dT_*\mu(y)$$

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Under the weaker growth assumption

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$$\Phi(\mu) = I_V(\mu) - I_V(\mu^*)$$

Corollary

μ^N converges a.s to μ^* in the weak topology of $\mathcal{M}_1(\mathbb{R})$

Non-centered Wishart matrix :

Consider the probability distribution on $\mathcal{M}_{N+\alpha,N}(\mathbb{C})$ $(\alpha \geq 0)$

$$\frac{1}{Z_N} e^{-N(M-A)^*(M-A)} dM,$$

where, for $a > 0$,

$$A = \begin{bmatrix} \sqrt{a} & & \\ & \ddots & \\ & & \sqrt{a} \\ & \mathbf{0}_\alpha & \end{bmatrix}.$$

3) Large deviations

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$$J(\mu_1, \mu_2) \text{ "} = \text{" } \iint \log \frac{1}{|x-y|} d\mu_1(x) d\mu_1(y) - \iint \log \frac{1}{|x-y|} d\mu_1(x) d\mu_2(y)$$
$$+ \iint \log \frac{1}{|x-y|} d\mu_2(x) d\mu_2(y) + \int \left(x - 2\sqrt{ax}\right) d\mu_1(x)$$

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Consider

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$$+\iint \log\frac{1}{|x-y|}d\mu_2(x)d\mu_2(y)+\int\left(x-2\sqrt{ax}\right)d\mu_1(x)$$

Then,

Theorem (H. , Kuijlaars, in preparation)

The sequence $(\mu^N)_N$ satisfies a large deviation principle on $\mathcal{M}_1(\mathbb{R}_+)$ at speed N^2 and good rate function

$$\Phi(\mu_1)=\inf_{\mu_2\in\mathcal{M}_{1/2}(\mathbb{R}_-),\, d\mu_2\leq\frac{\sqrt{a}}{\pi}|x|^{-1/2}dx} J(\mu_1,\mu_2)-J(\mu_1^*,\mu_2^*)$$

Corollary

μ^N converges a.s to μ_1^* in the weak topology of $\mathcal{M}_1(\mathbb{R}_+)$

Thank you for your attention !