An introduction to Random Matrices and the Deift-Zhou steepest descent approach to asymptotics of Orthogonal Polynomials

Marco Bertola, Dep. Mathematics and Statistics, Concordia University
Centre de recherches mathématiques (CRM), UdeM

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Outline

Random Matrices: definition and goals

The term is very general and indicates the study of particular *ensembles* of matrices endowed with a *probability measure*. Thus the matrix itself is a *random variable*.

The main objective typically is to study

- the statistical properties of the **spectra** (for square matrices ensembles) or **singular values** (for rectangular ensembles). Thus we need to develop an understanding of the *joint probability distribution functions (jpdf)* of the eigen/singular-values.
- the properties of said statistics when the size of the matrix ensemble tends to infinity (under suitable assumption on the probability measure).

RM RH AND OPs Asymptotics Pot. Th. DZ Eigenvalue statistics Connection to Orthogonal Polynomials

Let $\mathscr{M}$ be a space of matrices of given size:

Example 1.1

- Hermitean matrices ($M = \mathscr{M}^\dagger$) of size $n \times n$: $\mathscr{M} := \{M \in Mat(n,n;\mathbb{C}),\ M_{ij} = M_{ji}^\star\}$
- Orthogonal matrices ($M = M^T$) of size $n \times n$: $\mathscr{M} := \{M \in Mat(n,n;\mathbb{R}),\ M_{ij} = M_{ji}\}$;
- Symplectic matrices $M^T J = J M^T$, $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \otimes \mathbf{1}_n$ of size $2n \times 2n$.
- Rectangular matrices $n \times K$
- $\mathscr{M} = Mat(n \times n; \mathbb{C})$, etc.

The first three examples are called **Unitary**, **Orthogonal** and **Symplectic** ensembles (referring rather to the compact group that leaves the measure invariant).
Each of these spaces is a *vector space* and thus carries a natural Lebesgue measure (invariant by translations) which we shall denote by $\mathrm{d}M$. Since we shall focus on the case of Hermitean matrices (**Unitary ensemble**) we see that in this case

$$M_{ab} = X_{ab} + iY_{ab}\ ,\quad X_{ab} = X_{ba}\ ,\quad Y_{ab} = -Y_{ba} \tag{1}$$

$$\dim \mathscr{M} = \frac{n}{2}(n+1) + \frac{n}{2}(n-1) = n^2 \tag{2}$$

$$\mathrm{d}M := \prod_{a=1}^{n} \mathrm{d}X_{aa} \prod_{1 \le a < b \le n} \mathrm{d}X_{ab}\, \mathrm{d}Y_{ab} \tag{3}$$

Lemma 1.1

In each (square) case the Lebesgue measure is **invariant under conjugation**: $\mathrm{d}M = \mathrm{d}(CMC^{-1})$.

Exercise 1.1

Prove the lemma. Hint: the map is linear and so the Jacobian is certainly constant: show that it is unity.

We recall

Theorem 1.1

Any Hermitean matrix can be diagonalized by a Unitary matrix $U \in \mathscr{U}(n)$ and its eigenvalues are real

$$\mathscr{U}(n) := \{U \in GL_n(\mathbb{C}) \ , \quad U^\dagger U = UU^\dagger = \mathbf{1}_n\} \tag{4}$$

$$M = U^\dagger XU \ , \quad X = \mathrm{diag}(x_1, x_2, \ldots, x_n) \ , \quad x_j \in \mathbb{R}. \tag{5}$$

Remark 1.1

The diagonalization is **not unique** even if X is semisimple (i.e. with *distinct eigenvalues*) because we can decide on an ordering of the eigenvalues. In general there are $n!$ distinct diagonalizations. The matrix U can be multiplied on the left by an arbitrary diagonal matrix $D = \mathrm{diag}(\mathrm{e}^{i\theta_1}, \ldots, \mathrm{e}^{i\theta_n})$.

We thus have a cover

$$\mathscr{U}(n) \times \mathbb{R}^n \to \mathscr{M} \tag{6}$$

which is generically many to one and it is branched along the locus of non-semisimple matrices. It is however clear that for any measure which is a.c. to the Lebesgue measure, namely $\mathrm{d}\mu(M) = f(M)\,\mathrm{d}M$ (with $f(M)$ some measurable nonnegative function) this locus has **zero measure**. Thus we shall only consider the cover

$$\mathscr{U}(n) \times \mathbb{R}^n_\Delta \to \mathscr{M}_{ss} \ , \tag{7}$$

$$\mathbb{R}^n_\Delta := \{\mathbb{R} \ni x_i \neq x_j, \ i \neq j\} \tag{8}$$

Theorem 1.2

Any compact group G has a **Haar measure** $\mathrm{d}U$ *which is invariant under left/right translations*

$$dU = d(Ug) = d(gU) \ , \quad \forall g \in G \tag{9}$$

We shall not need or use the detailed form of the Haar measure for $\mathscr{U}(n)$, except for the abovementioned property.

THEOREM 1.3

The Lebesgue measure on $\mathscr{M}_{ss}$ can be written as

$$\mathrm{d}M = \Delta(X)^2 \prod_{i=1}^{n} \mathrm{d}x_i \,\mathrm{d}U\ , \qquad \Delta(X) := \prod_{1\le i<j\le n} (x_j - x_i) = \det\left[x_a^{b-1}\right]_{1\le a,b\le n} \tag{10}$$

Skip Proof **Proof** This is an example of Weyl's integration formula [2] We shall give a sketch of proof that can be modified to the other cases along the same logic. From $U^{-1} = U^\dagger$ we have that (here the dot $\dot{}$ denotes any vector field, i.e. any derivative)

$$\dot U^\dagger = (\dot{U^{-1}}) = -U^{-1}\dot U U^{-1} \tag{11}$$

It follows that $\dot U U^{-1}$ is an arbitrary anti-Hermitean matrix

$$U^{-1}\dot U = -\left(U^{-1}\dot U\right)^\dagger \tag{12}$$

$$M = UXU^\dagger \quad \Rightarrow \quad \dot M = \dot U X U^\dagger + U\dot X U^\dagger + UX\dot U^\dagger \tag{13}$$

$$Ad_U\left(\dot M\right) = \dot X + U^{-1}\dot U X - XU^{-1}\dot U = \dot X + \left[U^{-1}\dot U, X\right] \tag{14}$$

Here $h := \dot M$ is an arbitrary Hermitean matrix (in the tangent space of $T_M\mathscr{M}$) and $u := U^{-1}\,\mathrm{d}U$ is an arbitrary anti-Hermitean matrix (in the tangent space $T_{\mathbf{1}}\mathscr{U}$) and $\xi = \dot X$ is an arbitrary diagonal matrix. Thus the Jacobian of the change of coordinates from the U, X to the M is to be read off

$$Ad_U(h) = \xi + \left[u, X\right] \tag{15}$$

It is clear that the conjugation Ad_U does not affect the determinant, so it suffices to compute the determinant of the linear map

$$\Phi_\star := \mathbb{R}^n \oplus u(n) \to \mathscr{H}_n\ , \quad \Phi(\xi, u) = \xi + [u, X] \tag{16}$$

We can diagonalize this linear map taking the diagonal elementary matrices E_{aa} and the elementary antihermitean matrices

$$f_{ab} = E_{ab} - E_{ba}\ , \quad g_{ab} := i(E_{ab} + E_{ba})\ , \quad a < b \tag{17}$$

for $T\mathscr{M}$ we use the basis

$$E_{aa}\ , \quad s_{ab} = E_{ab} + E_{ba} = -ig_{ab}\ , \quad r_{ab} := i(E_{ab} - E_{ba}) = if_{ab} \tag{18}$$

And we see

$$\Phi_\star E_{aa} = E_{aa} \tag{19}$$

$$\Phi_\star f_{ab} = (x_b - x_a)(-i g_{ab}) = (x_b - x_a) s_{ab} \tag{20}$$

$$\Phi_\star g_{ab} = (x_b - x_a) i f_{ab} = (x_b - x_a) r_{ab} \tag{21}$$

We thus have diagonalized (relative to the choice of bases) the map and the determinant is thus immediately computed as the product of eigenvalues

$$\det \Phi_\star = \prod_{1 \le a < b \le b} (x_a - x_b)^2 \quad \square \tag{22}$$

REMARK 1.2

A similar computation shows that in the other two cases

$$\text{Orthogonal} \qquad \mathrm{d}M = |\Delta(X)|\, \mathrm{d}X\, \mathrm{d}U \tag{23}$$

$$\text{Symplectic} \qquad \mathrm{d}M = \Delta(X)^4\, \mathrm{d}X\, \mathrm{d}U \tag{24}$$

where $\mathrm{d}U$ is the Haar measure in the respective compact group ($O(n)$ or $Sp(2n)$). Since the exponent of the Vandermonde determinant $\Delta(X)$ is $\beta = 1,2,4$ (Orthogonal, Unitary, Symplectic ensembles), they are also universally known as the $\beta = 1,2,4$ ensembles.

UNITARILY-INVARIANT MEASURES AND JPDF'S OF EIGENVALUES

One can consider measures of the form

$$\mathrm{d}\mu(M) = F(M)\, \mathrm{d}M \ , \tag{25}$$

with $F : \mathscr{M} \to \mathbb{R}_+$ some suitable ($L^1(\mathrm{d}M)$) function of total integral 1. This can be viewed as (i.e. it can be *pulled back* to) a probability measure on $U(n) \times \mathbb{R}^n$ as (we use the same symbol)

$$\mathrm{d}\mu(U, \vec{x}) := \frac{1}{n!(2\pi)^n} F(U^\dagger X U) \Delta(X)^2\, \mathrm{d}X\, \mathrm{d}U \tag{26}$$

$$X = \mathrm{diag}(x_1, \dots, x_n), \quad \mathrm{d}X := \prod_{a=1}^n \mathrm{d}x_a \tag{27}$$

If we are interested only on the eigenvalues one can study the *reduced measure* (indicated by the same symbol)

$$\mathrm{d}\mu(\vec{x}) = \Delta^2(X)\, \mathrm{d}X \overbrace{\int_{\mathscr{U}(n)} \frac{F(U^\dagger X U)}{n!(2\pi)^n}\, \mathrm{d}U}^{=:\mu(\vec{x})} \tag{28}$$

where *a fortiori* $\mu(\vec{x})$ is a symmetric function of the n arguments.

The connection to Orthogonal Polynomials (in the simplest incarnation) becomes possible only when μ is the product of a single function of the individual eigenvalues.

$$\mu(\vec{x}) \propto \prod_{a=1}^{n} \mu(x_a) \tag{29}$$

Writing $\mu(x) = \mathrm{e}^{-V(x)}$ ($V(x)$ is called the **potential**) these measures can be thought as the reduction to the eigenvalues of the measure

$$\mathrm{d}\mu(M) = \frac{1}{Z}\mathrm{e}^{-\mathrm{Tr}V(M)}\,\mathrm{d}M = \frac{1}{Z}\mathrm{e}^{-\sum_{a=1}^{n} V(x_a)}\,\mathrm{d}M = \tag{30}$$

We stipulate from now on that this is the choice we are presented with, that is that the reduced jpdf on the eigenvalues is

$$\mu(\vec{x}) = \frac{1}{\mathscr{Z}} \prod_{1\le a<b\le n} (x_a - x_b)^2 \prod_{a=1}^{n} \mathrm{e}^{-V(x_a)}\,\mathrm{d}x_a \tag{31}$$

with $\mathscr{Z}$ the appropriate normalization constant.

DYSON'S THEOREM

We start with

LEMMA 1.2

Given any functions $f_j(x), h_j(x)$, $j = 1, \ldots, n$ and measure $\mathrm{d}\nu(x)$ we have (provided all integrals make sense)

$$\int_{\mathbb{R}^n} \det[f_a(x_b)]_{a,b} \det[h_a(x_b)]_{a,b} \prod_{a=1}^{n} \mathrm{d}\nu(x_a) = n! \det G \tag{32}$$

$$G_{ab} = \int_{\mathbb{R}} f_a(\xi) h_b(\xi)\,\mathrm{d}\nu(\xi) \tag{33}$$

Skip Proof **Proof**

$$LHS = \int_{\mathbb{R}^n} \prod_{j=1}^{n} \mathrm{d}\nu(x_j) \sum_{\sigma,\rho\in S_n} (-1)^{\sigma}(-1)^{\rho} \prod_{a=1}^{n} f_{\sigma(a)}(x_a) \prod_{b=1}^{n} h_{\rho(b)}(x_b) = \tag{34}$$

$$= \sum_{\sigma,\rho\in S_n} (-1)^{\sigma}(-1)^{\rho} \prod_{a=1}^{n} \int_{\mathbb{R}} \mathrm{d}\nu(x_a) f_{\sigma(a)}(x_a) h_{\rho(a)}(x_a) = \sum_{\sigma,\rho\in S_n} (-1)^{\sigma}(-1)^{\rho} \prod_{a=1}^{n} G_{\sigma(a),\rho(a)} = \tag{35}$$

$$= \int_{\mathbb{R}^n} \prod_{j=1}^{n} \mathrm{d}x_j \sum_{\sigma,\rho\in S_n} (-1)^{\sigma\rho} \prod_{b=1}^{n} G_{\sigma\rho^{-1}(b),b} = n! \sum_{\varepsilon\in S_n} (-1)^{\varepsilon} \prod_{b=1}^{n} G_{\varepsilon(b),b} = n! \det G \tag{36}$$

We now start analyzing the JPDF's of eigenvalues

$$\mu(\vec{x})\,\mathrm{d}X = \frac{1}{\mathscr{Z}} \prod_{1\le a<b\le n} (x_a - x_b)^2 \prod_{a=1}^{n} \mathrm{e}^{-V(x_a)}\,\mathrm{d}x_a \tag{37}$$

The Lemma 1.2 applies to this integral with $f_j(x) = g_j(x) = x^{j-1}\mathrm{e}^{-\frac{1}{2}V(x)}$ and hence we obtain

Corollary 1.1

The (reduced) **partition function** is

$$\mathscr{Z} = n!\det \mathfrak{M}_{ab}\ ,\quad \mathfrak{M}_{ab} = \int_{\mathbb{R}} x^{a+b}\mathrm{e}^{-V(x)}\ ,\quad 0\le a,b\le n-1. \tag{38}$$

Note that $\mathfrak{M}$ is a (principal submatrix of the) **Hankel matrix** of the *moments* of the measure $\mathrm{e}^{-V(x)}\,\mathrm{d}x$.

Lemma 1.3

We have

$$\frac{1}{\mathscr{Z}} \prod_{1\le a<b\le n} (x_a - x_b)^2 \prod_{a=1}^{n} \mathrm{e}^{-V(x_a)} = \frac{1}{n!}\det\left[K(x_a,x_b)\right]_{1\le a,b\le n} \tag{39}$$

where

$$K(x,y) = \mathrm{e}^{-\frac{V(x)+V(y)}{2}} \sum_{j,k=0}^{n-1} x^j y^k [\mathfrak{M}]^{-1}_{jk}\ , \tag{40}$$

Skip Proof **Proof of Lemma 1.3** Since $\Delta(X) = \det\left[x_a^{b-1}\right]_{1\le a,b\le n}$ we shall denote by $W(X)$ the Vandermonde matrix

$$W(X) := \begin{bmatrix} 1 & x_1 & \dots & x_1^{n-1} \\ \vdots & & & \\ 1 & x_n & \dots & x_n^{n-1} \end{bmatrix} \qquad \det W(X) = \prod_{a<b}(x_b - x_a) \tag{41}$$

The proposed expression is (we use $V(X) = \mathrm{diag}(V(x_1),\dots)$)

$$\frac{1}{n!}\det K(x_a,x_b) = \frac{1}{n!}\det \sum_{j,k} \mathrm{e}^{-\frac{V(x_a)}{2}} x_a^j \mathfrak{M}^{-1}_{jk} x_b^k \mathrm{e}^{-\frac{V(x_b)}{2}} = \tag{42}$$

$$= \frac{1}{n!}\det\left[\mathrm{e}^{-\frac{1}{2}V(X)} W\cdot\mathfrak{M}^{-1}\cdot W^T \mathrm{e}^{-\frac{1}{2}V(X)}\right] = \frac{1}{n!\det\mathfrak{M}}\Delta^2(X)\mathrm{e}^{-\mathrm{Tr}V(X)} = \frac{1}{\mathscr{Z}}\Delta^2(X)\mathrm{e}^{-\mathrm{Tr}V(X)} \quad \square \tag{43}$$

THE KERNEL

PROPOSITION 1.1

The Kernel $K(x,y)$ has the following properties

$$\int_{\mathbb{R}} K(x,z)K(z,y)\,\mathrm{d}z = K(z,y) \quad \text{(reproducibility)} \tag{44}$$

$$\int_{\mathbb{R}} K(x,x)\,\mathrm{d}x = n \quad \text{(normalization)} \tag{45}$$

Skip Proof

Proof **Reproducibility**. By definition of K (all indices summed from 0 to $n-1$)

$$\int_{\mathbb{R}} K(x,z)K(z,y)\,\mathrm{d}z = \mathrm{e}^{-\frac{V(x)+V(y)}{2}} \int_{\mathbb{R}} x^i \mathfrak{M}_{ia}^{-1} z^{a+b} \mathfrak{M}_{bj}^{-1} y^j \mathrm{e}^{-V(z)}\,\mathrm{d}z \tag{46}$$

We now extract the constants from the integral ...

$$\int_{\mathbb{R}} K(x,z)K(z,y)\,\mathrm{d}z = \mathrm{e}^{-\frac{V(x)+V(y)}{2}} x^i \mathfrak{M}_{ia}^{-1} \overset{\delta_{bj}}{\cancel{\mathfrak{M}_{bj}^{-1} y^j \mathfrak{M}_{ab}}} = \mathrm{e}^{-\frac{V(x)+V(y)}{2}} x^i \mathfrak{M}_{ij}^{-1} y^j = K(x,y) \tag{47}$$

Now simplify the matrices $\mathfrak{M}$: This concludes the proof of reproducibility. Recall that the blue integral is the definition of $\mathfrak{M}_{ab}$....

Normalization.

$$\int_{\mathbb{R}} K(x,x)\,\mathrm{d}x = \int_{\mathbb{R}} x^{i+j} \mathfrak{M}_{ij}^{-1} \mathrm{e}^{-V(x)}\,\mathrm{d}x = \mathfrak{M}_{ji}\mathfrak{M}_{ij}^{-1} = \delta_{ii} = n \tag{48}$$

This ends the proof. □

DYSON'S THEOREM I

THEOREM 1.4

Suppose that a kernel $K(x,y)$ has the properties of **reproducibility** *and* **normalization** *(to n). Then*

$$\textbf{(a)} \quad \int_{\mathbb{R}} \det[K(x_a,x_b)]_{a,b\le r}\,\mathrm{d}x_r = (n-r-1)\det[K(x_a,x_b)]_{a,b\le r-1} \tag{49}$$

$$\textbf{(b)} \quad \int_{\mathbb{R}^{n-r}} \det[K(x_a,x_b)]_{a,b\le n}\,\mathrm{d}x_{r+1}\dots\mathrm{d}x_n = (n-r)!\det[K(x_a,x_b)]_{a,b\le r-1} \tag{50}$$

Proof. Part **(b)** follows from **(a)** by induction. We expand the determinant along the last row (use the shorthand $K_{ab} := K(x_a,x_b)$)

$$\det[K_{ab}]_{a,b\le r} = K_{rr}\det[K_{ab}]_{a,b<r} + \sum_{j\le r-1}(-1)^{r+j}K_{jr}\det[K_{ab}]_{\substack{a\neq j\\ b<r}} \tag{51}$$

... and then expand each minor along the last column (save for the (rr) minor)...

$$K_{rr}\det[K_{ab}]_{a,b<r} + \sum_{j\le r-1}(-1)^{r+j}K_{jr}\sum_{i\le r-1}(-1)^{r-1+i}K_{ri}\det[K_{ab}]_{\substack{a\neq j<r\\ b\neq i<r}} \tag{52}$$

Rearrange the terms:

$$K_{rr}\ n\ \det[K_{ab}]_{a,b<r} + \sum_{j\le r-1}(-1)^{r+j}\sum_{i\le r-1}(-1)^{r-1+i}K_{jr}K_{ri}\ K_{ji}\ \det[K_{ab}]_{\substack{a\neq j<r\\ b\neq i<r}} \tag{53}$$

DYSON'S THEOREM II

Integrating w.r.t. x_r and using $\int K_{rr}\,\mathrm{d}x_r = n$, $\int K_{jr}K_{ri}\,\mathrm{d}x_r = K_{ji}$ the above becomes... ...and then finally simplify...

$$= n\det[K_{ab}]_{a,b<r} - \sum_{j\le r-1} \underbrace{\sum_{i\le r-1} (-1)^{i+j} K_{ji} \det[K_{ab}]_{\substack{a\neq j<r\\ b\neq i<r}}}_{\det[K_{ab}]_{a,b<r}} = (n-r-1)\det[K_{ab}]_{a,b<r} \tag{54}$$

Now, part **(b)** follows by induction. □.

REMARK 1.3

Dyson's theorem says that the JPDF and *all the marginals* (partial integrations thereof) are in the form of a **determinant** built out of the same kernel.
This is the prototype of the so-called **random point fields**. I refer to the review by Soshnikov [11] for more details and more general definitions.

REMARK 1.4

It is important that the whole statistical information is contained in the Kernel and hence the remainder of this lecture shall be on the connection of $K(x,y)$ with orthogonal polynomials.

Kernel and Orthogonal Polynomials

It can be shown that $\mathfrak{M}$ is (for any size) positive definite (and symmetric). Consider the Lower-Diagonal-Upper decomposition (keeping into account the symmetry)

$$\mathfrak{M} = LHL^T \quad \Rightarrow \quad \mathfrak{M}^{-1} = L^{-T}H^{-1}L^{-1} \tag{55}$$

where L is a lower **unipotent** matrix (with ones on the diagonal) and $H = \mathrm{diag}(h_0, \dots, h_{n-1})$ Then

$$K(x,y) = [1, x, \dots, x^{n-1}]\overbrace{L^{-T}H^{-1}L^{-1}}^{\mathfrak{M}^{-1}}[1, y, \dots y^{n-1}]^t \tag{56}$$

Definition 1.1

The polynomials

$$\begin{bmatrix} p_0(x) \\ p_1(x) \\ \vdots \\ p_{n-1}(x) \end{bmatrix} = L^{-1} \begin{bmatrix} 1 \\ x \\ \vdots \\ x^{n-1} \end{bmatrix} \tag{57}$$

are called **orthogonal polynomials** for the measure $\mathrm{e}^{-V(x)}\,\mathrm{d}x$

Properties

Using formula (55) and the definition of the orthogonal polynomials p_n we can rephrase the Kernel in the following form

Proposition 1.2

The Kernel $K(x,y)$ is written

$$K(x,y) = \mathrm{e}^{-\frac{V(x)+V(y)}{2}} \sum_{j=0}^{n-1} \frac{p_n(x)p_n(y)}{h_n} \tag{58}$$

Exercise 1.2 (Characterization of OPs)

The following properties are **exercises** and are **equivalent** to the above definition.

1. $\deg p_n(x) = n$ and $p_n(x) = x^n + \dots$;
2. $\displaystyle\int_{\mathbb{R}} p_n(x)p_m(x)\mathrm{e}^{-V(x)}\,\mathrm{d}x = \delta_{nm}h_n$; recall $\int_{\mathbb{R}} x^{a+b}\mathrm{e}^{-V(x)}\,\mathrm{d}x = \mathfrak{M}_{ab}$;
3. They solve a **three terms recurrence relation**

$$xp_n(x) = p_{n+1}(x) + \alpha_n p_n(x) + \frac{h_n}{h_{n-1}}p_{n-1}(x) \tag{59}$$

In addition we have

1. $h_n > 0$;
2. $\mathscr{Z} = n!\det\mathfrak{M} = n!\prod_{j=0}^{n-1} h_j$;

ORTHOGONAL POLYNOMIALS I

Since all statistics are expressed in terms of the Kernel and this, in turn, is expressed in terms of Orthogonal Polynomials, we increasingly focus on the latter.

THEOREM 1.5 (CHRISTOFFEL-DARBOUX FORMULA)

For any set of orthogonal polynomials we have

$$K(x,y) = \mathrm{e}^{-\frac{V(x)+V(y)}{2}} \sum_{j=0}^{n-1} \frac{p_j(x)p_j(y)}{h_j} = \frac{1}{h_n} \frac{p_n(x)p_{n-1}(y) - p_n(y)p_{n-1}(x)}{x-y} \tag{60}$$

Skip Proof **Proof** Use the three–term recurrence relation and write it as a telescopic sum $(p_{-1} \equiv 0)$:

$$\mathrm{e}^{\frac{V(x)+V(y)}{2}} (x-y)K(x,y) = \tag{61}$$

$$= \sum_{j=0}^{n-1} \frac{1}{h_j} \left[\left(p_{j+1}(x) + \alpha_j p_j(x) + \frac{h_j}{h_{j-1}} p_{j-1}(x) \right) p_j(y) - p_j(x) \left(p_{j+1}(y) + \alpha_j p_j(y) + \frac{h_j}{h_{j-1}} p_{j-1}(y) \right) \right] \tag{62}$$

$$= \sum_{j=0}^{n-1} \frac{1}{h_j} \left[\left(p_{j+1}(x) + \cancel{\alpha_j p_j(x)} + \frac{h_j}{h_{j-1}} p_{j-1}(x) \right) p_j(y) - p_j(x) \left(p_{j+1}(y) + \cancel{\alpha_j p_j(y)} + \frac{h_j}{h_{j-1}} p_{j-1}(y) \right) \right] \tag{63}$$

ORTHOGONAL POLYNOMIALS II

we cancel what is immediately obvious and simplify

$$= \sum_{j=0}^{n-1} \left[\frac{p_{j+1}(x)p_j(y)}{h_j} + \frac{p_{j-1}(x)p_j(y)}{h_{j-1}} - \frac{p_j(x)p_{j+1}(y)}{h_j} - \frac{p_j(x)p_{j-1}(y)}{h_{j-1}} \right] \tag{64}$$

the two pairs of terms with the same colors form a telescopic sum: only the last term survives (the first is zero due to $p_{-1} = 0$)

$$= \frac{p_n(x)p_{n-1}(y) - p_n(y)p_{n-1}(x)}{h_n} \qquad \square \tag{65}$$

Thus, if we need to study asymptotics for $n \to \infty$, this makes it very convenient because we only have two terms to control, rather than an expanding sum of terms.

RIEMANN–HILBERT APPROACH

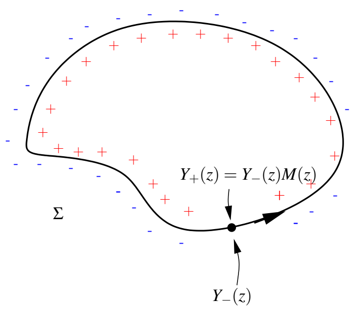
A **Riemann–Hilbert problem** is a **boundary–value problem** for a matrix–valued, piecewise analytic function $\Gamma(z)$. We will not enter in the details of smoothness. Everything is assumed smooth enough.

PROBLEM 2.1

Let Σ be an **oriented** (union of) curve(s) and $M(z)$ a (sufficiently smooth) matrix function defined on Σ. Find a function $Y(z)$ with the properties that

- $Y(z)$ is **analytic** on $\mathbb{C}\setminus\Sigma$;
- $\lim_{z\to\infty} Y(z) = \mathbf{1}$ (or some other normalization);
- for all $z\in\Sigma$, denoting by $Y(z)_\pm$ the (nontangential) boundary values of $Y(z)$ from the left/right of Σ, we have

$$Y_+(z) = Y_-(z)M(z)\ . \tag{66}$$



THEOREM 2.1 (SOKHOTSKY-PLEMELJI FORMULA)

Let $h(w)$ be α–Hölder on Σ and

$$f(z) := \frac{1}{2i\pi}\int_\Sigma \frac{h(w)\,\mathrm{d}w}{w-z} \tag{67}$$

Then $f_+(w) - f_-(w) = h(w)$ and $f_+(w)+f_-(w) =: H(h)(w)$ exists (the Cauchy principal value).

In the 90's Fokas, Its and Kitaev [7] proved the following crucial theorem establishing the relationship between orthogonal polynomials and RHPs and paving the way for a fruitful area of mathematics.

PROBLEM 2.2 (THE RHP FOR ORTHOGONAL POLYNOMIALS)

Find a 2×2 matrix–valued function $Y(z) = Y_n(x)$ with the properties

1. $Y(z)$ is analytic in $\mathbb{C}_\pm := \{\pm\Im(z) > 0\}$;
2. The boundary values of $Y(z)$ on $\Sigma = \mathbb{R}$ (oriented in the natural direction) satisfy

$$Y_+(x) = Y_-(x)\overbrace{\begin{bmatrix} 1 & \mathrm{e}^{-nV(x)} \\ 0 & 1 \end{bmatrix}}^{=:M(x)} \tag{68}$$

3. In the sectors $\arg(z)\in(0,\pi)$ and $\arg(z)\in(\pi,2\pi)$ the function $Y(z)$ has the expansion

$$Y(z) = (\mathbf{1}+\mathscr{O}(z^{-1}))\begin{bmatrix} z^n & 0 \\ 0 & z^{-n}\end{bmatrix} = (\mathbf{1}+\mathscr{O}(z^{-1}))z^{n\sigma_3}\ ,\quad \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1\end{bmatrix} \tag{69}$$

The above expansion is *uniform*[a] in the sense that for any $R>0$ there exists $C>0$ such that for $|z|>R$, $z\notin\mathbb{R}$,

$$\left\| Y(z)z^{-n\sigma_3} - \mathbf{1}\right\| < C\frac{1}{|z|} \tag{70}$$

[a] This is not the strongest form of the problem but it is sufficient for us.

In what follows, $V(z)$ shall be a real-analytic function; to simplify further, we shall take it as a **polynomial** (of even degree and positive leading coefficient).

THEOREM 2.2 (FOKAS-ITS-KITAEV)

The **unique** *solution of the RH Problem 2.2 is as follows*

$$Y(z) := Y_n(z) := \left[\begin{array}{cc} p_n(z) & \frac{1}{2i\pi}\int \frac{p_n(x)\mathrm{e}^{-nV(x)}\,\mathrm{d}x}{x-z} \\ \frac{-2i\pi}{h_{n-1}}p_{n-1}(z) & \frac{-1}{h_{n-1}}\int \frac{p_{n-1}(x)\mathrm{e}^{-nV(x)}\,\mathrm{d}x}{x-z} \end{array}\right] \tag{71}$$

where $p_n(z), p_{n-1}(z)$ are the orthogonal polynomials for the measure $\mathrm{e}^{-nV(x)}\,\mathrm{d}x$ on $\mathbb{R}$ and h_n the corresponding squared norms, exactly as in the previous definition.

To prove the theorem: **Uniqueness**

- Show that $\det Y(z)$ has **no jump** on $\mathbb{R}$ (so it is entire);
- Show that $\det Y(z) \to 1$ as $|z| \to \infty$ and hence (Liouville's thm.) it is identically one. Thus any solution $Y(z)$ is invertible and with analytic inverse;
- If $\widetilde{Y}$ is another solution, then show that $R := \widetilde{Y}Y^{-1}$ has **no jumps** on $\mathbb{R}$ and hence it is entire;
- Using the asymptotic behavior (69) show that $R \to \mathbf{1}_2$ and hence (by Liouville's thm. again) $R \equiv \mathbf{1}$.

Then one shows directly that the proposed expression (using **Sokhotsky–Plemelji**'s formula) satisfies the conditions. Et voilà! □

RELATION WITH THE KERNEL

PROPOSITION 2.1

The Kernel $K(x,y)$ is expressed as

$$K(x,y) = \mathrm{e}^{-\frac{n}{2}(V(x)+V(y))}\sum_{j=0}^{n-1}\frac{p_n(x)p_n(y)}{h_n} = \frac{1}{2i\pi(x-y)}\left(Y_\pm^{-1}(y)Y_\pm(x)\right)_{21}\mathrm{e}^{-\frac{n}{2}(V(x)+V(y))} \tag{72}$$

Proof. Use the Christoffel-Darboux formula and explicit form of Y, together with its inverse (the determinant is 1). The choice of boundary value is irrelevant because the terms involved in the expression are only the polynomials. □

SOME INTERESTING PROPERTIES (NOT PROVED HERE): THE SPECTRAL CURVE I

Consider the simple case of $V(x) = \sum_{j=1}^{d} \frac{t_j}{j} x^j$ (a polynomial potential). Let Q be the **Jacobi matrix** (tridiagonal, symmetric) for the three-term recustion relation of

$$\pi_n(x) := \frac{p_n(x)}{\sqrt{h_n}} \tag{73}$$

and define

$$\Pi_n(z) := \mathrm{diag}\left(\frac{1}{\sqrt{h_n}}, \frac{\sqrt{h_{n-1}}}{-2i\pi}\right) Y(z) \mathrm{e}^{nV(z)\mathrm{diag}(0,1)} = \begin{bmatrix} \pi_n(z) & \frac{1}{2i\pi}\mathrm{e}^{nV(z)} \int \frac{\pi_n(x)\mathrm{e}^{-nV(x)}\,\mathrm{d}x}{x-z} \\ \pi_{n-1}(z) & \frac{\mathrm{e}^{nV(z)}}{2i\pi} \int \frac{\pi_{n-1}(x)\mathrm{e}^{-nV(x)}\,\mathrm{d}x}{x-z} \end{bmatrix} \tag{74}$$

Then [1]

Differential equation

We have $\frac{1}{n}\partial_z \Pi_n(z) = \mathscr{D}_n(z)\Pi_n(z)$ with $\mathscr{D}_n$ a polynomial 2×2 matrix

$$\mathscr{D}_n(z) = \begin{bmatrix} 0 & 0 \\ 0 & V'(z) \end{bmatrix} + \begin{bmatrix} \left(\frac{V'(Q)-V'(z)}{Q-z}\right)_{n,n} & \left(\frac{V'(Q)-V'(z)}{Q-z}\right)_{n,n-1} \\ \left(\frac{V'(Q)-V'(z)}{Q-z}\right)_{n-1,n} & \left(\frac{V'(Q)-V'(z)}{Q-z}\right)_{n-1,n-1} \end{bmatrix} \begin{bmatrix} 0 & \gamma_n \\ -\gamma_n & 0 \end{bmatrix} \tag{75}$$

with $\gamma_n := \sqrt{h_n/h_{n-1}}$.

SOME INTERESTING PROPERTIES (NOT PROVED HERE): THE SPECTRAL CURVE II

Spectral Curve (in terms of the Jacobi matrix)

$$\det(\lambda - \mathscr{D}_n(z)) = \lambda^2 - \lambda V'(z) + \frac{1}{n}\sum_{j=1}^{n}\left(\frac{V'(Q)-V'(z)}{Q-z}\right)_{jj} \tag{76}$$

Spectral Curve (in terms of the Random Matrix)

$$\det(\lambda - \mathscr{D}_n(z)) = \lambda^2 - \lambda V'(z) + \frac{1}{n}\left\langle \mathrm{Tr}\frac{V'(M)-V'(z)}{M-z}\right\rangle_{n\times n} \tag{77}$$

where

$$\langle F(M)\rangle_{n\times n} := \mathbb{E}(F) = \frac{1}{\mathscr{Z}}\int_{\mathscr{M}} \mathrm{d}M \mathrm{e}^{-n\mathrm{Tr}V(M)} F(M) \tag{78}$$

ASYMPTOTICS OF OP'S FOR $n \to \infty$

Summarizing, we shall consider the Hermitean Matrix model with measure

$$\mathrm{d}\mu(M) = \frac{1}{\mathscr{Z}_n} \mathrm{e}^{-\Lambda \mathrm{Tr} V(M)}\, \mathrm{d}M = \frac{1}{\mathscr{Z}_n} \mathrm{e}^{-\Lambda \sum_{j=1}^{n} V(x_j)}\, \mathrm{d}M \tag{79}$$

- Here Λ is a scaling parameter that we shall take to be exactly n (the dimension).
- The limit we shall consider for the statistics (i.e. the Kernel) is $n \to \infty$ and $\Lambda \to \infty$. More generally one may take a limit where $\Lambda = \frac{n}{T}$ and $T > 0$ is some constant.
- Show the essential steps of the **Deift–Zhou** [4] steepest descent method to obtain strong asymptotic formulæ for the orthogonal polynomials.
- We shall tacitly consider $V(z)$ to be a **polynomial** (e.g. $V(z) = z^2$) but all can be extended to real-analytic potentials as long as it grows at infinity

$$\liminf_{|x|\to\infty} \frac{V(x)}{\ln(1+|x|)} = +\infty \tag{80}$$

- Prove the **Sine-kernel Universality in the Bulk**

$$\lim_{n\to\infty} \frac{1}{n\rho(x_0)} K\left(x_0 + \frac{\xi}{n\rho(x_0)}, x_0 + \frac{\eta}{n\rho(x_0)}\right) = \frac{\sin(\pi(\eta-\xi))}{\pi(\eta-\xi)} \tag{81}$$

DISCLAIMER

The steepest descent method in full detail can easily occupy a semester long course. Here we have only one hour!

EQUILIBRIUM MEASURES

Given $V(x)$ as above

THEOREM 4.1 (E.G. IN SAFF–TOTIK'S BOOK, CH. 1 [10])

There is a unique **probability measure** $\rho(x)\,\mathrm{d}x$ *minimizing*

$$\mathscr{F}[\mathrm{d}\mu] := \int_{\mathbb{R}} V(x)\,\mathrm{d}\mu(x) + \int_{\mathbb{R}^2} \ln\frac{1}{|x-y|}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \tag{82}$$

The minimizer $\rho(x)\,\mathrm{d}x$ *is characterized by*

$$V(x) + 2\int_{\mathbb{R}} \ln\frac{1}{|x-y|}\rho(y)\,\mathrm{d}y + \ell \geq 0 \qquad x \in \mathbb{R} \tag{83}$$

$$V(x) + 2\int_{\mathbb{R}} \ln\frac{1}{|x-y|}\rho(y)\,\mathrm{d}y + \ell \equiv 0 \qquad x \in \operatorname{supp}\rho \tag{84}$$

The constant ℓ *is called* **(modified) Robin's constant**.

Theorem 4.2 (Deift et al.)

Suppose $V(x)$ is also real-analytic: then $\operatorname{supp}\rho$ *is a* **finite union of compact intervals**.

A simple proof is available in Arno's notes (using Shiffer's variations). It can also be shown that if $V(x)$ is **convex** (concave upwards) then there is only one interval of support. Since additional technical complications arise when there are several intervals, we shall assume that the support is indeed only one single interval

$$V''(x) > 0 \quad \Rightarrow \quad \operatorname{supp}\rho = [a,b] \tag{85}$$

Definition 4.1 (The g–function)

$$g(z) := \int_a^b \ln(z-y)\rho(y)\,\mathrm{d}y \tag{86}$$

where $g(z)$ is defined as analytic on $\mathbb{C}$ minus the cut from $-\infty$ to b, with the principal branch of $\ln$; for z approaching $\mathbb{R}$ above/below:

$$\ln(z_\pm - y) = \ln|z-y| \pm i\pi\chi_{y\ge z} \tag{87}$$

So that for $z = x \in \mathbb{R}$

$$g_\pm(x) = \int_a^b \ln|x-y|\rho(y)\,\mathrm{d}y \pm i\pi\chi_{x\le b}\int_x^b \rho(y)\,\mathrm{d}y \tag{88}$$

Note that the minimizer conditions (83, 84) in Thm. 4.1 read

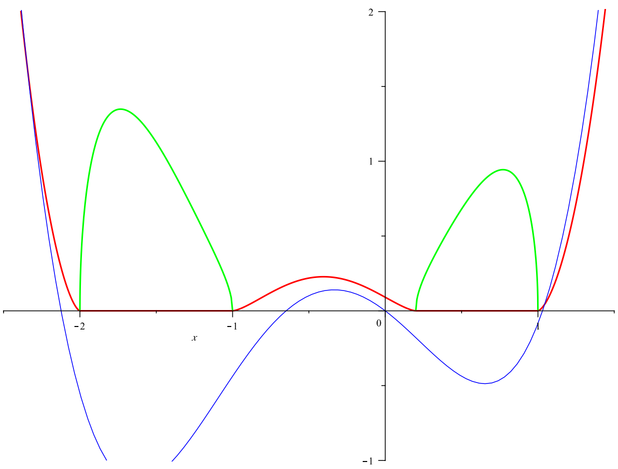
$$V(x) + 2\int_{\mathbb{R}} \ln\frac{1}{|x-y|}\rho(y)\,\mathrm{d}y + \ell = V(x) - 2\Re g(x) + \ell \begin{cases} \ge 0 & x \in \mathbb{R} \\ \equiv 0 & x \in [a,b] = \operatorname{supp}\rho \end{cases} \tag{89}$$

Definition 4.2

The (complex) effective potential

$$\varphi(z) := V(z) - 2g(z) + \ell \tag{90}$$

For $z \in \mathbb{R}$ this represents the electrostatic potential of the charge distribution $\rho(x)\,\mathrm{d}x$ plus the external potential, in equilibrium.



In blue a quartic potential; in red $\Re\varphi$; in green the equilibrium density

PROPERTIES AND SIGN DISTRIBUTION

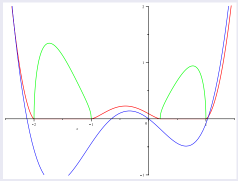
$$g(z) := \int_a^b \ln(z-x)\rho(x)\,\mathrm{d}x\,, \quad \ln(z_\pm - x) = \ln|z-x| \pm i\pi\chi_{x\ge z} \qquad (91)$$

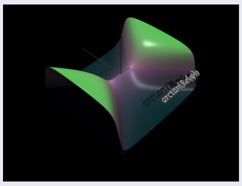
$$g_\pm(x) = \int_a^b \ln|x-y|\rho(y)\,\mathrm{d}y \pm i\pi\chi_{x\le b}\int_x^b \rho(y)\,\mathrm{d}y \qquad (92)$$

$$\varphi(z) := V(z) - 2g(z) + \ell \qquad (93)$$

PROPERTIES OF $g(z)$ AND $\varphi(z)$

- $g(z) = \ln z + \mathscr{O}(1/z)$ as $z \to \infty$;
- $\frac{-\varphi_+ + \varphi_-}{2} = g_+(x) - g_-(x) = 2\pi i \int_x^\infty \rho(s)\,\mathrm{d}s$ for $x \in \mathbb{R}$;
- $-\frac{1}{2}\Im\varphi_+ = \Im g_+(x) = \pi i \int_{-\infty}^x \rho(s)\,\mathrm{d}s - i\pi$ is **decreasing** on $[a,b]$;
- $\frac{\varphi_+ + \varphi_-}{2} = \Re\varphi = V - g_+ - g_- + \ell \equiv 0$ for $x \in [a,b]$;
- $\frac{\varphi_+ + \varphi_-}{2} = \Re\varphi = V - g_+ - g_- + \ell \ge 0$ for $x \notin [a,b]$.





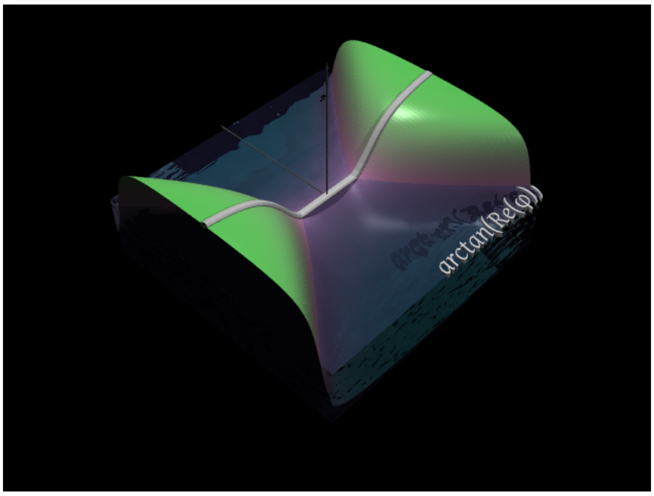
The second bullet implies (using Cauchy Riemann: $u_y = -v_x$) that $\Re\varphi_+$ is **decreasing** in the vertical direction because $\Im\varphi_+ \nearrow$. Since $\Im\varphi_- = -\Im\varphi_+$, then $\Im\varphi_- \searrow$ and hence $\Re\varphi_-$ **decreases also going below** $\mathbb{R}$.

THE SIGN DISTRIBUTION: LAND AND SEA

The above remarks paint a picture of the **sign distribution** (i.e. the regions of equal signs) of $\Re\varphi$ in the complex plane. Here is a typical picture (for $V(x) = x^2$).

DEFINITION 4.3

The **land** is where $\Re\varphi > 0$; the **sea** (or water) is where $\Re\varphi < 0$.



The properties of the minimizer as such that the **real axis never sinks undewater**.

DIRECT CONTRUCTION OF $g(z)$ IN THE ONE–CUT CASE

Assuming that we know existence of the equilibrium measure (and sufficient smoothness) we want to find the solution of the **scalar** RHP

$$V(x) - g_+(x) - g_-(x) + \ell = 0 \quad \Rightarrow \quad g'_+(x) + g_-(x) = V'(x) \ , \quad x \in [a,b] \tag{94}$$

The following analysis is *perfunctory*:

let $R(z) := \sqrt{(z-a)(z-b)}$ be the holomorphic function on $\mathbb{C} \setminus [a,b]$ with $R(z) \sim z$ at infinity. Then (from the argument principle)

$$R_+(x) = -R_-(x). \tag{95}$$

Dividing (94) by R_+ we have

$$\frac{1}{R_+}(g'_+ + g'_-) = \left(\frac{g'}{R}\right)_+ - \left(\frac{g'}{R}\right)_- = \frac{V'}{R_+} \tag{96}$$

Thus the function $f := g'/R$ is analytic on $\mathbb{C} \setminus [a,b]$ and

$$f_+(x) - f_-(x) = \frac{V'(x)}{R_+(x)} \qquad x \in [a,b] \tag{97}$$

This RHP is solved with the **Sokhotsky-Plemelji** formula

$$f(z) = \int_a^b \frac{V'(x)\,\mathrm{d}x}{R_+(x)(x-z)2i\pi} \quad \Rightarrow \quad g'(z) = R(z)\int_a^b \frac{V'(x)\,\mathrm{d}x}{R_+(x)(x-z)2i\pi} \tag{98}$$

On the other hand we had $g(z) = \ln(z) + \mathscr{O}(z^{-1})$ and hence $g'(z) = \frac{1}{z} + \mathscr{O}(z^{-2})$. The expansion of the proposed expression at $z = \infty$ is

$$g'(z) = R(z)\int_a^b \frac{V'(x)\,\mathrm{d}x}{R_+(x)(x-z)2i\pi} = -\int_a^b \frac{V'(x)\,\mathrm{d}x}{R_+(x)2i\pi} + \frac{1}{z}\left(\frac{b+a}{2}\int_a^b \frac{V'(x)\,\mathrm{d}x}{R_+(x)2i\pi} - \int_a^b \frac{xV'(x)\,\mathrm{d}x}{R_+(x)2i\pi}\right) + \dots \tag{99}$$

This gives the following two equations (*moment conditions*) for the two unknowns a, b

$$-\int_a^b \frac{V'(x)\,\mathrm{d}x}{R_+(x)2i\pi} = 0 \qquad -\int_a^b \frac{xV'(x)\,\mathrm{d}x}{R_+(x)2i\pi} = 1 \tag{100}$$

For V polynomial, both integrals are computed explicitly and the equations become algebraic.

EXAMPLE 4.1

For $V(x) = \frac{t}{2}x^2 + \frac{\kappa}{4}x^4$ one obtains (**exercise!**) $b = -a$ and

$$a = \left(\frac{-2t + \sqrt{4t^2 + 48\kappa}}{3\kappa}\right)^{\frac{1}{2}} \tag{101}$$

Another form is as follows: if γ is a counterclockwise contour surrounding $[a,b]$ then the residue theorem yields

$$g'(z) = R(z)\int_a^b \frac{V'(x)\,\mathrm{d}x}{R_+(x)(x-z)2i\pi} = -\frac{1}{2}R(z)\oint_\gamma \frac{V'(x)\,\mathrm{d}x}{R(x)(x-z)2i\pi} = \tag{102}$$

$$= \frac{V'(z)}{2} - \frac{1}{2}R(z)\oint_{|x|>|z|} \frac{V'(x)\,\mathrm{d}x}{R(x)(x-z)2i\pi} = \tag{103}$$

$$= \frac{V'(z)}{2} - \frac{1}{2}R(z)\oint_{|x|>|z|} \frac{(V'(x)-V'(z))\,\mathrm{d}x}{R(x)(x-z)2i\pi} = \frac{V'(z)}{2} - M(z)R(z) \tag{104}$$

where $M(z)$ is patently a polynomial of degree at most $\deg V - 2$.

Since the equilibrium density is $\rho(x) = i\frac{g'_+(x)}{\pi}$ we see that

$$\rho(x) = \frac{1}{\pi}M(x)\sqrt{|x-a|\,|x-b|} \tag{105}$$

and hence $M(z)$ must remain positive for $x \in [a,b]$.

EXAMPLE 4.2

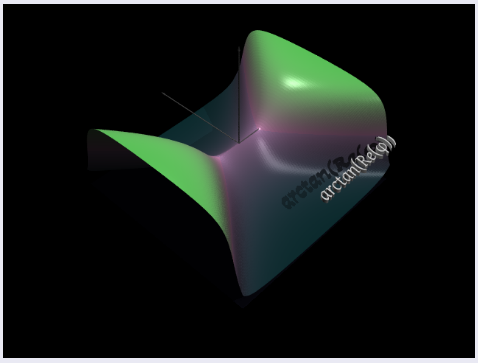
For $V = \frac{x^2}{2}$ the OP's involved are the Hermite polynomials: the equilibrium density is

$$\rho(x) = \frac{1}{\pi}\sqrt{2-x^2}\ , \quad x \in [-2,2] \tag{106}$$

and the complex effective potential φ

$$\varphi = \frac{z\sqrt{z^2-2}}{2} - \ln\left(\frac{z+\sqrt{z^2-2}}{2}\right) \tag{107}$$

The plot of $\arctan(\Re\varphi)$ is below: note that $\Re\varphi = \equiv 0$ on the support $\operatorname{supp}\rho = [-2,2]$.



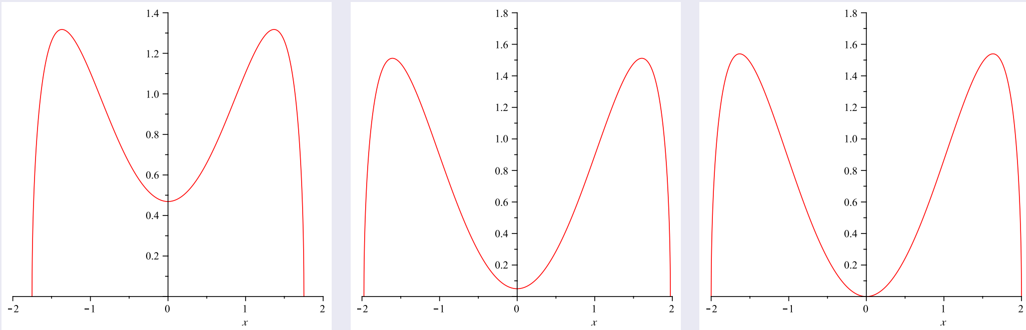
Example 4.3

In the above example $V(x) = \frac{t}{2}x^2 + \frac{\kappa}{4}x^4$ one finds (**exercise**)

$$\rho(x) = \frac{1}{\pi}M(x)\sqrt{x^2 + \frac{2t - \sqrt{4t^2+48\kappa}}{3\kappa}} \tag{108}$$

$$M(x) = \frac{\kappa}{2}x^2 + \frac{2t+\sqrt{t^2+12\kappa}}{6} \ , \tag{109}$$

and one can verify (**exercise**) that $M(x)$ vanishes within the interval of support when $t - 2\sqrt{\kappa}$ and becomes even *negative* for $t < -2\sqrt{\kappa}$. This signals that the interval of support for $t = -2\sqrt{\kappa}$ is about to "split" into two and the *assumption* that the support is only one interval is about to become invalid.



Regular potentials

For later reference we make the

Definition 4.4

The (real-analytic) potential V is **regular** (in the potential-theoretic sense) if

- $\rho(x)$ (the equilibrium measure) is **strictly positive** in the interior of the support;
- as x approaches an endpoint c of the support (from the interior) $\lim_{x\to c} \frac{\rho(x)}{\sqrt{|x-c|}} = G_c > 0$;
- $V - 2\Re g + \ell > 0$ (strictly) outside of the support.

In particular, for regular potentials, near the endpoint b (of $[a,b]$) one has

$$\varphi(z) = \frac{2}{3}G(z-b)^{\frac{3}{2}}(\mathbf{1} + \mathscr{O}(z-b)) \tag{110}$$

with the cut of the root extending along the support. To see this it suffices to recall from (104)

$$\frac{\varphi'(z)}{2} = \frac{V'(z)}{2} - g'(z) = M(z)R(z) \quad \Rightarrow \quad \varphi(z) = 2\int_b^z M(w)R(w)\,\mathrm{d}w \tag{111}$$

with the contour of integration in $\mathbb{C}\setminus(-\infty, b]$.

Warning 1

We shall assume (in due time) that V is regular in the above sense.

Philosophy of the Steepest descent method: the small norm theorem

The steepest descent method of Deift and Zhou is based upon the application of the following prototype theorem (called the "small norm theorem") The proof is not difficult but takes too much time.

Theorem 5.1 (Small norm theorem)

Suppose a RHP is posed on a (collection of) contour(s) Σ for a matrix $\mathscr{E}(z)$

$$\mathscr{E}(z)_+ = \mathscr{E}(z)_-(\mathbf{1}+\delta G(z)) , \quad z \in \Sigma , \quad \mathscr{E}(z) = \mathbf{1} + \mathscr{O}(z^{-1}) \quad \text{as } z \to \infty. \tag{112}$$

(with $\det(\mathbf{1}+\delta G) \equiv 1$). Denote by N_p the norms in $L^p(\Sigma, |dz|)$ of the matrix $\delta G(z)$. Then

1. *There is a constant C_Σ such that if $N_\infty < C_\Sigma^{-1}$ the solution of the RHP exists;*[a]
2. *Then*
$$\|\mathscr{E}(z) - \mathbf{1}\| \le \frac{1}{2\pi \mathrm{dist}(z,\Sigma)} \left(N_1 + \frac{C_\Sigma {N_2}^2}{1 - C_\Sigma N_\infty} \right) , \quad \forall z \in \mathbb{C} \tag{113}$$
and if the jump $\delta G(z)$ is analytic in a neighborhood of Σ the denominator can be replaced by $1 + \mathrm{dist}(z,\Sigma)$.

[a] The constant turns out to be the operator norm of the Cauchy boundary operator in $L^2(\Sigma)$.

The reason of the name is because if the norms $1, 2$ are small, then the solution $\mathscr{E}(z)$ is close to the identity (**pointwise!**). In practice the jump δG depends on some parameter (like n) and typically all norms N_p tend to zero. Skip Proof

Sketch of proof I

Exercise 5.1

The RHP is equivalent to the following singular–integral equation

$$\mathscr{E}(z) = \mathbf{1} + \frac{1}{2i\pi} \int_\Sigma \frac{\mathscr{E}_-(w)\delta G(w)\,\mathrm{d}w}{w-z} \tag{114}$$

Hint: show that the rhs has the correct jump (use Sokhotski–Plemelji) and normalization.

Then we have

$$\mathscr{E}(z) - \mathbf{1} = \frac{1}{2i\pi} \int_\Sigma \frac{\delta G(w)\,\mathrm{d}w}{w-z} + \frac{1}{2i\pi} \int_\Sigma \frac{(\mathscr{E}_-(w) - \mathbf{1})\delta G(w)\,\mathrm{d}w}{w-z} \tag{115}$$

We take the boundary value on the right $_-$:

$$\overbrace{\mathscr{E}_-(z) - \mathbf{1}}^{f(z)} = \overbrace{\frac{1}{2i\pi} \int_\Sigma \frac{\delta G(w)\,\mathrm{d}w}{w-z_-}}^{\delta h} + \overbrace{\frac{1}{2i\pi} \int_\Sigma \frac{(\mathscr{E}_-(w) - \mathbf{1})\delta G(w)\,\mathrm{d}w}{w-z_-}}^{=:\mathscr{L}(f)} \tag{116}$$

The term δh is explicitly given. The equation turns into...

$$(\mathrm{Id} - \mathscr{L})(f) \ = \delta h \tag{117}$$

$$\mathscr{L}(f) := \frac{1}{2i\pi} \int_\Sigma \frac{f(w)\delta G(w)\,\mathrm{d}w}{w-z_-} \tag{118}$$

SKETCH OF PROOF II

This is considered an equation in $L^2(\Sigma)$; the solvability is guaranteed if the **operator norm** of $\mathscr{L}$ is smaller than 1. Then the solution is simply

$$\mathscr{E}_- - 1 = f = (\mathrm{Id} - \mathscr{L})^{-1}(\delta h) = \sum_{m=0}^{\infty} \mathscr{L}^m(\delta h) \tag{119}$$

$$\|f\| \le \frac{\|\delta h\|}{1 - |||\mathscr{L}|||} \tag{120}$$

We need estimates of these norms....
Since $\mathscr{L}$ is multiplication (on the right) by δG followed by the Cauchy boundary value, its norm is estimated as

$$|||\mathscr{L}||| \le \overbrace{\|\delta G\|_\infty}^{N_\infty} C_\Sigma \tag{121}$$

where C_Σ is the norm of the Cauchy operator on Σ (known to be finite).

The first result is that the solution of the RHP exists if $N_\infty < C_\Sigma^{-1}$.

EXERCISE 5.2

Show that if Σ is a circle (of any radius) then $C_\Sigma = 1$.

SKETCH OF PROOF III

The norm of δh is also

$$\delta h = \frac{1}{2i\pi} \int_\Sigma \frac{\delta G(w)\,\mathrm{d}w}{w - z_-} \quad \Rightarrow \quad \|\delta h\| \le C_\Sigma \overbrace{\|\delta G\|}^{N_2} \tag{122}$$

so that ...

$$\|\mathscr{E}_- - 1\|_\Sigma \le \frac{\|\delta h\|}{1 - |||\mathscr{L}|||} \le \frac{C_\Sigma N_2}{1 - C_\Sigma N_\infty} \tag{123}$$

Next, we have to estimate pointwise $\mathscr{E}(z)$ for $z \notin \Sigma$....
Denote by $\|M\| = \sqrt{\mathrm{Tr} M^\dagger M}$ the matrix Hilbert–Schmidt norm; then

$$\|\mathscr{E}(z) - 1\| \le \left\| \frac{1}{2i\pi} \int_\Sigma \frac{\delta G(w)\,\mathrm{d}w}{w - z} \right\| + \left\| \frac{1}{2i\pi} \int_\Sigma \frac{(\mathscr{E}_-(w) - \mathbf{1})\delta G(w)\,\mathrm{d}w}{w - z} \right\| \tag{124}$$

$$\le \frac{1}{2\pi} \frac{N_1}{\mathrm{dist}(z,\Sigma)} + \frac{1}{2\pi} \frac{\|\mathscr{E}_- - 1\|_\Sigma N_2}{\mathrm{dist}(z,\Sigma)} = \frac{1}{2\pi \mathrm{dist}(z,\Sigma)} \left(N_1 + \frac{C_\Sigma {N_2}^2}{1 - C_\Sigma N_\infty} \right) = \tag{125}$$

Et voilà! $\square$.

PHILOSOPHY OF THE STEEPEST DESCENT METHOD: **use** OF THE SMALL NORM THEOREM

Consider the solution of the RHP for our polynomials $Y_n(z)$; it can be transformed into an **equivalent** RHP with a solution $W(z)$ with jumps on some contours Σ

$$W_+(z) = W_-(z)M(z)\ ,\quad z\in\Sigma\ ,\quad W(z) = \mathbf{1}+\mathscr{O}(z^{-1})\quad z\to\infty. \tag{126}$$

Suppose we can find an "approximate solution" (and explicit) $\widetilde{W}$, where by "approximate" it means that its jumps $\widetilde{M}$ are "close" to M in the sense $M\widetilde{M}^{-1} = \mathbf{1}+\delta F$. Then consider

THE error matrix

$$\mathscr{E}(z) := W(z)\widetilde{W}^{-1}(z) \tag{127}$$

The jumps of $\mathscr{E}$ are

$$\mathscr{E}_+ = W_+\widetilde{W}_+^{-1} = W_-M\widetilde{M}^{-1}\widetilde{W}_-^{-1} = \mathscr{E}_-\overbrace{\widetilde{W}_-\widetilde{M}^{-1}\widetilde{W}_-^{-1}}^{\mathbf{1}+\delta G} \tag{128}$$

If δG satisfies (as dependent on n) the conditions of the small norm theorem, then we can rightfully consider the (hopefully explicit) $\widetilde{W}$ as an approximation. The small-norm theorem also gives the order of approximation.

The steepest descent method is the implementation of this "philosophy", with many devilish details. The method was started in [4] and then applied to orthogonal polynomials in [6]; many followers....

RHP FOR OPS: DEIFT–ZHOU METHOD

This is a short outline of a collection of ideas and methods developed in the 90's by Percy Deift and Xin Zhou [4] and then applied to orthogonal polynomials by D, Z, Kriecherbauer, McLaughlin, Venakides in [6]. We recall the characterization of the Orthogonal polynomials in terms of a RHP.

THEOREM 5.2 (FOKAS-ITS-KITAEV[7])

The matrix

$$Y(z) := Y_n(z) := \left[\begin{array}{cc} p_n(z) & \frac{1}{2i\pi}\int\frac{p_n(x)e^{-\Lambda(V(x)}\,dx}{x-z} \\ \frac{-2i\pi}{h_{n-1}}p_{n-1}(z) & \frac{-1}{h_{n-1}}\int\frac{p_{n-1}(x)e^{-\Lambda(V(x)}\,dx}{x-z}\end{array}\right] \tag{129}$$

satisfies

$$\text{(RHP)}\quad \left\{\begin{array}{l} Y_+(z) = Y_-(z)\left[\begin{array}{cc}1 & e^{-\Lambda V(z)}\\ 0 & 1\end{array}\right]\ ,\quad z\in\mathbb{R} \\ Y(z) = \left(\mathbf{1}+\mathscr{O}\left(\frac{1}{z}\right)\right)z^{n\sigma_3}\end{array}\right. \tag{130}$$

$$\sigma_3 := \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \tag{131}$$

The conditions **(RHP)** *characterize OPs and* $(Y_n)_{11}(z) = p_n(z)$

MASSAGING THE PROBLEM I

Before finding the approximation we need to "deform" the contours of the jumps (similar idea to contour deformation of analytic integrals using Cauchy's theorem). Define

$$W(z) := e^{-\frac{n}{2}\ell\sigma_3} Y(z) e^{-n\left(g(z)-\frac{\ell}{2}\right)\sigma_3} \tag{132}$$

Since $g(z)$ has jumps on the real axis only, the jumps of W are modified. Let's see how:

$$W_+ = e^{-\frac{n}{2}\ell\sigma_3} Y_+ e^{-n\left(g_+-\frac{\ell}{2}\right)\sigma_3} \tag{133}$$

Using the jump of Y:

$$W_+ = e^{-\frac{n}{2}\ell\sigma_3} Y_- \begin{bmatrix} 1 & e^{-\Lambda V} \\ 0 & 1 \end{bmatrix} e^{-n\left(g_+-\frac{\ell}{2}\right)\sigma_3} \tag{134}$$

We now insert ...

$$W_+ = \underbrace{e^{-\frac{n}{2}\ell\sigma_3} Y_- e^{-n\left(g_- -\frac{\ell}{2}\right)\sigma_3}}_{W_-} e^{n\left(g_- -\frac{\ell}{2}\right)\sigma_3} \begin{bmatrix} 1 & e^{-\Lambda V} \\ 0 & 1 \end{bmatrix} e^{-n\left(g_+-\frac{\ell}{2}\right)\sigma_3} \tag{135}$$

Simplifying the jump we obtain:

$$W_+(z) = W_-(z) \begin{bmatrix} e^{n(g_- - g_+)} & e^{-n(V-g_- -g_+ +\ell)} \\ 0 & e^{-n(g_- - g_+)} \end{bmatrix} \tag{136}$$

MASSAGING THE PROBLEM II

Using $\varphi := V - 2g + \ell$ we can rewrite

$$W_+(z) = W_-(z) \begin{bmatrix} e^{\frac{n}{2}(\varphi_+ - \varphi_-)} & e^{-\frac{n}{2}(\varphi_+ + \varphi_-)} \\ 0 & e^{-\frac{n}{2}(\varphi_+ - \varphi_-)} \end{bmatrix} \tag{137}$$

Massaging (cont'd)

So we now have the new jumps

$$W(z) := \mathrm{e}^{-\frac{n}{2}\ell\sigma_3} Y(z) \mathrm{e}^{-n\left(g(z)-\frac{\ell}{2}\right)\sigma_3} \tag{138}$$

$$W_+(z) = W_-(z) \begin{bmatrix} \mathrm{e}^{\frac{n}{2}(\varphi_+-\varphi_-)} & \mathrm{e}^{-\frac{n}{2}(\varphi_++\varphi_-)} \\ 0 & \mathrm{e}^{-\frac{n}{2}(\varphi_+-\varphi_-)} \end{bmatrix} \tag{139}$$

We need the asymptotic at infinity: recall that

$$g(z) = \ln z + \mathcal{O}(z^{-1}) \quad \Rightarrow \quad \mathrm{e}^{ng(z)} = z^n(1+\mathcal{O}(z^{-1})) \tag{140}$$

and hence

$$W(z) = \mathrm{e}^{-\frac{n}{2}\ell\sigma_3} Y(z) \mathrm{e}^{-n\left(g(z)-\frac{\ell}{2}\right)\sigma_3} = \mathrm{e}^{-\frac{n}{2}\ell\sigma_3}(\mathbf{1}+....)z^{n\sigma_3}z^{-n\sigma_3}(\mathbf{1}+...)\mathrm{e}^{n\frac{\ell}{2}\sigma_3} = \mathbf{1}+\mathcal{O}(z^{-1}) \tag{141}$$

Summary of RHP for W

So we now have the new jumps

RHP for W

$$W(z) := \mathrm{e}^{-\frac{n}{2}\ell\sigma_3} Y(z) \mathrm{e}^{-n\left(g(z)-\frac{\ell}{2}\right)\sigma_3} \tag{142}$$

$$W_+(z) = W_-(z) \begin{bmatrix} \mathrm{e}^{\frac{n}{2}(\varphi_+-\varphi_-)} & \mathrm{e}^{-\frac{n}{2}(\varphi_++\varphi_-)} \\ 0 & \mathrm{e}^{-\frac{n}{2}(\varphi_+-\varphi_-)} \end{bmatrix} \tag{143}$$

$$W(z) = \mathbf{1} + \mathcal{O}(z^{-1}) \tag{144}$$

Remark 5.1

This RHP is much better suited to application of the small-norm theorem.

We now have a miracle!

Matrix algebra Miracle

$$\begin{bmatrix} \mathrm{e}^a & \mathrm{e}^b \\ 0 & \mathrm{e}^{-a} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \mathrm{e}^{-a-b} & 1 \end{bmatrix} \begin{bmatrix} 0 & \mathrm{e}^b \\ -\mathrm{e}^{-b} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \mathrm{e}^{a-b} & 1 \end{bmatrix} \tag{145}$$

We will use this miracle with

$$a = \frac{n}{2}(\varphi_+ - \varphi_-), \quad b = -\frac{n}{2}(\varphi_+ + \varphi_-) \tag{146}$$

Matrix algebra Miracle

$$\begin{bmatrix} e^a & e^b \\ 0 & e^{-a} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ e^{-a-b} & 1 \end{bmatrix} \begin{bmatrix} 0 & e^b \\ -e^{-b} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ e^{a-b} & 1 \end{bmatrix} \tag{147}$$

$$a = \frac{n}{2}(\varphi_+ - \varphi_-)\ , \quad b = -\frac{n}{2}(\varphi_+ + \varphi_-) \tag{148}$$

The jump for W thus factorizes

$$\begin{bmatrix} e^{\frac{n}{2}(\varphi_+ - \varphi_-)} & e^{-\frac{n}{2}(\varphi_+ + \varphi_-)} \\ 0 & e^{-\frac{n}{2}(\varphi_+ - \varphi_-)} \end{bmatrix} = \tag{149}$$

$$= \begin{bmatrix} 1 & 0 \\ e^{n\varphi_-} & 1 \end{bmatrix} \begin{bmatrix} 0 & e^{-\frac{n}{2}(\varphi_+ + \varphi_-)} \\ -e^{\frac{n}{2}(\varphi_+ + \varphi_-)} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ e^{n\varphi_+} & 1 \end{bmatrix} \tag{150}$$

The "opening of lenses"

The jump of W can now be written as

$$W_+(z) = W_-(z) \begin{bmatrix} 1 & 0 \\ e^{n\varphi_-} & 1 \end{bmatrix} \begin{bmatrix} 0 & e^{-\frac{n}{2}(\varphi_+ + \varphi_-)} \\ -e^{\frac{n}{2}(\varphi_+ + \varphi_-)} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ e^{n\varphi_+} & 1 \end{bmatrix} \tag{151}$$

Bringing the rightmost red matrix to the left-hand side:

$$W_+(z) \begin{bmatrix} 1 & 0 \\ -e^{n\varphi_+} & 1 \end{bmatrix} = W_-(z) \begin{bmatrix} 1 & 0 \\ e^{n\varphi_-} & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{152}$$

Crucial observation 1

On the support of ρ, $x \in [a,b]$ we have (remember!) $\varphi_+ + \varphi_- \equiv 0$ and hence the blue matrix is simply

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{153}$$

Crucial observation 2

The red matrices admit analytic continuation in the upper (lower) half planes (respectively). This allows us to re-define W

Definition 5.1 (The matrix Φ)

Outside of the lense(s), $\Phi \equiv W$. In the **lense(s)**

$$\Phi(z) := W(z)\begin{bmatrix}1 & 0\\ -e^{n\varphi} & 1\end{bmatrix} =: W(z)M_L^{-1}(z)\ , \qquad z\in \text{upper lens}$$

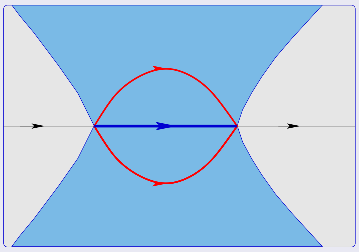
$$\Phi(z) := W(z)\begin{bmatrix}1 & 0\\ e^{n\varphi} & 1\end{bmatrix} =: W(z)M_L(z)\ , \qquad z\in \text{lower lens}$$

$$\Phi(z) = W(z)\ , \quad z\in \text{elsewhere}$$

$$\Phi_+(z) = \Phi_-(z)i\sigma_2\ , \quad z\in[a,b]$$

$$\Phi_+ = \Phi_-\begin{bmatrix}1 & 0\\ e^{n\varphi} & 1\end{bmatrix}\ , \quad z\in \ \text{upper}\backslash\text{lower rim}$$

$$\Phi(z) = \mathbf{1}+\mathscr{O}(z^{-1})\ , \quad z\to\infty$$



So the jumps of Φ are: The asymptotic at infinity is the same because $\Phi\equiv W$ outside **On the support** $z\in[a,b]$**:**

$$W_+(z)\begin{bmatrix}1 & 0\\ -e^{n\varphi_+} & 1\end{bmatrix} = W_-(z)\begin{bmatrix}1 & 0\\ e^{n\varphi_-} & 1\end{bmatrix}\begin{bmatrix}0 & e^{-\frac{n}{2}(\varphi_++\varphi_-)}\\ -e^{\frac{n}{2}(\varphi_++\varphi_-)} & 0\end{bmatrix} \tag{154}$$

$$\Phi_+(z) = \Phi_-\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} = \Phi_- i\sigma_2 \tag{155}$$

Upper rim of the lens:

$$\Phi_+ = W_+ = \Phi_-\begin{bmatrix}1 & 0\\ e^{n\varphi} & 1\end{bmatrix} \tag{156}$$

Lower rim of the lens:

$$\Phi_+ = W\begin{bmatrix}1 & 0\\ e^{n\varphi} & 1\end{bmatrix} = \Phi_-\begin{bmatrix}1 & 0\\ e^{n\varphi} & 1\end{bmatrix} \tag{157}$$

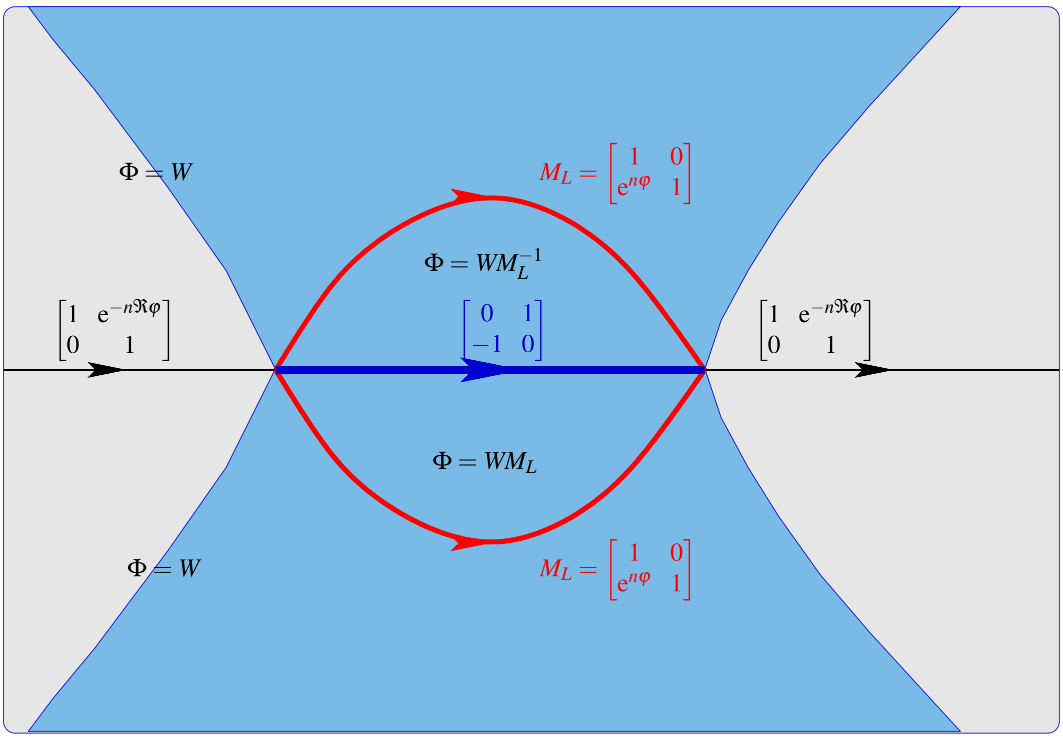
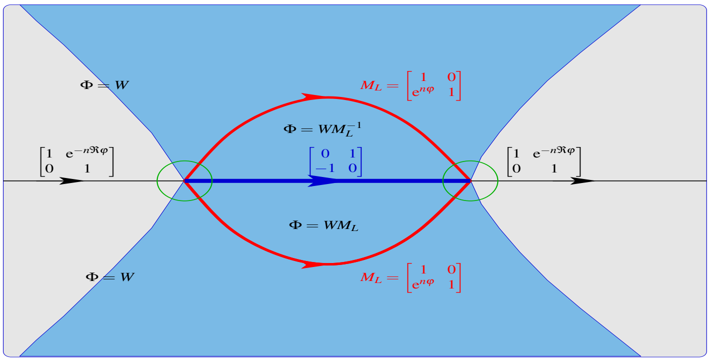


Figure: The final steps in the transformation chain, with the lens and $\varphi(z) := V(z) - 2g(z) + \ell$; in blue the typical shape (near the support) of the regions where $\Re\varphi < 0$.

Jumps of Φ: analysis of orders I



The (black) jump on $\mathbb{R} \setminus [a,b]$ has only one element off-diagonal $e^{-n\Re\varphi}$: the sign-distribution tells that $\Re\varphi > 0$ there (**on land**). **Therefore** this term tends to zero **exponentially fast** (as $n \to \infty$). This is true in any L^p **except** in L^∞ because $\Re\varphi(b) = \Re\varphi(a) = 0$. (fly in the ointment!) The (red) jumps on the rims also have only one element off-diagonal $e^{n\varphi}$: the sign-distribution tells that $\Re\varphi < 0$ there (**underwater**) . **Therefore** this term tends to zero **exponentially fast** (as $n \to \infty$). This is true in any L^p **except** in L^∞ because $\Re\varphi(b) = \Re\varphi(a) = 0$. (fly in the ointment!) The only jump that is definitely not close to the identity is the blue one (on the support of ρ). If it were not for the L^∞ norm-problem, the **small norm theorem** could be used to argue that we can disregard the black and red jumps.

Jumps of Φ: analysis of orders II

Remark 5.2

The argument cannot be made at this point: one needs to add to fixed (small) disks around the endpoints. It will be shown by Arno that **inside** these disks the RHP can be solved **exactly** . The *local solution* is called the **Parametrix**. In the generic case this can be constructed with the aid of *Airy functions*, but in non-generic situations (transitions of genus etc.) one needs special functions (*Painlevé*).

Warning 2

We shall ignore this problem here!

STEP 1: THE OUTER PARAMETRIX (MODEL PROBLEM) I

Find $\Psi(z)$ with the same jump as Φ on the support and same asymptotics

$$\Psi_+(z) = \Psi_-(z)\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} = \Psi_-(z) i\sigma_2 \tag{158}$$

$$\Psi(z) = \left(\mathbf{1} + \mathscr{O}\left(\frac{1}{z}\right)\right), \quad |z|\to\infty \tag{159}$$

And also $\Psi(z)$ should have the "minimal" growth at $z = a,b$ compatibly with the jump.
Solution The jump matrix is constant and can be diagonalized:

$$i\sigma_2 = F e^{\frac{i\pi}{2}\sigma_3} F^{-1}, \quad F := \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i\\ i & -1\end{bmatrix} = \frac{\sigma_3+\sigma_2}{\sqrt{2}} = F^{-1} \tag{160}$$

Thus the matrix $\widehat{\Psi} := F^{-1}\Psi F$ has jump:

$$\widehat{\Psi}_+ = \widehat{\Psi}_- e^{\frac{i\pi}{2}\sigma_3}, \quad \widehat{\Psi}(z) = \mathbf{1} + \mathscr{O}(z^{-1}), z\to\infty \tag{161}$$

This RHP decouples into two **scalar** RHPs; taking logarithms the solution is easily written using S-P formula (this step is *perfunctory*[1])

$$\widehat{\Psi}(z) = \exp\left[\frac{i\pi\sigma_3}{2}\int_a^b \frac{\mathrm{d}w}{2i\pi(w-z)}\right] = \exp\left[\frac{\sigma_3}{4}\ln\left(\frac{z-b}{z-a}\right)\right] = \left(\frac{z-b}{z-a}\right)^{\frac{\sigma_3}{4}} \tag{162}$$

The **solution** is

STEP 1: THE OUTER PARAMETRIX (MODEL PROBLEM) II

$$\Psi(z) = \frac{1}{2}\begin{bmatrix}1 & -i\\ i & -1\end{bmatrix}\left(\frac{z-b}{z-a}\right)^{\frac{1}{4}\sigma_3}\begin{bmatrix}1 & -i\\ i & -1\end{bmatrix} \tag{163}$$

REMARK 5.3

The matrix Ψ has also the symmetry

$$(\Psi(\bar z))^\dagger = \sigma_3\Psi^{-1}(z)\sigma_3 \ \textbf{(exercise)} \tag{164}$$

from which it follows that

$$\Psi_{11,+} = \overline{\Psi_{22,-}} = \overline{\Psi_{12,+}} \tag{165}$$

[1] I like this adjective!

STEP 2: THE ERROR

We are now in a position of using the **small-norm Theorem**: define the error matrix by

$$\mathcal{E}(z) := \Phi(z)\Psi^{-1}(z) \tag{166}$$

It solves a RHP without jumps on $[a,b]$ and elsewhere we have

$$\mathcal{E}_+(z) = \mathcal{E}_-(z)\Psi(z)M(z)\Psi(z)^{-1} \tag{167}$$

with $M(z)$ being the jump matrix for Φ. We know that $M(z) = \mathbf{1} + \delta G(z)$ and $|\delta G(z)| \to 0$ exponentially fast (at least in any L^p, $p < \infty$ norm).
On the face of it, it seems that $\mathcal{E}(z)$ will tend to the identity matrix *exponentially fast* as $n \to \infty$: this is not so, due to the backlash from the neglected problem.

THE GENERIC ESTIMATE

Generically (to be defined below) the error matrix tends to $\mathbf{1}$ as n^{-1}. The "generic" case corresponds to **regular potentials** V

DEFINITION 5.2

The (real-analytic) potential V is **regular** (in the potential-theoretic sense) if

- $\rho(x)$ (the equilibrium measure) is **strictly positive** in the interior of the support;
- as x approaches an endpoint c of the support (from the interior) $\lim_{x\to c} \frac{\rho(x)}{\sqrt{|x-c|}} > 0$;
- $V - 2\Re g + \ell > 0$ (strictly) outside of the support.

REAPING THE HARVEST

Recall the chain of transformations:

$$Y \longrightarrow W = \mathrm{e}^{-\frac{n}{2}\ell\sigma_3} Y \mathrm{e}^{-n(g-\frac{\ell}{2})\sigma_3} \longrightarrow \Phi = \begin{cases} W(\mathbf{1} - \mathrm{e}^{n\varphi}\sigma_-) & \text{upper lens} \\ W & \text{outside} \\ W(\mathbf{1} + \mathrm{e}^{n\varphi}\sigma_-) & \text{lower lens} \end{cases} \tag{168}$$

Then we have argued (with some devilish details missing at the endpoints)

$$\Phi(z) = \mathcal{E}(z)\Psi(z) = \mathcal{E}(z)\frac{\sigma_3 + \sigma_2}{\sqrt{2}}\left(\frac{z-b}{z-a}\right)^{\frac{1}{4}\sigma_3}\frac{\sigma_3+\sigma_2}{\sqrt{2}} \tag{169}$$

and $\mathcal{E}(z)$ is (uniformly) close to the identity. Unrolling the sequence of transformations: **outside of the lenses** ($\Phi = W = \mathcal{E}\Psi$)

$$Y_n(z) = \mathrm{e}^{\frac{n}{2}\ell\sigma_3}\mathcal{E}_n(z)\Psi(z)\mathrm{e}^{n(g-\frac{\ell}{2})\sigma_3} \tag{170}$$

ON CLOSED SUBSETS OF $\mathbb{C}\setminus[a,b]$ (I.E. **outside the lenses**)

$$p_n(z) = (Y_n)_{11}(z) = \mathrm{e}^{ng(z)}(\mathcal{E}\Psi)_{11} = \Psi_{11}(z)\mathrm{e}^{ng(z)}(1 + \mathcal{O}(n^{-1})) \tag{171}$$

$$= \frac{1}{2}\left[\left(\frac{z-b}{z-a}\right)^{\frac{1}{4}} + \left(\frac{z-a}{z-b}\right)^{\frac{1}{4}}\right]\mathrm{e}^{ng(z)} \tag{172}$$

Inside the lenses I

$$Y \longrightarrow W = e^{-\frac{n}{2}\ell\sigma_3} Y e^{-n(g-\frac{\ell}{2})\sigma_3} \longrightarrow \Phi = \begin{cases} W(\mathbf{1}-e^{n\varphi}\sigma_-) & \text{upper lens} \\ W & \text{outside} \\ W(\mathbf{1}+e^{n\varphi}\sigma_-) & \text{lower lens} \end{cases} \tag{173}$$

Inside the lenses (but away from the endpoints we have (upper/lower lens)

$$Y = e^{\frac{n}{2}\ell\sigma_3} W e^{n\left(g-\frac{\ell}{2}\right)\sigma_3} = e^{\frac{n}{2}\ell\sigma_3}\Phi(\mathbf{1}\pm\sigma_- e^{n\varphi})e^{n\left(g-\frac{\ell}{2}\right)\sigma_3} = \tag{174}$$

$$= e^{\frac{n}{2}\ell\sigma_3}\mathscr{E}\Psi\begin{bmatrix} 1 & 0 \\ \pm e^{n\varphi} & 1\end{bmatrix} e^{n\left(g_\pm-\frac{\ell}{2}\right)\sigma_3} \tag{175}$$

Thus, matrix multiplication gives (recall φ is purely imaginary on $x\in(a,b)$ and $\varphi_+ = -\varphi_-$)

$$p_n(x) = Y_{11} = (\Psi_{11,\pm} \pm \Psi_{12,\pm}e^{n\varphi_\pm})e^{ng_\pm} = \left(\Psi_{11,\pm}e^{-\frac{n}{2}\varphi_\pm} \pm \Psi_{12,\pm}e^{\frac{n}{2}\varphi_\pm}\right)\underbrace{e^{n\left(g_\pm+\frac{\varphi_\pm}{2}\right)}}_{e^{\frac{n}{2}(V+\ell)}} \tag{176}$$

Factor out a term $e^{\frac{n}{2}\varphi_\pm}\ldots$ $(\varphi = V-2g+\ell)$ Recalling that the RHP for Ψ implies that $\Psi_{11,\pm} = \pm\overline{\Psi_{12,\pm}}$ we see that whichever boundary value we take it is the same result, that is....

$$p_n(x) = Y_{11} = 2\Re\left(\Psi_{11,+}e^{-\frac{n}{2}\varphi_+}\right)e^{\frac{n}{2}(V+\ell)} = \Re\left[\left(e^{\frac{i\pi}{4}}\left|\frac{x-b}{x-a}\right|^{\frac{1}{4}} + e^{-\frac{i\pi}{4}}\left|\frac{x-a}{x-b}\right|^{\frac{1}{4}}\right)e^{ng_+}\right] \tag{177}$$

Recalling that $g_+(x) = i\pi\int_x^b \rho(s)\,\mathrm{d}s$ we have obtained:

Inside the lenses II

Asymptotic on the support

$$p_n(x) = \Re\left[\left(e^{\frac{i\pi}{4}}\left|\frac{x-b}{x-a}\right|^{\frac{1}{4}} + e^{-\frac{i\pi}{4}}\left|\frac{x-a}{x-b}\right|^{\frac{1}{4}}\right)e^{in\pi\int_x^b\rho(s)\,\mathrm{d}s}\right] \tag{178}$$

SUMMARY OF ASYMPOTICS

ASYMPTOTIC ON THE SUPPORT

$$p_n(x) = \Re\left[\left(e^{\frac{i\pi}{4}}\left|\frac{x-b}{x-a}\right|^{\frac{1}{4}} + e^{-\frac{i\pi}{4}}\left|\frac{x-a}{x-b}\right|^{\frac{1}{4}}\right)e^{in\pi\int_x^b \rho(s)\,ds}\right] \tag{179}$$

ON CLOSED SUBSETS OF $\mathbb{C}\setminus[a,b]$ (I.E. **outside the lenses**)

$$p_n(z) = (Y_n)_{11}(z) = e^{ng(z)}(\mathscr{E}\Psi)_{11} = \Psi_{11}(z)e^{ng(z)}(1+\mathscr{O}(n^{-1})) \tag{180}$$

$$= \frac{1}{2}\left[\left(\frac{z-b}{z-a}\right)^{\frac{1}{4}} + \left(\frac{z-a}{z-b}\right)^{\frac{1}{4}}\right]e^{ng(z)} \tag{181}$$

REMARK 5.4

Potential theory arguments (without any RHP) can give the following **weak asymptotics** for z outside of the (convex hull of the) support of the equilibrium measure:

$$\lim_{n\to\infty}\frac{1}{n}\ln|p_n(z)| = \Re g(z) \tag{182}$$

In a way, the RHP and the DZ method have been able to turn the weak asymptotic into strong, using the same data.

UNIVERSALITY IN THE BULK I

This refers to the behavior of the kernel $K(x,y)$ for x,y in the **bulk** i.e. in the *interior* of the support of the equilibrium measure, when $|x-y| = \mathscr{O}(n^{-1})$.
Specifically: let $x_0 \in (a,b)$ be fixed and define, for some constant $C>0$ (to be chosen later)

$$x = x_0 + \frac{\xi}{nC}\ ,\quad y = x_0 + \frac{\eta}{nC} \tag{183}$$

Recall that

$$K(x,y) = \frac{e^{-n\frac{V(x)+V(y)}{2}}}{2i\pi(x-y)}\left(Y_+^{-1}(y)Y_+(x)\right)_{21} \tag{184}$$

and that

$$Y_+(x) = e^{\frac{n}{2}\ell\sigma_3}\mathscr{E}\Psi_+\begin{bmatrix}1 & 0\\ e^{n\varphi_+} & 1\end{bmatrix}e^{n\left(g_+-\frac{\ell}{2}\right)\sigma_3} \tag{185}$$

Then

$$Y_+^{-1}(y)Y_+(x) = \underbrace{e^{-n\left(g_+-\frac{\ell}{2}\right)\sigma_3}\begin{bmatrix}1 & 0\\ -e^{n\varphi_+} & 1\end{bmatrix}\Psi_+^{-1}\mathscr{E}^{-1}}_{y}\underbrace{\mathscr{E}\Psi_+\begin{bmatrix}1 & 0\\ e^{n\varphi_+} & 1\end{bmatrix}e^{n\left(g_+-\frac{\ell}{2}\right)\sigma_3}}_{x} = \tag{186}$$

$$\simeq e^{-n\left(g_+(y)-\frac{\ell}{2}\right)\sigma_3}\begin{bmatrix}1 & 0\\ e^{n\varphi_+(x)}-e^{n\varphi_+(y)} & 1\end{bmatrix}e^{n\left(g_+(x)-\frac{\ell}{2}\right)\sigma_3} \tag{187}$$

Universality in the bulk II

We simplify in the obvious way. Taking the element (2,1)... Then

$$e^{-n\frac{V(x)+V(y)}{2}}(Y_+^{-1}(y)Y_+(x))_{21} \simeq e^{n\left(-\frac{1}{2}V(y)+g_+(y)-\frac{\ell}{2}\right)}\left(e^{n\varphi_+(x)}-e^{n\varphi_+(y)}\right)e^{n\left(-\frac{1}{2}V(x)+g_+(x)-\frac{\ell}{2}\right)} \tag{188}$$

Then

$$e^{-n\frac{V(x)+V(y)}{2}}(Y_+^{-1}(y)Y_+(x))_{21} \simeq e^{-\frac{n}{2}\varphi_+(y)}\left(e^{n\varphi_+(x)}-e^{n\varphi_+(y)}\right)e^{-\frac{n}{2}\varphi_+(x)} = \tag{189}$$

$$= e^{\frac{n}{2}(\varphi_+(x)-\varphi_+(y))} - e^{-\frac{n}{2}(\varphi_+(x)-\varphi_+(y))} \tag{190}$$

But $\varphi_+(x) = -2g_+(x) = -2i\pi\int_x^b \rho(s)\,ds$ and so

$$\varphi_+(x)-\varphi_+(y) = -2i\pi\int_x^y \rho(s)\,ds \tag{191}$$

$$e^{-n\frac{V(x)+V(y)}{2}}(Y_+^{-1}(y)Y_+(x))_{21} \simeq \exp\left(-i\pi n\int_x^y \rho(s)\,ds\right) - \exp\left(i\pi n\int_x^y \rho(s)\,ds\right) = 2i\sin\left(n\pi\int_y^x \rho(s)\,ds\right) \tag{192}$$

Now look at the expressions for x, y (183)
Then

$$e^{-n\frac{V(x)+V(y)}{2}}(Y_+^{-1}(y)Y_+(x))_{21} \simeq 2i\sin\left(\pi n\int_x^y \rho(s)\,ds\right) \simeq 2i\sin\left(\pi\frac{\rho(x_0)}{C}(\eta-\xi)\right) \tag{193}$$

Universality in the bulk III

Clearly it is convenient to choose $C = \rho(x_0)$ (the local density of eigenvalues). So we have (*perfunctorily*) proved:

Theorem 5.3 (Sine-kernel Universality in the Bulk)

$$\lim_{n\to\infty}\frac{1}{n\rho(x_0)}K\left(x_0+\frac{\xi}{n\rho(x_0)}, x_0+\frac{\eta}{n\rho(x_0)}\right) = \frac{\sin(\pi(\eta-\xi))}{\pi(\eta-\xi)} \tag{194}$$

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