Asymptotic Analysis of Random Matrices and Orthogonal Polynomials

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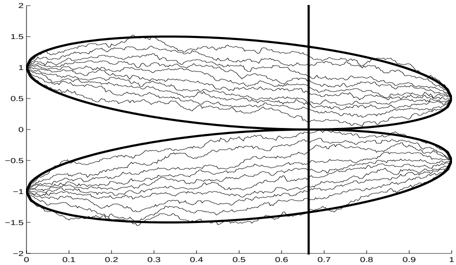
Two starting and ending points

- In case of **two (or more) starting** and **two (or more) end points**, the positions of non-intersecting Brownian motions, are not a MOP ensemble in the sense that we discussed.
- There is an extension using **MOPs of mixed type** that applies here.
- There is still a RH problem but with jump condition

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

  and a Christoffel-Darboux formula for the correlation kernel.

Critical separation



- **Two tangent ellipses and limiting density at each time consists of two semicircle laws.**
- **New scaling limits at the point where ellipses meet, called tacnode.**

Delvaux, K, Zhang (2011)
Adler, Ferrari, Van Moerbeke (2012)
Johansson (arXiv)

Random matrix model with external source

- **Hermitian matrix model with external source**

$$\frac{1}{Z_n} e^{-n \operatorname{Tr}(V(M)-AM)}\, dM$$

- **Assume n is even and external source is**

$$A = \operatorname{diag}(\underbrace{a, \ldots, a}_{n/2 \text{ times}}, \underbrace{-a, \ldots, -a}_{n/2 \text{ times}}), \qquad a > 0.$$

- **Assume V is an even polynomial**

Asymptotic analysis in this case is taken from the paper
P. Bleher, S. Delvaux, and A.B.J. Kuijlaars,
Random matrix model with external source and a vector equilibrium problem,
Comm. Pure Appl. Math. 64 (2011), 116–160

Reminder: MOP ensemble

- **Eigenvalues are MOP ensemble with weights**

$$w_1(x) = e^{-n(V(x)-ax)}, \qquad w_2(x) = e^{-n(V(x)+ax)}$$

and $\vec{n} = (n/2, n/2)$**.**

- **Eigenvalue correlation kernel is**

$$K_n(x,y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & w_1(y) & w_2(y) \end{pmatrix} Y_+^{-1}(y) Y_+(x) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

where Y **is the solution of a RH problem.**

Reminder: RH problem

RH-Y1 $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{3\times 3}$ **is analytic.**

RH-Y2 Y **has boundary values for** $x \in \mathbb{R}$**, denoted by** $Y_\pm(x)$**, and**

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & e^{-n(V(x)-ax)} & e^{-n(V(x)+ax)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

RH-Y3 **As** $z \to \infty$**,**

$$Y(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right) \begin{pmatrix} z^n & 0 & 0 \\ 0 & z^{-n/2} & 0 \\ 0 & 0 & z^{-n/2} \end{pmatrix}$$

Simplifying assumptions (for convenience)

- n is a multiple of four.
- The eigenvalues of M accumulate as $n \to \infty$ on at most 2 intervals.
- We are in a non-critical situation.

By second assumption and symmetry, limiting support of eigenvalues is either one interval

$$[-q, q], \qquad q > 0$$

or union of two symmetric intervals

$$[-q, -p] \cup [p, q], \qquad q > p > 0$$

- The assumption on support is satisfied if

$$x \mapsto V(\sqrt{x}) \quad \text{is convex on } [0, \infty).$$

First transformation

Define X by

$$X(z) = Y(z) \times \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -e^{-2naz} \\ 0 & 0 & 1 \end{pmatrix}, & \textbf{for } \operatorname{Re} z > 0, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -e^{2naz} & 1 \end{pmatrix}, & \text{for } \operatorname{Re} z < 0. \end{cases}$$

- **Jump for $x > 0$,**

$$X_-^{-1}(x) X_+(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^{-2nax} \\ 0 & 0 & 1 \end{pmatrix} Y_-(x)^{-1} Y_+(x) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -e^{2nax} \\ 0 & 0 & 1 \end{pmatrix}$$

Jump for $x > 0$ (cont.)

$$X_-^{-1}(x)X_+(x) =$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^{-2nax} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & e^{-n(V(x)-ax)} & e^{-n(V(x)+ax)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -e^{2nax} \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & e^{-n(V(x)-ax)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

RH problem for X

RH-X1 $X : \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R}) \to \mathbb{C}^{3\times 3}$ **is analytic.**

RH-X2 X **has boundary values for** $x \in \mathbb{R} \cup i\mathbb{R}$**, and**

$$X_+(x) = X_-(x) \begin{pmatrix} 1 & e^{-n(V(x)-ax)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \textbf{for } x > 0,$$

$$X_+(x) = X_-(x) \begin{pmatrix} 1 & 0 & e^{-n(V(x)+ax)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \textbf{for } x < 0,$$

$$X_+(z) = X_-(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & e^{-2naz} \\ 0 & -e^{2naz} & 1 \end{pmatrix} \quad \textbf{for } z \in i\mathbb{R}.$$

RH-X3 $X(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right) \begin{pmatrix} z^n & 0 & 0 \\ 0 & z^{-n/2} & 0 \\ 0 & 0 & z^{-n/2} \end{pmatrix}$ **as** $z \to \infty$**.**

Equilibrium problem

- **Recall: in RH analysis for orthogonal polynomials an important role is played by the equilibrium measure.**
- **This is the probability measure μ_V on $\mathbb{R}$ that minimizes**

$$\iint \log \frac{1}{|x-y|} d\mu(x) d\mu(y) + \int V(x) d\mu(x)$$

- **The g-function** $\boxed{g(z) = \int \log(z-s) d\mu_V(s)}$ **is used to normalize the RH problem at infinity.**
- **For a 3×3 RH problem we need two equilibrium measures and two g functions**

Vector equilibrium problem

Minimize

$$\iint \log \frac{1}{|x-y|} d\mu_1(x) d\mu_1(y) + \iint \log \frac{1}{|x-y|} d\mu_2(x) d\mu_2(y)$$
$$- \iint \log \frac{1}{|x-y|} d\mu_1(x) d\mu_2(y) + \int (V(x) - a|x|)\, d\mu_1(x)$$

among all vectors of measures (μ_1, μ_2) **such that**

(a) μ_1 **is a measure on** $\mathbb{R}$ **with total mass** 1**,**

(b) μ_2 **is a measure on** $i\mathbb{R}$ **with total mass** $1/2$**,**

(c) $\mu_2 \leq \sigma$ **where** σ **has constant density**

$$\frac{d\sigma}{|dz|} = \frac{a}{\pi}, \qquad z \in i\mathbb{R}.$$

Results 1

Proposition

There is a unique minimizer (μ_1, μ_2) and it satisfies

(a) **The support of μ_1 is bounded and consists of a finite union of intervals on the real line**

$$\text{supp}(\mu_1) = \bigcup_{j=1}^{N} [a_j, b_j].$$

(b) **The support of μ_2 is the full imaginary axis and there exists $c \geq 0$ such that**

$$\text{supp}(\sigma - \mu_2) = (-i\infty, -ic] \cup [ic, i\infty).$$

- **We assumed at most two intervals:** $N \leq 2$

Results 2

Theorem

$$\lim_{n\to\infty} \frac{1}{n} K_n(x,x) = \frac{d\mu_1}{dx}, \qquad x \in \mathbb{R},$$

where μ_1 is the first component of the minimizer of the vector equilibrium problem.
We find sine kernel in the bulk and Airy kernel at regular edge points.

Variational conditions

Notation: $$U^{\mu}(x) = \int \log \frac{1}{|x-s|} d\mu(s)$$

- **Equilibrium measures are characterized by variational conditions**

$$2U^{\mu_1}(x) = U^{\mu_2}(x) - V(x) + a|x| + \ell, \qquad x \in \operatorname{supp}(\mu_1),$$
$$2U^{\mu_1}(x) \geq U^{\mu_2}(x) - V(x) + a|x| + \ell, \qquad x \in \mathbb{R} \setminus \operatorname{supp}(\mu_1),$$

for some ℓ, and

$$2U^{\mu_2}(z) = U^{\mu_1}(z), \qquad z \in \operatorname{supp}(\sigma - \mu_2),$$
$$2U^{\mu_2}(z) \leq U^{\mu_1}(z), \qquad z \in i\mathbb{R} \setminus \operatorname{supp}(\sigma - \mu_2).$$

- **Variational conditions can be reformulated in terms of g-functions**

$$g_1(z) = \int \log(z-s) d\mu_1(s), \qquad g_2(z) = \int \log(z-s) d\mu_2(s)$$

Three regular cases

We distinguish three regular, non-critical cases

Case I: $N = 2$ and $c = 0$. **In this case**

$$\text{supp}(\mu_1) = [-q, -p] \cup [p, q], \quad \text{supp}(\sigma - \mu_2) = i\mathbb{R}.$$

Constraint is not active.

Case II: $N = 2$ and $c > 0$. **In this case**

$$\text{supp}(\mu_1) = [-q, -p] \cup [p, q], \quad \text{supp}(\sigma - \mu_2) = (-i\infty, -ic] \cup [ic, i\infty)$$

Constraint is active on $[-ic, ic]$.

Case III: $N = 1$ and $c > 0$. **In this case**

$$\text{supp}(\mu_1) = [-q, q], \quad \text{supp}(\sigma - \mu_2) = (-i\infty, -ic] \cup [ic, i\infty)$$

We put $p = 0$ **in case III.**

Riemann surface

- **Define three-sheeted Riemann surface**

$$\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$$

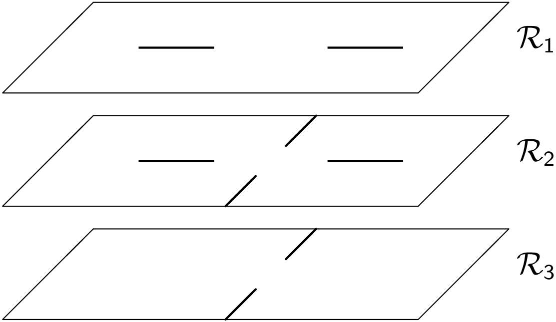
$$\mathcal{R}_1 = \overline{\mathbb{C}} \setminus \mathrm{supp}(\mu_1),$$

$$\mathcal{R}_2 = \mathbb{C} \setminus (\mathrm{supp}(\mu_1) \cup \mathrm{supp}(\sigma - \mu_2)),$$

$$\mathcal{R}_3 = \mathbb{C} \setminus \mathrm{supp}(\sigma - \mu_2).$$

- **Compact Riemann surface of genus $N - 2$ or $N - 1$**
- **Genus = 0 in our cases I and III,**
- **Genus = 1 in our case II.**

Riemann surface in Case II



Meromorphic function

Define $\boxed{F_j(z) = g_j'(z) = \int \frac{d\mu_j(s)}{z-s}}$ **for** $z \in \mathbb{C} \setminus \mathrm{supp}(\mu_j)$

Proposition

The function

$$\xi_1(z) = V'(z) - F_1(z), \qquad z \in \mathcal{R}_1$$

has a meromorphic continuation to the full Riemann surface. On other sheets it is given by

$$\begin{aligned} \xi_2(z) &= \pm a + F_1(z) - F_2(z), \quad && z \in \mathcal{R}_2, \quad && \pm \mathrm{Re}\, z > 0, \\ \xi_3(z) &= \mp a + F_2(z), && z \in \mathcal{R}_3, && \pm \mathrm{Re}\, z > 0. \end{aligned}$$

The only pole is at the point at infinity on the first sheet. This is a pole of order $\deg V - 1$.

Recall RH problem for X

RH-X1 $X : \mathbb{C} \setminus (\mathbb{R} \cup i\mathbb{R}) \to \mathbb{C}^{3\times 3}$ **is analytic.**

RH-X2 X **has boundary values for** $x \in \mathbb{R} \cup i\mathbb{R}$**, and**

$$X_+(x) = X_-(x) \begin{pmatrix} 1 & e^{-n(V(x)-ax)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \textbf{for } x > 0,$$

$$X_+(x) = X_-(x) \begin{pmatrix} 1 & 0 & e^{-n(V(x)+ax)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \textbf{for } x < 0,$$

$$X_+(z) = X_-(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & e^{-2naz} \\ 0 & -e^{2naz} & 1 \end{pmatrix} \quad \textbf{for } z \in i\mathbb{R}.$$

RH-X3 $X(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right) \begin{pmatrix} z^n & 0 & 0 \\ 0 & z^{-n/2} & 0 \\ 0 & 0 & z^{-n/2} \end{pmatrix}$ **as** $z \to \infty$**.**

Second transformation $X \mapsto T$

We use g**-functions to define** T

$$T(z) = \begin{pmatrix} e^{n\ell} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} X(z)$$

$$\times \begin{cases} \begin{pmatrix} e^{-n(g_1(z)+\ell)} & 0 & 0 \\ 0 & e^{n(g_1(z)-g_2(z))} & 0 \\ 0 & 0 & e^{ng_2(z)} \end{pmatrix} & \textbf{for } \operatorname{Re} z > 0, \\ \begin{pmatrix} e^{-n(g_1(z)+\ell)} & 0 & 0 \\ 0 & e^{ng_2(z)} & 0 \\ 0 & 0 & e^{n(g_1(z)-g_2(z))} \end{pmatrix} & \textbf{for } \operatorname{Re} z < 0. \end{cases}$$

- **Then** $T(z) = I + \mathcal{O}(1/z)$ **as** $z \to \infty$**.**

Jumps on the real line

The jumps for T on the real axis are

$$T_+ = T_- \begin{pmatrix} e^{-n(g_{1,+}-g_{1,-})} & 1 & 0 \\ 0 & e^{n(g_{1,+}-g_{1,-})} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \textbf{on } [p,q],$$

$$T_+ = T_- \begin{pmatrix} 1 & e^{n(g_{1,+}+g_{1,-}-g_2-V+ax+\ell)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \textbf{on } \mathbb{R}^+ \setminus [p,q],$$

$$T_+ = T_- \begin{pmatrix} e^{-n(g_{1,+}-g_{1,-})} & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & e^{n(g_{1,+}-g_{1,-})} \end{pmatrix} \quad \textbf{on } [-q,-p],$$

$$T_+ = T_- \begin{pmatrix} 1 & 0 & e^{n(g_{1,+}+g_{1,-}-g_{2,+}-V-ax+\ell)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \textbf{on } \mathbb{R}^- \setminus [-q,-p],$$

Simplication of jumps

Use $\boxed{2\varphi_1(z) = V(z) \mp az - 2g_1(z) + g_2(z) - \ell}$

$$T_+ = T_- \begin{pmatrix} e^{2n\varphi_{1,+}} & 1 & 0 \\ 0 & e^{2n\varphi_{1,-}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \textbf{on } [p,q],$$

$$T_+ = T_- \begin{pmatrix} 1 & e^{-2n\varphi_1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \textbf{on } \mathbb{R}^+ \setminus [p,q],$$

$$T_+ = T_- \begin{pmatrix} e^{2n\varphi_{1,+}} & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & e^{2n\varphi_{1,-}} \end{pmatrix} \qquad \textbf{on } [-q,-p],$$

$$T_+ = T_- \begin{pmatrix} 1 & 0 & e^{-2n\varphi_1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \textbf{on } \mathbb{R}^- \setminus [-q,-p],$$

Jumps on the imaginary axis

The jumps for T on the imaginary axis are

$$T_+ = T_- \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & e^{-n(g_{2,+}-g_{2,-}+2az)} \\ 0 & -e^{n(g_{2,+}-g_{2,-}+2az)} & 1 \end{pmatrix} \quad \text{on } i\mathbb{R} \setminus (-ic, ic),$$

$$T_+ = T_- \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & e^{n(g_1-g_{2,+}-g_{2,-})} \end{pmatrix} \quad \text{on } (-ic, ic)$$

- **The right lower 2×2 block is**

$$\begin{pmatrix} 0 & e^{-n(g_{2,+}-g_{2,-}+2az)} \\ -e^{n(g_{2,+}-g_{2,-}+2az)} & e^{n(g_1-g_{2,+}-g_{2,-})} \end{pmatrix}$$

which reduces to above forms. For $3,3$ entry

$$g_1 - g_{2,+} - g_{2,-} = -U^{\mu_1} + 2U^{\mu_2} \quad + \text{ variational (in)equalities}$$

Simplication of jumps

Use $\boxed{2\varphi_2(z) = -g_1(z) + 2g_2(z) \mp 2az}$

$$T_+ = T_- \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & e^{2n\varphi_{2,-}} \\ 0 & -e^{2n\varphi_{2,+}} & 1 \end{pmatrix} \quad \textbf{on } i\mathbb{R} \setminus (-ic, ic),$$

$$T_+ = T_- \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & e^{-2n\varphi_2} \end{pmatrix} \quad \textbf{on } (-ic, ic)$$

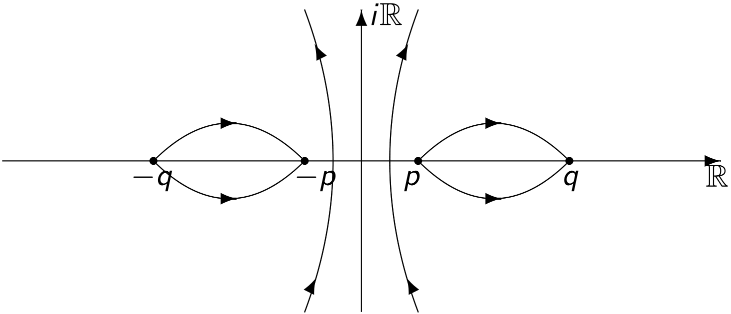
- **First jump matrix has factorization (we only show** 2×2 **block)**

$$\begin{pmatrix} 0 & e^{2n\varphi_{2,-}} \\ -e^{2n\varphi_{2,+}} & 1 \end{pmatrix} = \begin{pmatrix} 1 & e^{2n\varphi_{2,-}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -e^{2n\varphi_{2,+}} & 1 \end{pmatrix}$$

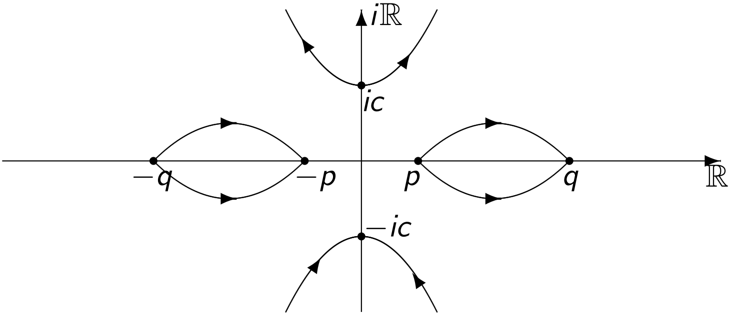
Third transformation

- **In the third transformation $T \mapsto S$ we open up lenses around $\text{supp}(\mu_1)$ and $\text{supp}(\sigma - \mu_2)$.**
- **We use factorizations of the jump matrices for T on these parts.**
- **The lenses look different in the three cases.**

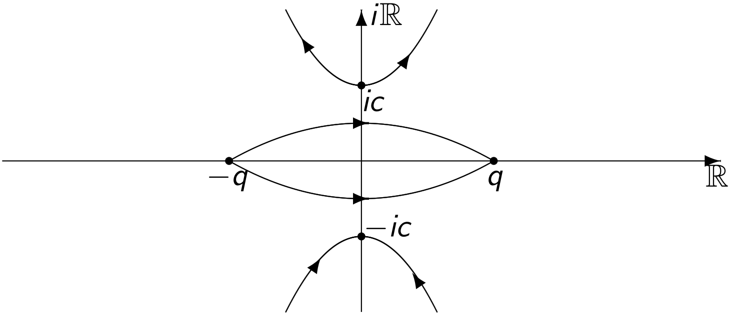
Opening of lenses in case I



Opening of lenses in case II



Opening of lenses in case III



Third transformation

We define S in lenses around $[-q,-p] \cup [p,q]$

$$S = T \begin{pmatrix} 1 & 0 & 0 \\ -e^{2n\varphi_1} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$ **in upper part of lens around $[p,q]$,**

$$S = T \begin{pmatrix} 1 & 0 & 0 \\ e^{2n\varphi_1} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$ **in lower part of lens around $[p,q]$,**

$$S = T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -e^{2n\varphi_1} & 0 & 1 \end{pmatrix},$$ **in upper part of lens around $[-q,-p]$,**

$$S = T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ e^{2n\varphi_1} & 0 & 1 \end{pmatrix},$$ **in lower part of lens around $[-q,-p]$,**

Third transformation (cont.)

We define S **in lenses around** $(-i\infty, -ic] \cup [ic, i\infty)$

$$S = T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & e^{2n\varphi_2} & 1 \end{pmatrix}, \qquad \text{in left part of lens around } (-i\infty, -ic] \cup [ic, i\infty),$$

$$S = T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^{2n\varphi_2} \\ 0 & 0 & 1 \end{pmatrix}, \qquad \text{in right part of lens around } (-i\infty, -ic] \cup [ic, i\infty),$$

and

$$S = T \qquad \text{elsewhere}$$

RH problem for S, part 1

RH-S1 **S is analytic on $\mathbb{C} \setminus \Sigma_S$.**

RH-S2 **The jumps for S on the real axis are**

$$S_+ = S_- \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \textbf{on } [p, q],$$

$$S_+ = S_- \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad \textbf{on } [-q, -p],$$

$$S_+ = S_- \begin{pmatrix} 1 & e^{-2n\varphi_1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \textbf{on } \mathbb{R}^+ \setminus [p, q],$$

$$S_+ = S_- \begin{pmatrix} 1 & 0 & e^{-2n\varphi_1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \textbf{on } \mathbb{R}^- \setminus [-q, -p],$$

RH problem for S, part 2

RH-S2 **The jumps for S on the imaginary axis**

$$S_+ = S_- \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & e^{-2n\varphi_2} \end{pmatrix}$$

on $(-ic, ic)$ and, in case III, outside the lens around $(-q, q)$

$$S_+ = S_- \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -e^{2n(\varphi_1 - \varphi_2)} & -1 & e^{-2n\varphi_2} \end{pmatrix}$$

on $(-ic, ic)$ but inside the lens around $(-q, q)$ (only in case III)

- **All entries in red are exponentially decaying as $n \to \infty$!!**

RH problem for S, part 3

RH-S2 **The jumps for S on the lips of the lenses**

$$S_+ = S_- \begin{pmatrix} 1 & 0 & 0 \\ e^{2n\varphi_1} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ **on lips of lens around $[p, q]$,**

$$S_+ = S_- \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ e^{2n\varphi_1} & 0 & 1 \end{pmatrix}$$ **on lips of lens around $[-q, -p]$,**

$$S_+ = S_- \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -e^{2n\varphi_2} & 1 \end{pmatrix}$$ **on left lip of lens around** $(-i\infty, -ic] \cup [ic, i\infty)$,

$$S_+ = S_- \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & e^{2n\varphi_2} \\ 0 & 0 & 1 \end{pmatrix}$$ **on right lip of lens around** $(-i\infty, -ic] \cup [ic, i\infty)$,

RH-S3 $S(z) = I + \mathcal{O}(1/z)$ **as** $z \to \infty$**.**

Parametrices

- **Global parametrix N should satisfy**

RH-N1 **N is analytic on** $\mathbb{C} \setminus ([-q,-p] \cup [p,q] \cup [-ic,ic])$,

RH-N2 **The jumps for N are**

$$N_+ = N_- \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \textbf{on } (p,q),$$

$$N_+ = N_- \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \textbf{on } (-q,-p),$$

$$N_+ = N_- \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad \textbf{on } (-ic,ic).$$

RH-N3 $N(z) = I + \mathcal{O}(1/z)$ **as** $z \to \infty$

Parametrices

- **Construction of N is not entirely straightforward.**
- **It can be done with the help of the Riemann surface.**

- **Local parametrices P around each of the endpoints $\pm q$, $\pm p$ (not in case III), $\pm ic$ (not in case I) are constructed with the help of Airy functions**

Final transformation

The final transformation $S \mapsto R$ is

$$R(z) = \begin{cases} S(z)N(z)^{-1} & \textbf{away from the branch points,} \\ S(z)P(z)^{-1} & \textbf{near the branch points.} \end{cases}$$

- **All jump conditions in the RH problem for R satisfy**

$$R_+ = R_-(I + \mathcal{O}(1/n)), \qquad \textbf{as } n \to \infty$$

- **It follows that**

$$R(z) = I + \mathcal{O}\left(\frac{1}{n(|z|+1)}\right) \qquad \textbf{as } n \to \infty,$$

uniformly for $z \in \mathbb{C} \setminus \Sigma_R$.

Conclusion of proof

- **Following the transformations in the steepest descent analysis one may now prove the limiting mean eigenvalue density**

$$\lim_{n\to\infty} \frac{1}{n} K_n(x,x) = \rho(x) := \frac{d\mu_1(x)}{dx}$$

in the same as for one matrix model

- **We also find the local scaling limit for**

$$\lim_{n\to\infty} \frac{1}{n\rho(x_0)} \widehat{K}_n\left(x_0 + \frac{x}{n\rho(x_0)}, x_0 + \frac{y}{n\rho(x_0)}\right) = \frac{\sin \pi(x-y)}{\pi(x-y)}$$

whenever $\rho(x_0) > 0$, **where**

$$\widehat{K}_n(x,y) = \frac{e^{\frac{n}{2}(V(y)-ay+g_2(y))}}{e^{\frac{n}{2}(V(x)-ax+g_2(x))}} K_n(x,y),$$

is an equivalent kernel, which generates the same determinantal process as K_n**.**

Explicit calculations

- **Recall: meromorphic function on Riemann surface**

$$\begin{aligned} \xi_1(z) &= V'(z) - F_1(z), && z \in \mathcal{R}_1, \\ \xi_2(z) &= \pm a + F_1(z) - F_2(z), && z \in \mathcal{R}_2, \pm \operatorname{Im} z > 0, \\ \xi_3(z) &= \mp a + F_2(z), && z \in \mathcal{R}_3, \pm \operatorname{Im} z > 0 \end{aligned}$$

- **They are solutions of a cubic equation**

$$\xi^3 - V'(z)\xi^2 + p_1(z)\xi + p_0(z) = 0$$

with polynomial coefficients (spectral curve).

- **We can make explicit calculations for low degree V.**

Quadratic potential

- **Suppose** $V(z) = \frac{1}{2}z^2$
- **Then spectral curve is (Pastur's equation)**

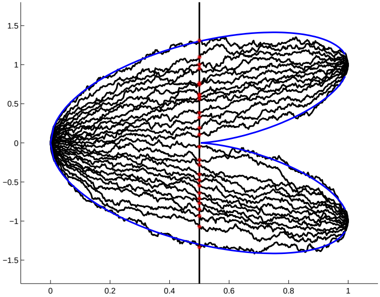
$$\xi^3 - z\xi^2 + (1 - a^2)\xi + a^2 z = 0.$$

- **Always four branch points, so Case II does not happen**
- **If** $a > 1$ **then four real branch points, and we are in Case I.**
- **If** $0 < a < 1$ **then two branch points on imaginary axis, and we are in Case III.**

Pearcey transition

- **Transition from Case I to Case III is Pearcey transition as in the case for non-intersecting Brownian motions.**
- **Density vanishes at the origin**

$$\rho(x) \sim |x|^{1/3}$$



Quartic potential

- **Suppose** $V(z) = \frac{1}{4}z^4 - \frac{t}{2}z^2$.
- **Spectral curve (McLaughlin's equation)**

$$\xi^3 - (z^3 - tz)\xi^2 + p_1(z)\xi + p_0(z) = 0.$$

with

$$p_1(z) = z^2 + \alpha, \qquad p_0(z) = a^2 z^3 + \beta z$$

- **Two undetermined parameters.**

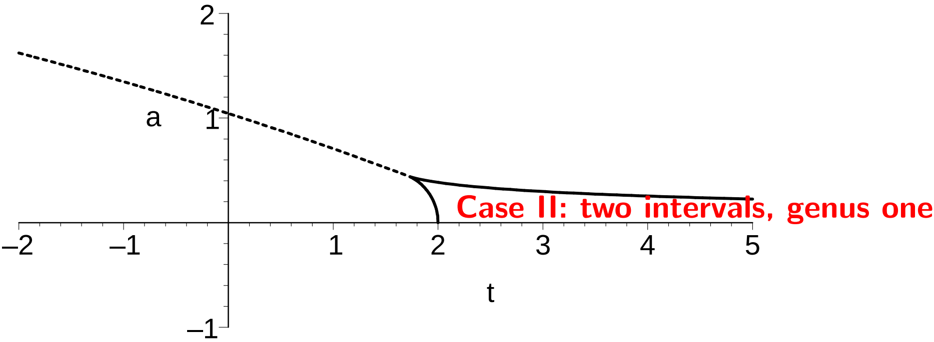
Discriminant analysis

- **Discriminant of spectral curve w.r.t. ξ is a degree 12 polynomial in z.**
- **Branch points are among the zeros of this polynomial. Other zeros have higher even multiplicity.**
- **In Case II there should be a 6 fold zero at 0. This implies $\alpha = \beta = 0$.**
- **Zero is an 8 fold zero if $\alpha = \beta = 0$ and**

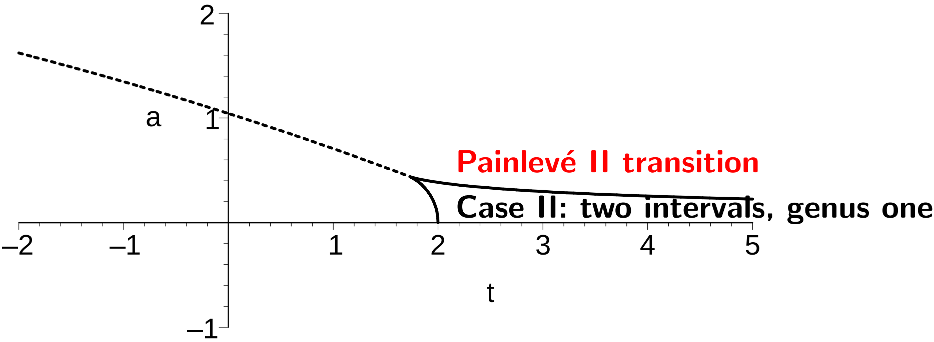
$$-27a^4 + (18t - 4t^3)a^2 - 4 + t^2 = 0$$

- **The Case II region in $t - a$ plane is bounded by two branches of this curve.**

Phase diagram

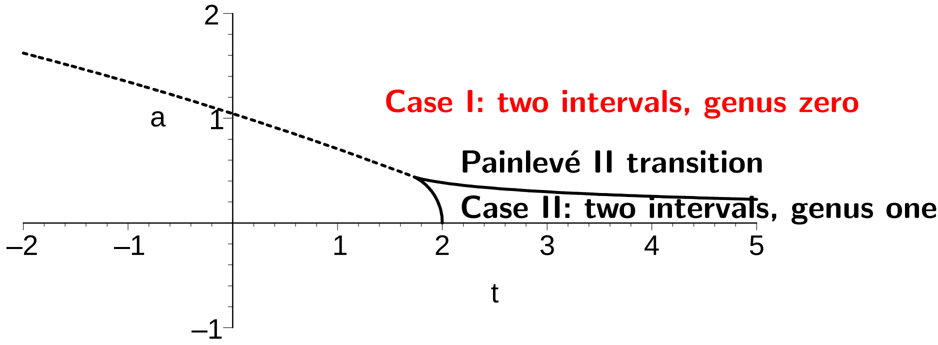


Phase diagram



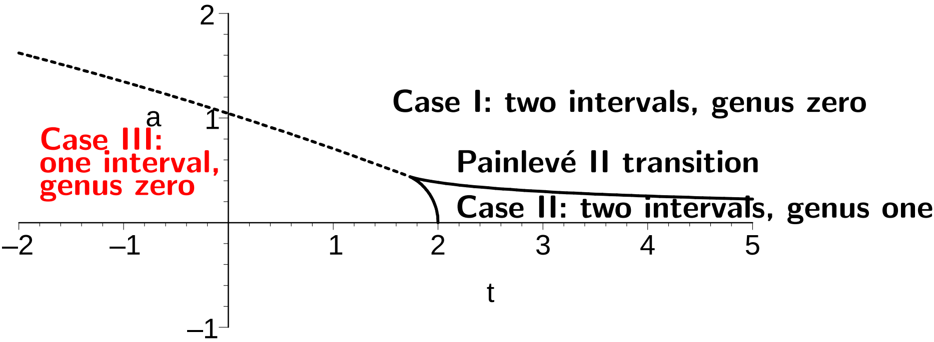
- **Transition from Case II to one of the other cases gives a change in genus. It is a Painlevé II transition**

Phase diagram



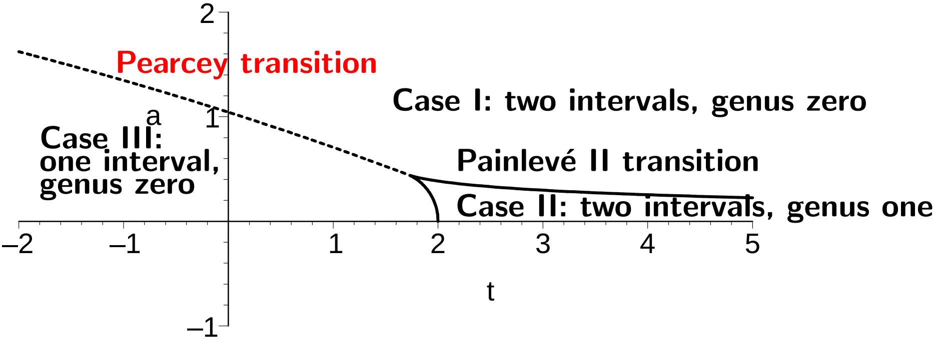
- **If you move up in phase diagram you go to Case I.**

Phase diagram



- If you move the left in phase diagram you go to Case III.

Phase diagram

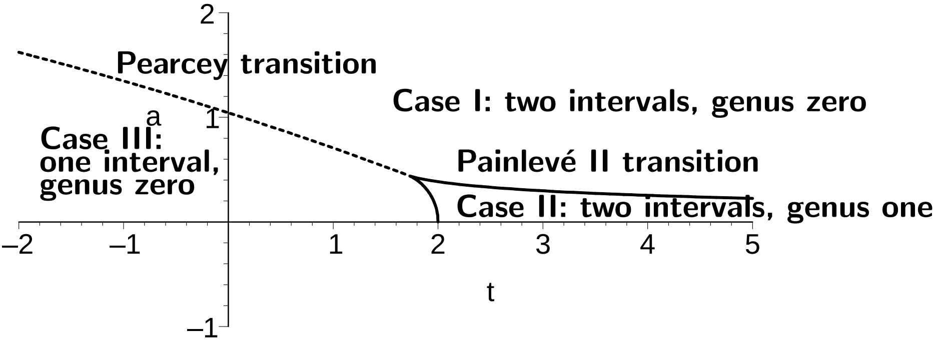


- **Within the genus zero region there is a transition from Case I to Case III.**
- **This happens on the curve**

$$54a^4 + (72t - t^3)a^2 - (t^4 - 16t^2 + 64) = 0$$

- **This is a Pearcey transition**

Phase diagram



- One special point in the phase diagram

$$t_c = 3^{1/2} \quad \textbf{and} \quad a_c = 3^{-3/4}$$

- For these values there is a 10-fold zero of the discriminant at 0.
- New local eigenvalue behavior at 0. Scaling limits are unknown.

That's all

Thank you for your attention