Asymptotic Analysis of Random Matrices and Orthogonal Polynomials

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Les Houches, 5-9 March 2012

Multiple orthogonal polynomials

- **Given weight functions** $w_1, \ldots, w_r$ **on the real line and** $\vec{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$**. Notation** $|\vec{n}| = n_1 + \cdots + n_r$
- **The type II multiple orthogonal polynomial (MOP) is a monic polynomial** $P_{\vec{n}}$ **of degree** $|\vec{n}|$ **such that**

$$\begin{cases} \int P_{\vec{n}}(x) x^k w_1(x)\, dx = 0, & k = 0, 1, \ldots, n_1 - 1, \\ \int P_{\vec{n}}(x) x^k w_2(x)\, dx = 0, & k = 0, 1, \ldots, n_2 - 1, \\ \quad \vdots & \quad \vdots, \\ \int P_{\vec{n}}(x) x^k w_r(x)\, dx = 0, & k = 0, 1, \ldots, n_r - 1, \end{cases}$$

- **These are** $|\vec{n}|$ **conditions for the** $|\vec{n}|$ **free coefficients of** $P_{\vec{n}}$**. In typical cases there is existence and uniqueness, but not always.**

Type I multiple orthogonality

- **Type I multiple orthogonal polynomials are r polynomials $A_{\vec{n}}^{(1)}$, $A_{\vec{n}}^{(2)}, \cdots A_{\vec{n}}^{(r)}$, of degrees**

$$\deg A_{\vec{n}}^{(j)} \leq n_j - 1, \qquad j = 1, \ldots, r$$

- **They are such that the linear form**

$$Q_{\vec{n}}(x) = A_{\vec{n}}^{(1)}(x) w_1(x) + \cdots + A_{\vec{n}}^{(r)}(x) w_r(x)$$

satisfies

$$\begin{cases} \int x^k Q_{\vec{n}}(x)\, dx = 0, & k = 0, 1, \ldots, |\vec{n}| - 2, \\ \int x^k Q_{\vec{n}}(x)\, dx = 1, & k = |\vec{n}| - 1. \end{cases}$$

Block Hankel matrix

- **Moments** $\mu_j^{(i)} = \int x^j w_i(x) dx$
- $n \times m$ **Hankel matrix for** i**th weight**

$$H_{n,m}^{(i)} = \left(\mu_{j+k-2}^{(i)}\right)_{j=1,\dots,n,k=1,\dots,m}$$

- **Block Hankel matrix**

$$H_{\vec{n}} = \begin{bmatrix} H_{n,n_1}^{(1)} & \cdots & H_{n,n_r}^{(r)} \end{bmatrix}, \qquad n = |\vec{n}|$$

- **Conditions for type I MOPs give linear system with matrix** $H_{\vec{n}}$**.**
- **Conditions for type II MOP give linear system with matrix** $H_{\vec{n}}^T$**.**
- **Both type of MOPs exist if and only if**

$$\det H_{\vec{n}} \neq 0.$$

Riemann-Hilbert problem (case $r = 2$)

In the RH problem we look for a 3×3 matrix valued function $Y(z)$ satisfying

RH-Y1 $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{3\times 3}$ **is analytic.**

RH-Y2 Y **has boundary values for $x \in \mathbb{R}$, denoted by $Y_\pm(x)$, and**

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w_1(x) & w_2(x) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \text{for } x \in \mathbb{R}.$$

RH-Y3 **As $z \to \infty$,**

$$Y(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right) \begin{pmatrix} z^{n_1+n_2} & 0 & 0 \\ 0 & z^{-n_1} & 0 \\ 0 & 0 & z^{-n_2} \end{pmatrix}$$

Solution in terms of type II MOPs

Theorem (Van Assche, Geronimo, K (2001))

RH problem has a unique solution if and only if the type II MOP $P_{\vec{n}}$ uniquely exists.
In that case the first row of Y is given by

$$\begin{pmatrix} P_{\vec{n}}(z) & \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{P_{\vec{n}}(s)w_1(s)}{s-z}ds & \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{P_{\vec{n}}(s)w_2(s)}{s-z}ds \\ * & * & * \\ * & * & * \end{pmatrix}$$

- **Other rows are filled using $P_{\vec{n}-\vec{e}_1}$ and $P_{\vec{n}-\vec{e}_2}$ (if they exist).**

Inverse of Y

- **The type I MOPs are in the inverse of Y.**

$$Y^{-1}(z) = \begin{pmatrix} -\int_{-\infty}^{\infty} \frac{Q_{\vec{n}}(s)}{s-z}\,ds & * & * \\ 2\pi i A_{\vec{n}}^{(1)}(z) & * & * \\ 2\pi i A_{\vec{n}}^{(2)}(z) & * & * \end{pmatrix}$$

where $Q_{\vec{n}} = A_{\vec{n}}^{(1)} w_1 + A_{\vec{n}}^{(2)} w_2$

- **Other columns contain type I MOPs with multi-indices $\vec{n} + \vec{e}_1$ and $\vec{n} + \vec{e}_2$.**

Biorthogonal ensembles

- **Probability density function on $\mathbb{R}^n$ of the form**

$$\frac{1}{Z_n} \det\left[f_i(x_j)\right]_{i,j=1}^n \cdot \det\left[g_i(x_j)\right]_{i,j=1}^n ,$$

- **Normalization constant**

$$Z_n = \int_{\mathbb{R}^n} \det\left[f_i(x_j)\right]_{i,j=1}^n \cdot \det\left[g_i(x_j)\right]_{i,j=1}^n \; dx_1 \cdots dx_n \neq 0$$

- **By Andréief (1883) identity**

$$Z_n = n! \det M_n, \qquad M_n = \left[\int_{-\infty}^{\infty} f_i(x) g_j(x)\, dx\right]_{i,j=1}^n$$

- **Corollary:** $\det M_n \neq 0$

Correlation kernel

- **Biorthogonal ensemble is a determinantal point process with correlation kernel**

$$K_n(x,y) = \sum_{i=1}^{n}\sum_{j=1}^{n} \left(M_n^{-1}\right)_{ji} f_i(x) g_j(y).$$

- **Representation as determinant**

$$K_n(x,y) = -\frac{1}{\det M_n} \det \begin{bmatrix} & M_n & & \begin{matrix} f_1(x) \\ \vdots \\ f_n(x) \end{matrix} \\ g_1(y) & \cdots & g_n(y) & 0 \end{bmatrix}$$

- **Perform elementary row and column transformations to transform M_n to the identity matrix I_n**

Correlation kernel (cont.)

- **After transformation** $M_n \mapsto I_n$

$$K_n(x,y) = -\det \begin{bmatrix} & I_n & & \begin{matrix}\phi_1(x)\\ \vdots \\ \phi_n(x)\end{matrix} \\ \psi_1(y) & \cdots & \psi_n(y) & 0 \end{bmatrix}$$

with functions ϕ_j **and** ψ_j **satisfying**

$$\int_{-\infty}^{\infty} \phi_i(x)\psi_j(x)\,dx = \delta_{i,j} \qquad \textbf{(biorthogonality)}$$

- **Also single sum** $\quad K_n(x,y) = \sum_{j=1}^{n} \phi_j(x)\psi_j(y)$

Correlation kernel (cont.)

- **Characterization:**
  K_n **is the kernel of the projection operator onto the linear span of** $f_1, \ldots, f_n$**, whose kernel is the orthogonal complement of the linear span of** $g_1, \ldots, g_n$**.**
- **Operator**

$$K_n : h \mapsto K_n h, \qquad K_n h(x) = \int K_n(x,y) h(y)\, dy$$

- **Characterization**

$$K_n h = h \quad \textbf{if } h = \alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n,$$

$$K_n h = 0 \quad \textbf{if } \int h(x) g_j(x) dx = 0 \textbf{ for } j = 1, \ldots, n.$$

MOP ensembles

Definition

A **multiple orthogonal polynomial (MOP) ensemble** **is a biorthogonal ensemble with functions**

$$\begin{aligned} f_i(x) &= x^{i-1}, && \textbf{for } i = 1, \dots, n, \\ g_i(y) &= y^{i-1} w_1(y), && \textbf{for } i = 1, \dots, n_1, \\ g_{n_1+i}(y) &= y^{i-1} w_2(y), && \textbf{for } i = 1, \dots, n_2, \\ &\vdots \\ g_{n_1+\cdots+n_{r-1}+i}(y) &= y^{i-1} w_r(y), && \textbf{for } i = 1, \dots, n_r. \end{aligned}$$

Here $w_1, \dots, w_r$ **are given functions, and** $n_1, \dots, n_r$ **are non-negative integers such that**

$$n = n_1 + \cdots + n_r.$$

Block Hankel matrix

- **In a MOP ensemble the matrix M_n is the block Hankel matrix**

$$M_n = H_{\vec{n}} = \begin{bmatrix} H^{(1)}_{n,n_1} & \cdots & H^{(r)}_{n,n_r} \end{bmatrix}, \qquad n = |\vec{n}|$$

- $\det H_{\vec{n}} \neq 0$ **and so the MOPs exist.**
- **The RH problem has a unique solution.**

Christoffel Darboux formula

Theorem (Bleher-K (2004) for $r = 2$, Daems-K (2004))

The correlation kernel K_n for the MOP ensemble is given by

$$K_n(x,y) = \frac{1}{2\pi i(x-y)} \times$$

$$\begin{pmatrix} 0 & w_1(y) & \cdots & w_r(y) \end{pmatrix} Y_+^{-1}(y) Y_+(x) \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Proof for case $r = 2$

- **Assume** $r = 2$**. Let** $L_n(x, y)$ **be the right-hand side**

$$L_n(x,y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & w_1(y) & w_2(y) \end{pmatrix} Y_+^{-1}(y) Y_+(x) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

We show

(a) $L_n h = h$ **if** h **is a polynomial of degree** $\leq n-1$**,**

(b) $L_n h = 0$ **if** $\int h(y) y^{j-1} w_i(y)\, dy = 0$ **for** $j = 1, \ldots, n_i$**, and** $i = 1, 2$**.**

Proof of (a)

- **Let** h **be a polynomial of degree** $\leq n-1$**.**

$$L_n(x,y)h(y) = \frac{h(y)}{2\pi i(x-y)} \begin{pmatrix} 0 & w_1(y) & w_2(y) \end{pmatrix} Y_+^{-1}(y) Y_+(x) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{h(y)-h(x)}{2\pi i(x-y)} \begin{pmatrix} 0 & w_1(y) & w_2(y) \end{pmatrix} Y_+^{-1}(y) Y_+(x) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$+ \frac{h(x)}{2\pi i(x-y)} \begin{pmatrix} 0 & w_1(y) & w_2(y) \end{pmatrix} Y_+^{-1}(y) Y_+(x) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

- $\int L_n(x,y)h(y)dy$ **splits into two integrals.**

Proof of (a), first integral

- **First integral has**

$$\underbrace{\frac{h(y)-h(x)}{2\pi i(x-y)}}_{\substack{\textbf{polynomial in y}\\ \textbf{of degree } \leq n-2}} \quad \underbrace{\begin{pmatrix}0 & w_1(y) & w_2(y)\end{pmatrix} Y_+^{-1}(y)}_{\substack{\textbf{vector with linear forms}\\ \textbf{of type I MOPs}}} \quad Y_+(x) \begin{pmatrix}1\\0\\0\end{pmatrix}$$

- **Integral with respect to y is 0 for every x because of type I multiple orthogonality.**

Proof of (a), second integral

- **Second integral is**

$$\frac{h(x)}{2\pi i} \int_{-\infty}^{\infty} \begin{pmatrix} 0 & w_1(y) & w_2(y) \end{pmatrix} Y_+^{-1}(y) Y_+(x) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \frac{dy}{x-y}$$

- **From jump condition in RH problem**

$$\begin{pmatrix} 0 & w_1(y) & w_2(y) \end{pmatrix} Y_+^{-1}(y) = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \left(Y_-^{-1}(y) - Y_+^{-1}(y) \right)$$

- **It remains to prove**

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \left[\frac{Y_-^{-1}(y) - Y_+^{-1}(y)}{x-y} Y_+(x) \right]_{1,1} dy = 1.$$

Proof of (a), second integral (cont.)

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\left[\frac{Y_-^{-1}(y)-Y_+^{-1}(y)}{x-y}Y_+(x)\right]_{1,1}dy = 1.$$

- **Replace** $x \in \mathbb{R}$ **by** z **with** $\operatorname{Im} z > 0$.
- $y \mapsto \left[\frac{Y_-^{-1}(y)}{z-y}Y(z)\right]_{1,1}$ **is analytic in lower half plane and is** $O(y^{-n-1})$ **as** $y \to \infty$. **By Cauchy's theorem**
$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\left[\frac{Y_-^{-1}(y)}{z-y}Y(z)\right]_{1,1}dy = 0$$
- $y \mapsto \left[\frac{Y_+^{-1}(y)}{z-y}Y(z)\right]_{1,1}$ **has pole in upper half plane and same behavior at infinity. By residue calculation**
$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\left[\frac{Y_+^{-1}(y)}{z-y}Y(z)\right]_{1,1}dy = -1$$
- **Subtract the two results and then let** $z \to x \in \mathbb{R}$.

Proof of (b)

- **Assume** $\int h(y) y^{j-1} w_i(y) dy = 0$ **for** $j = 1, \ldots, n_j$, $i = 1, 2$. **We have to prove** $L_n h(x) = 0$
- **We have that** $L_n h(x) =$

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} h(y) \begin{pmatrix} 0 & w_1(y) & w_2(y) \end{pmatrix} \frac{Y_+^{-1}(y) - Y_+^{-1}(x)}{x-y} Y_+(x) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} dy$$

$$+ \frac{1}{2\pi i} \int_{-\infty}^{\infty} h(y) \begin{pmatrix} 0 & w_1(y) & w_2(y) \end{pmatrix} \frac{Y_+^{-1}(x)}{x-y} Y_+(x) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} dy.$$

- **Second integral is obviously zero.**
- **In first integral we can take out** $Y_+(x) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Proof of (b), (cont.)

- We are left to evaluate

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} h(y)\begin{pmatrix}0 & w_1(y) & w_2(y)\end{pmatrix}\frac{Y_+^{-1}(y)-Y_+^{-1}(x)}{x-y}dy$$

- Second row of Y^{-1} has polynomials of degree $\leq n_1$
- Third row of Y^{-1} has polynomials of degree $\leq n_2$
- Hence for every x, the entries of

$$\begin{pmatrix}0 & w_1(y) & w_2(y)\end{pmatrix}\frac{Y_+^{-1}(y)-Y_+^{-1}(x)}{x-y}$$

  take the form

  $w_1(y)$(poly of deg $\leq n_1-1$)$+w_2(y)$(poly of deg $\leq n_2-1$)

- This is in the linear span of $g_1,\ldots,g_n$ and the integral is zero.

Examples of MOP ensembles

- **Non-intersecting Brownian motions**
- **Non-intersecting squared Bessel paths**
- **Random matrix model with external source**
- **Two matrix model**

Non-intersecting Brownian motions

- **Brownian motion transition probability density**

$$p_t(x,y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}}$$

- **Biorthogonal ensemble of non-intersecting Brownian motions**

$$\frac{1}{Z_n} \det\left[p_t(a_i, x_j)\right]_{i,j=1}^n \cdot \det\left[p_{T-t}(x_i, b_j)\right]_{i,j=1}^n$$

with Z_n depending on $a_1, \ldots, a_n$ and $b_1, \ldots . b_n$.

- **In confluent limit where all $a_j \to a$ both Z_n and the first determinant tend to 0.**
- **Take limit using L'Hôpital's rule. First determinant becomes**

$$\det\left[\frac{\partial^{i-1}}{\partial a^{i-1}} p_t(a, x_j)\right]_{i,j=1}^n .$$

Non-intersecting Brownian motions

- **From** $p_t(a,x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-a)^2}{2t}}$ **we get**

$$\frac{\partial^{i-1}}{\partial a^{i-1}} p_t(a,x) = \left(\begin{array}{c} \textbf{polynomial in } x \\ \textbf{of degree } i-1 \end{array} \right) e^{-\frac{1}{2t}(x^2-2ax)}$$

- **Apply appropriate row operations to the determinant and take out common factors from each column**

$$\det\left[\frac{\partial^{i-1}}{\partial a^{i-1}} p_t(a,x_j)\right]_{i,j=1}^{n} \propto \det\left[x_j^{i-1}\right]_{i,j=1}^{n} \cdot \prod_{j=1}^{n} e^{-\frac{1}{2t}(x_j^2-2ax_j)}$$

- **Similarly when all** $b_j \to b$ **we get a second factor**

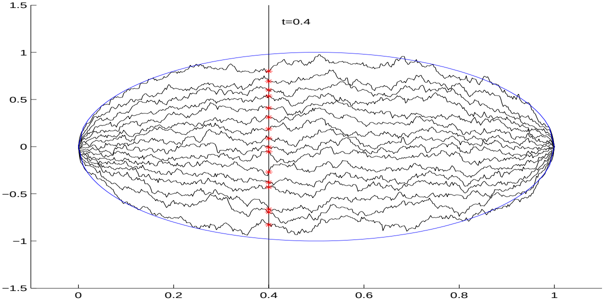
$$\propto \det\left[x_j^{i-1}\right]_{i,j=1}^{n} \cdot \prod_{j=1}^{n} e^{-\frac{1}{2(T-t)}(x_j^2-2bx_j)}$$

Non-intersecting Brownian motions

- **In** **fully confluent limit** **all** $a_j \to a$, **all** $b_j \to b$, **we find an OP ensemble with quadratic potential** $(=$ **GUE)**

$$\frac{1}{Z_n}\left(\det\left[x_j^{i-1}\right]_{i,j=1}^{n}\prod_{j=1}^{n}e^{-\frac{1}{2t}(x_j^2-2ax_j)}\right)\left(\det\left[x_j^{i-1}\right]_{i,j=1}^{n}\prod_{j=1}^{n}e^{-\frac{1}{2(T-t)}(x_j^2-2bx_j)}\right)$$

$$=\frac{1}{Z_n}\Delta(x)^2\prod_{j=1}^{n}e^{-V(x_j)},\qquad V(x)=\frac{T}{t(T-t)}\left(\frac{x^2}{2}-((1-\tfrac{t}{T})a+\tfrac{t}{T}b)x\right)$$



Non-intersecting Brownian motions

- **If n_1 of the b_j's tend to b_1 and the remaining n_2 tend to b_2, then we have to treat the first n_1 rows separately from the last n_2 rows in taking the confluent limit.**
- **The second determinant now becomes**

$$\det\left[g_i(x_j)\right]_{i,j=1}^{n}$$

with functions

$$g_i(x) = x^{i-1} e^{-\frac{1}{2(T-t)}(x^2-2b_1x)}, \qquad i = 1, \ldots, n_1,$$
$$g_{n_1+i}(x) = x^{i-1} e^{-\frac{1}{2(T-t)}(x^2-2b_2x)}, \qquad i = 1, \ldots, n_2.$$

- **Together with** $\Delta(x) \prod_{j=1}^{n} e^{-\frac{1}{2t}(x_j^2-2ax_j)}$ **we now find a MOP ensemble with two weights and $\vec{n} = (n_1, n_2)$.**

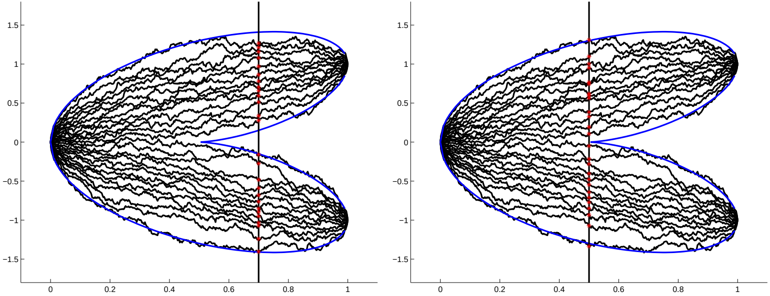
Non-intersecting Brownian motions

- **Two Gaussian weights**

$$w_i(x) = e^{-V_i(x)}, \qquad V_i(x) = \frac{T}{t(T-t)}\left(\frac{x^2}{2} - c_i x\right),$$

where $c_i = (1 - \frac{t}{T})a + \frac{t}{T} b_i$ **for** $i = 1, 2$**.**

- **Associated MOPs are multiple Hermite polynomials**



Non-intersecting squared Bessel paths

- **Squared Bessel processes is diffusion process on $[0, \infty)$ with transition probability density**

$$p_t(x, y) = \frac{1}{2t} \left(\frac{y}{x}\right)^{\alpha/2} e^{-\frac{1}{2t}(x+y)} I_\alpha \left(\frac{\sqrt{xy}}{t}\right), \qquad x, y > 0,$$

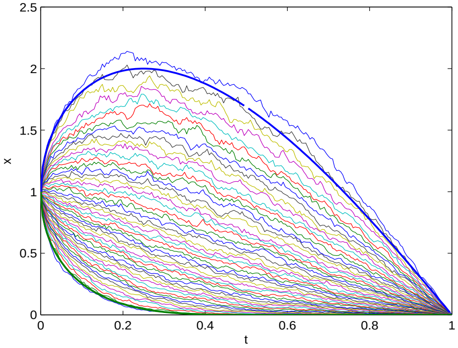
- I_α **is the modified Bessel function of order $\alpha > -1$.**
- **In confluent limit all $a_j \to a$, all $b_j \to 0$, this leads to a MOP ensemble with two weights**

$$w_1(x) = x^{\alpha/2} e^{-\frac{Tx}{2t(T-t)}} I_\alpha \left(\frac{\sqrt{ax}}{t}\right)$$

$$w_2(x) = x^{(\alpha+1)/2} e^{-\frac{Tx}{2t(T-t)}} I_{\alpha+1} \left(\frac{\sqrt{ax}}{t}\right)$$

and $n_1 = \lceil n/2 \rceil$, $n_2 = \lfloor n/2 \rfloor$

Non-intersecting squared Bessel paths



- **For $a \to 0$ this further reduces to a** **Laguerre unitary ensemble** **(LUE)**

Random matrix model with external source

- **Hermitian matrix model with external source**

$$\frac{1}{Z_n} e^{-\operatorname{Tr}(V(M)-AM)}\, dM$$

- **External source** A **is a given Hermitian** $n \times n$ **matrix**
- **Joint p.d.f. for eigenvalues**

$$P(x_1, \ldots, x_n) \propto \Delta(x)^2 \prod_{j=1}^{n} e^{-V(x_j)} \int_{U(n)} e^{\operatorname{Tr} AUXU^{-1}}\, dU$$

where $A = \operatorname{diag}(a_1, \ldots, a_n)$**,** $X = \operatorname{diag}(x_1, \ldots, x_n)$**.**

- **The integral over the unitary group can be done by the Harish-Chandra / Itzykson-Zuber formula.**

Random matrix model with external source

- **If all a_i and all x_j are distinct then**

$$\int_{U(n)} e^{\mathrm{Tr}\, AUXU^{-1}} dU \propto \frac{\det\left[e^{a_i x_j}\right]_{i,j=1}^{n}}{\Delta(a)\Delta(x)}$$

- **P.d.f. for eigenvalues**

$$\propto \Delta(x) \prod_{j=1}^{n} e^{-V(x_j)} \frac{\det\left[e^{a_i x_j}\right]_{i,j=1}^{n}}{\Delta(a)}$$

- **If some a_i's coincide, we take the confluent limit.**
- **If n_1 of the a_j's tend to c_1 and $n_2 = n - n_1$ to c_2, then we find MOP ensemble with two weights**

$$w_1(x) = e^{-(V(x)-c_1 x)}, \qquad w_2(x) = e^{-(V(x)-c_2 x)}.$$

Random matrix model with external source

- **MOP ensemble with weights**

$$w_1(x) = e^{-(V(x)-c_1x)}, \qquad w_2(x) = e^{-(V(x)-c_2x)}$$

  and $\vec{n} = (n_1, n_2)$**.**
- **In Gaussian case** $V(x) = \frac{1}{2}x^2$**, the eigenvalues in the external source model have the same joint distribution has the positions of non-intersecting Brownian motions with one starting position and two ending positions.**
- **If** V **is non-Gaussian then we have something else.**

Two matrix model

- **The Hermitian two matrix model**

$$\frac{1}{Z_n} e^{-\operatorname{Tr}(V(M_1)+W(M_2)-\tau M_1 M_2)}\, dM_1\, dM_2$$

is a probability measure on pairs (M_1, M_2) of $n \times n$ Hermitian matrices.

- V **and** W **are polynomial potentials**
- $\tau \neq 0$ **is a coupling constant**

Determinantal point process

- **Explicit formula for joint p.d.f. of the eigenvalues of M_1 and M_2**

$$\frac{1}{(n!)^2} \det \begin{pmatrix} K_{11}(x_i, x_j) & K_{12}(x_i, y_j) \\ K_{21}(y_i, x_j) & K_{22}(y_i, y_j) \end{pmatrix}$$

with 4 **kernels that are expressed in terms of** **biorthogonal polynomials**

- **Two sequences** $(p_j)_j$ **and** $(q_k)_k$ **of monic polynomials that satisfy if** $j \neq k$**,**

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_j(x) q_k(y) e^{-(V(x)+W(y)-\tau xy)} dxdy = h_k^2 \delta_{j,k}.$$

Mehta-Shukla (1994), Eynard-Mehta (1998)
Ercolani-McLaughlin (2001)
Bertola-Eynard-Harnad (2002-04)

Kernels

- **The kernels are expressed in terms of these biorthogonal polynomials and transformed functions**

$$Q_j(x) = \int_{-\infty}^{\infty} q_j(y) e^{-(V(x)+W(y)-\tau xy)} dy,$$

$$P_k(y) = \int_{-\infty}^{\infty} p_k(x) e^{-(V(x)+W(y)-\tau xy)} dx,$$

as follows:

$$K_{11}(x_1, x_2) = \sum_{k=0}^{n-1} \frac{1}{h_k^2} p_k(x_1) Q_k(x_2), \qquad K_{12}(x, y) = \sum_{k=0}^{n-1} \frac{1}{h_k^2} p_k(x) q_k(y),$$

$$K_{21}(y, x) = \sum_{k=0}^{n-1} \frac{1}{h_k^2} P_k(y) Q_k(x) - e^{-(V(x)+W(y)-\tau xy)}, \qquad K_{22}(y_1, y_2) = \sum_{k=0}^{n-1} \frac{1}{h_k^2} P_k(y_1) q_k(y_2)$$

Biorthogonality

- **Biorthogonality condition for p_n**

$$\int_{-\infty}^{\infty} p_n(x)Q_k(x)dx = 0 \qquad \textbf{for } k = 0, 1, \ldots, n-1$$

where $Q_k(x) = e^{-V(x)} \int_{-\infty}^{\infty} q_k(y)e^{-(W(y)-\tau xy)}dy$.

- **Equivalently, we may replace $q_k(y)$ by y^{k-1}**

$$w_k(x) = e^{-V(x)} \int_{-\infty}^{\infty} y^{k-1}e^{-(W(y)-\tau xy)}dy,$$

and $\int_{-\infty}^{\infty} p_n(x)w_k(x)dx = 0$ **for** $k = 1, \ldots, n$.

- **We integrate by parts if $k \geq \deg W$.**

Biorthogonality

- **Calculation for $W(y) = \frac{1}{4}y^4$, $k \geq 4$.**

$$
\begin{aligned}
w_k(x) &= e^{-V(x)} \int_{-\infty}^{\infty} y^{k-1} e^{-\left(\frac{1}{4}y^4 - \tau xy\right)} dy \\
&= -e^{-V(x)} \int_{-\infty}^{\infty} y^{k-4} e^{\tau xy} d\left(e^{-\frac{1}{4}y^4}\right) \\
&= e^{-V(x)} \int_{-\infty}^{\infty} \left((k-4)y^{k-5} + \tau xy^{k-4}\right) e^{-\left(\frac{1}{4}y^4 - \tau xy\right)} dy \\
&= (k-4)w_{k-4}(x) + \tau x w_{k-3}(x).
\end{aligned}
$$

- **This leads to multiple orthogonality**

Multiple orthogonality

Proposition (K-McLaughlin (2005))

Suppose $\deg W = r + 1$. **Then the biorthogonal polynomial** p_n **is a multiple orthogonal polynomial with** r **weights** $w_1, \ldots, w_r$, **and near-diagonal multi-index** $(n_1, \ldots, n_r)$.
If n **is a multiple of** r **then** $n_j = \frac{n}{r}$ **for every** j.

- **The eigenvalues of** M_1, **when averaged over** M_2, **are a MOP ensemble with** r **weights.**
- **There is a RH problem of size** $(r + 1) \times (r + 1)$.
- **Asymptotic analysis of this RH problem was done for** $W(y) = \frac{1}{4}y^4$ **by Duits-K (2009) and for** $W(y) = \frac{1}{4}y^4 + \frac{\alpha}{2}y^2$ **by Duits-K-Mo (2012)**