Asymptotic Analysis of Random Matrices and Orthogonal Polynomials

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Point processes

- A configuration $\mathcal{X}$ is a subset of $\mathbb{R}$ with $\#(\mathcal{X} \cap [a, b]) < +\infty$ for every bounded interval $[a, b] \subset \mathbb{R}$.
- A (locally finite) point process on $\mathbb{R}$ is a probability measure on the space of all configurations.
- A point process $\mathbb{P}$ is an n-point process if

$$\mathbb{P}(\#\mathcal{X} = n) = 1.$$

  - If $P(x_1, \ldots, x_n)$ is a probability density function on $\mathbb{R}^n$ which is invariant under permutation of coordinates,

$$P(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = P(x_1, \ldots, x_n)$$

  then P defines an n-point process.

Correlation functions

- **The 1-point correlation function** $\rho_1(x)$ **of** $\mathcal{X}$ **satisfies**

$$\int_A \rho_1(x)dx = \mathbb{E}[\#(\mathcal{X} \cap A)]$$

  $\rho_1(x)$ **is the particle density.**

- **The 2-point correlation function** $\rho_2(x, y)$ **is such that**
  - **for disjoint sets** A **and** B

$$\int_A \int_B \rho_2(x, y)dxdy = \mathbb{E}\left[\#(x, y) \in \mathcal{X}^2 \mid x \in A,\ y \in B\right],$$

  - **for any set** A

$$\int_A \int_A \rho_2(x, y)dxdy = \mathbb{E}\left[\#(x, y) \in \mathcal{X}^2 \mid x \in A,\ y \in A,\ x < y\right].$$

Higher order correlation functions

- **The** ***k*-point correlation function** ρ_k **(if it exists) has**
  - **for disjoint sets** A_j

$$\int_{A_1} \cdots \int_{A_k} \rho_k(x_1, \ldots, x_k) dx_1 \cdots dx_k = \mathbb{E}\left[\#(x_1, \ldots, x_k) \in \mathcal{X}^k \mid x_j \in A_j\right],$$

  - **for a single set** A

$$\int_{A} \cdots \int_{A} \rho_k(x_1, \ldots, x_k) dx_1 \cdots dx_k = \mathbb{E}\left[\#(x_1, \ldots, x_k) \in (\mathcal{X} \cap A)^k \mid x_1 < \cdots < x_k\right].$$

Marginal densities

- **For an invariant pdf $P(x_1, \ldots, x_n)$ on $\mathbb{R}^n$ the n-point process has correlation functions**

$$\rho_k(x_1, \ldots, x_k) = \frac{n!}{(n-k)!} \underbrace{\int \cdots \int}_{n-k \text{ times}} P(x_1, \ldots, x_n) dx_{k+1} \cdots dx_n$$

Determinantal point process

- **A point process with correlation functions ρ_k is determinantal (fermionic) if there exists a kernel $K(x, y)$ such that**

$$\rho_k(x_1, \dots, x_k) = \det \left[K(x_i, x_j)\right]_{i,j=1}^{k}$$

for every k and every $x_1, \dots, x_k$.

  - **K is called the correlation kernel.**

Biorthogonal ensembles

- **An n-point process is a biorthogonal ensemble if there exist two sequences of functions $f_1, \ldots, f_n$ and** $g_1, \ldots, g_n$

$$P(x_1, x_2, \ldots, x_n) = \frac{1}{Z_n} \det[f_i(x_j)]_{i,j=1}^{n} \cdot \det[g_i(x_j)]_{i,j=1}^{n}.$$

- **This is a determinantal point process with correlation kernel**

$$K_n(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{n} f_i(x) g_j(y) \left[M^{-1}\right]_{j,i}$$

where M is the matrix

$$M = (M_{i,j}), \qquad M_{i,j} = \int f_i(x) g_j(x) dx$$

Biorthogonal functions

- **We may find** $\phi_j \in \mathrm{span}\{f_1, \ldots, f_j\}$, $\psi_k \in \mathrm{span}\{g_1, \ldots, g_k\}$, **such that**

$$\int_{-\infty}^{\infty} \phi_j(x)\psi_k(x)dx = \delta_{jk}.$$

- **Then**

$$K_n(x,y) = \sum_{j=1}^{n} \phi_j(x)\psi_j(y)$$

and

$$P(x_1, \ldots, x_n) = \frac{1}{n!} \det[K_n(x_i, x_j)]_{i,j=1}^{n}.$$

- **An OP ensemble has** $f_j(x) = g_j(x) = \sqrt{w(x)}\; x^{j-1}$
- **Other examples come from non-intersecting paths.**

The Karlin-McGregor theorem (1959)

- **Let $p_t(a;x)$ be the transition probability density of a one-dimensional strong Markov process with continuous sample paths.**
- **Consider n independent copies $X_1(t), \ldots, X_n(t)$ conditioned so that**

$$X_j(0) = a_j$$

where $a_1 < a_2 < \cdots < a_n$ are given values. Let E_1, ..., E_n be Borel sets so that $\sup E_j < \inf E_{j+1}$ for $j = 1, \ldots, n-1$.

- **Then**

$$\int_{E_1} \cdots \int_{E_n} \det\left[p_t(a_i, x_j)\right]_{i,j=1}^{n} dx_1 \cdots dx_n$$

is equal to the probability that $X_j(t) \in E_j$ for $j = 1, \ldots, n$ in such a way that the paths have not intersected in the time interval $[0, t]$

Proof of the Karlin-McGregor theorem, step 1

- **Write**

$$p_t(a_i, E_j) = \int_{E_j} p_t(a_i, x_j)dx_j$$

so that we have the determinant

$$\det \left[p_t(a_i, E_j)\right]_{i,j=1}^n .$$

- **Expand the determinant**

$$\begin{aligned}\det \left[p_t(a_i, E_j)\right]_{i,j=1}^n &= \sum_\sigma \operatorname{sgn}(\sigma) \prod_{j=1}^n p_t(a_j, E_{\sigma(j)}) \\ &= \sum_\sigma \operatorname{sgn}(\sigma) P(A_\sigma),\end{aligned}$$

where for a permutation σ, we use A_σ to denote the event that $X_j(t) \in E_{\sigma(j)}$ for every $j = 1, \ldots, n$.

Proof of the Karlin-McGregor theorem, step 2

- **We decompose**

$$A_\sigma = B_\sigma \cup C_\sigma$$

where

  - B_σ **is the event that** $X_j(t) \in E_{\sigma(j)}$ **for** $j = 1, \dots, n$ **and the paths have not intersected in the time interval** $[0, t]$**, and**
  - $C_\sigma = A_\sigma \setminus B_\sigma$**.**

- **If** $\sigma \neq \mathrm{id}$ **then** $P(B_\sigma) = 0$ **(because of continuous sample paths). Hence**

$$\det \left[p_t(a_i, E_j) \right]_{i,j=1}^n = P(B_{\mathbf{id}}) + \sum_\sigma \mathrm{sgn}(\sigma) P(C_\sigma).$$

- **It remains to show that** $\displaystyle\sum_\sigma \mathrm{sgn}(\sigma) P(C_\sigma) = 0$**.**

Proof of the Karlin-McGregor theorem, step 3

- **For a transposition $\tau = (i, i')$, we use $C_{\sigma,\tau}$ to denote the event**
  (1) $X_j(t) \in E_{\sigma(j)}$ **for every** $j = 1, \ldots, n$, **and**
  (2) **there is** $s \in (0, t]$ **so that**
  1. **the paths do not intersect in the time interval** $(0, s)$,
  2. $X_i(s) = X_{i'}(s)$, **and**
  3. **if** $X_j(s) = X_{j'}(s)$, **for some** $1 \leq j < j' \leq n$, **then** $i \leq j$, **and if** $i = j$, **then** $i' \leq j'$.
- **We have a disjoint union** $C_\sigma = \bigcup_\tau C_{\sigma,\tau}$ **so that**

$$P(C_\sigma) = \sum_\tau P(C_{\sigma,\tau}).$$

- **Crucial observation**

$$P(C_{\sigma,\tau}) = P(C_{\sigma\circ\tau,\tau}).$$

This follows from the strong Markov property.

Proof of the Karlin-McGregor theorem, step 4

- **Now we have**

$$\sum_\sigma \operatorname{sgn}(\sigma) P(C_\sigma) = \sum_\sigma \sum_\tau \operatorname{sgn}(\sigma) P(C_{\sigma,\tau})$$
$$= \sum_\tau \sum_\sigma \operatorname{sgn}(\sigma) P(C_{\sigma\circ\tau,\tau})$$

- **Make a "change of variables"** $\sigma \mapsto \sigma \circ \tau^{-1}$

$$\sum_\sigma \operatorname{sgn}(\sigma) P(C_\sigma) = \sum_\tau \sum_\sigma \operatorname{sgn}(\sigma \circ \tau^{-1}) P(C_{\sigma,\tau})$$
$$= -\sum_\sigma \sum_\tau \operatorname{sgn}(\sigma) P(C_{\sigma,\tau})$$
$$= -\sum_\sigma \operatorname{sgn}(\sigma) P(C_\sigma)$$

- **Thus** $\sum_\sigma \operatorname{sgn}(\sigma) P(C_\sigma) = 0$**, which completes the proof.**

Consequences

- **In the situation of the Karlin-McGregor theorem, if we condition on the event that the paths have not intersected in $[0, t]$, then the positions of the paths at time t have joint pdf** $\frac{1}{Z_n} \det \left[p_t(a_i, x_j) \right]_{i,j=1}^{n}$
  - **This is NOT a determinantal point process. (We need two determinants).**
- **Also condition at a later time $T > t$.**
  - **Starting positions $a_1 < a_2 < \cdots < a_n$ at time 0**
  - **End positions $b_1 < b_2 < \cdots < b_n$ at time T**
  - **Non intersecting paths in full time interval $[0, T]$**
- **Then the positions at time $t \in (0, T)$ have joint pdf**

$$\frac{1}{Z_n} \det \left[p_t(a_i, x_j) \right]_{i,j=1}^{n} \det \left[p_{T-t}(x_i, b_j) \right]_{i,j=1}^{n}$$

- **Biorthogonal ensemble with $f_j(x) = p_t(a_j, x)$, $g_j(x) = p_{T-t}(x, b_j)$.**

Non-intersecting path ensembles

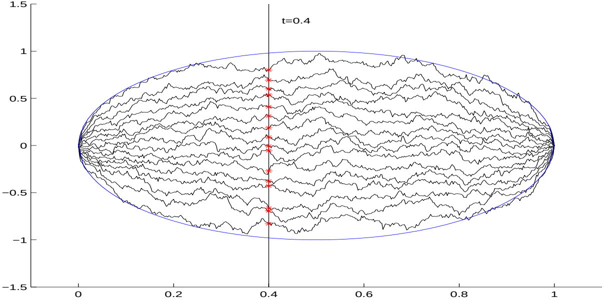
- **Let $p_t(a; x)$ be the transition probability density of a one-dimensional strong Markov process with continuous sample paths.**
- **Consider n independent copies $X_1(t), \dots, X_n(t)$ conditioned so that**
  - $X_j(0) = a_j$, $X_j(T) = b_j$ **where** $a_1 < \cdots < a_n$, $b_1 < \cdots < b_n$ **are given values,**
  - **The paths do not intersect in time interval $(0, T)$.**
- **Then the joint p.d.f. for the positions of the paths at time $t \in (0, T)$ is equal to**

$$\frac{1}{Z_n} \det \left[p_t(a_i, x_j) \right]_{i,j=1}^{n} \cdot \det \left[p_{T-t}(x_j, b_i) \right]_{i,j=1}^{n}$$

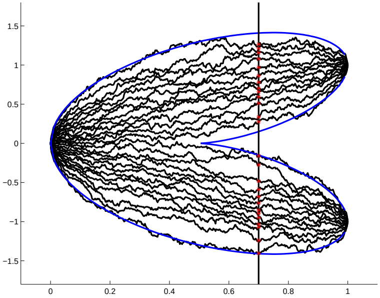
- **This is a determinantal point process.**

Confluent case

- **Take** **Brownian motion** **in the limit** $a_j \to a, \quad b_j \to b$**.**
- **This leads to same p.d.f. (after centering and scaling) as for the eigenvalues of GUE.**

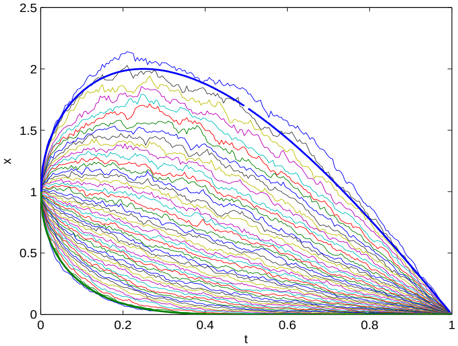


Two different endpoints



- **This is not an OP ensemble!**
- **Still Sine kernel in the bulk and Airy kernel at the edges**
- **Pearcey kernels at the cusp point (double scaling limit)** **Bleher-Kuijlaars (2007)**

Non-intersecting squared Bessel paths



- **Squared Bessel paths are always positive.**
- **Sine kernel in the bulk, Airy kernel at soft edges, and Bessel kernel at the hard edge**
- **New kernel at critical time**

Kuijlaars-Martínez Finkelshtein-Wielonsky (2011)

Matrix Riemann-Hilbert problem for OPs

- **Given weight $w = e^{-V}$ on $\mathbb{R}$ and $n \in \mathbb{N}$, find 2×2 matrix valued function $Y(z)$ such that**

RH-Y1 $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ **is analytic.**

RH-Y2 Y **has boundary values for $x \in \mathbb{R}$, denoted by $Y_{\pm}(x)$, and**

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & e^{-V(x)} \\ 0 & 1 \end{pmatrix}, \qquad x \in \mathbb{R}.$$

RH-Y3 **As $z \to \infty$,**

$$Y(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$$

Fokas, Its, Kitaev RH problem for OP

Theorem (Fokas, Its, Kitaev (1992))

The Riemann-Hilbert problem has the unique solution

$$Y(z) = \begin{pmatrix} \gamma_n^{-1} p_n(z) & \frac{1}{2\pi i}\gamma_n^{-1} \int\limits_{\mathbb{R}} \frac{p_n(s)w(s)}{s-z} ds \\ -2\pi i \gamma_{n-1} p_{n-1}(z) & -\gamma_{n-1} \int\limits_{\mathbb{R}} \frac{p_{n-1}(s)w(s)}{s-z} ds \end{pmatrix}$$

- p_n **is the orthonormal polynomial w.r.t.** $e^{-V(x)}dx$
- γ_n **is the leading coefficient of** p_n.

OP kernel in terms of the RH problem

- **OP kernel is**

$$K_n(x,y) = \frac{\sqrt{e^{-V(x)}}\sqrt{e^{-V(y)}}}{2\pi i(x-y)} \left[Y_+^{-1}(y)Y_+(x)\right]_{2,1}$$

$$= \frac{\sqrt{e^{-V(x)}}\sqrt{e^{-V(y)}}}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} Y_+^{-1}(y)Y_+(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Airy function

- **The Airy equation** $y''(z) = zy(z)$ **has the special solution**

$$\mathrm{Ai}(z) = \frac{1}{2\pi i}\int_C e^{-\frac{1}{3}t^3+zt}\,dt$$

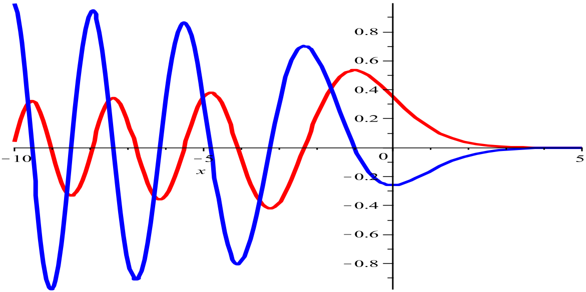
where C **is a contour in the complex** t**-plane that starts at infinity at angle** $\arg t = -2\pi/3$ **and ends at angle** $\arg t = 2\pi/3$**.**

- **This solution is characterized by its asymptotics as** $z \to \infty$ **in the sector** $-\pi < \arg z < \pi$**,**

$$\mathrm{Ai}(z) = \frac{1}{2\sqrt{\pi}z^{1/4}}e^{-\frac{2}{3}z^{3/2}}\left(1+\mathcal{O}(z^{-3/2})\right),$$

$$\mathrm{Ai}'(z) = -\frac{z^{1/4}}{2\sqrt{\pi}}e^{-\frac{2}{3}z^{3/2}}\left(1+\mathcal{O}(z^{-3/2})\right).$$

Plot



- **Plot of** Ai (**red**) **and its derivative** Ai′ (**blue**).

Other solutions

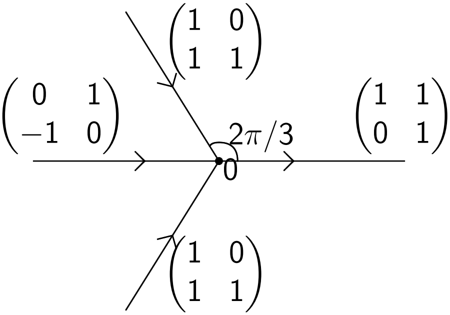
- **Airy function** Ai **is recessive in the sector** $-\pi/3 < \arg z < \pi/3$.
- **Other special solutions are**

$$\mathrm{Ai}(e^{2\pi i/3}z), \qquad \mathrm{Ai}(e^{-2\pi i/3}z)$$

- $\mathrm{Ai}(e^{2\pi i/3}z)$ **is recessive in** $-\pi < \arg z < -\pi/3$;
- $\mathrm{Ai}(e^{-2\pi i/3}z)$ **is recessive in** $\pi/3 < \arg z < \pi$.
- **The three solutions are related by (we use** $\omega = e^{2\pi i/3}$**)**

$$\mathrm{Ai}(z) + \omega\,\mathrm{Ai}(\omega z) + \omega^2\,\mathrm{Ai}(\omega^2 z) = 0.$$

Airy Riemann-Hilbert problem



RH-A1 $A : \mathbb{C} \setminus \Sigma \to \mathbb{C}^{2\times 2}$ **is analytic.**

RH-A2 $A_+(z) = A_-(z) v_A(z)$ **for** $z \in \Sigma$ **with jump matrices** v_A **as in figure.**

RH-A3 **As** $z \to \infty$, **we have**

$$A(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right) \begin{pmatrix} z^{-1/4} & 0 \\ 0 & z^{1/4} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} e^{-\frac{2}{3}z^{3/2}} & 0 \\ 0 & e^{\frac{2}{3}z^{3/2}} \end{pmatrix}$$

Solution of Airy Riemann-Hilbert problem

- **The unique solution of RH-A1, RH-A2, RH-A3 is given by**

$$A(z) = \sqrt{2\pi} \begin{pmatrix} \mathrm{Ai}(z) & -\omega^2 \mathrm{Ai}(\omega^2 z) \\ -i \mathrm{Ai}'(z) & i\omega \mathrm{Ai}'(\omega^2 z) \end{pmatrix}, \qquad 0 < \arg z < \frac{2\pi}{3},$$

$$A(z) = \sqrt{2\pi} \begin{pmatrix} -\omega \mathrm{Ai}(\omega z) & -\omega^2 \mathrm{Ai}(\omega^2 z) \\ i\omega^2 \mathrm{Ai}'(\omega z) & i\omega \mathrm{Ai}'(\omega^2 z) \end{pmatrix}, \qquad \frac{2\pi}{3} < \arg z < \pi,$$

$$A(z) = \sqrt{2\pi} \begin{pmatrix} -\omega^2 \mathrm{Ai}(\omega^2 z) & \omega \mathrm{Ai}(\omega z) \\ i\omega \mathrm{Ai}'(\omega^2 z) & -i\omega^2 \mathrm{Ai}'(\omega z) \end{pmatrix}, \qquad -\pi < \arg z < -\frac{2\pi}{3},$$

$$A(z) = \sqrt{2\pi} \begin{pmatrix} \mathrm{Ai}(z) & \omega \mathrm{Ai}(\omega z) \\ -i \mathrm{Ai}'(z) & -i\omega^2 \mathrm{Ai}'(\omega z) \end{pmatrix}, \qquad -\frac{2\pi}{3} < \arg z < 0.$$

Airy kernel

- The **Airy kernel**

$$K^{Airy}(x,y) = \frac{\mathrm{Ai}(x)\,\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\,\mathrm{Ai}(y)}{x-y} = \int_0^\infty \mathrm{Ai}(x+s)\,\mathrm{Ai}(y+s)ds$$

can be expressed in terms of the solution of the Airy RH problem

$$K^{Airy}(x,y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} A_+^{-1}(y) A_+(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \textbf{if } x,y > 0,$$

$$K^{Airy}(x,y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} -1 & 1 \end{pmatrix} A_+^{-1}(y) A_+(x) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \textbf{if } x,y < 0,$$

and a mixture of these formulas if x and y have opposite signs.

Scaling limit

- **In appropriate scaling limit, the OP kernel**

$$K_n(x,y) = \frac{\sqrt{e^{-V(x)}}\sqrt{e^{-V(y)}}}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} Y_+^{-1}(y) Y_+(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

tends to the Airy kernel

$$K^{Airy}(x,y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} A_+^{-1}(y) A_+(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- **This will be the goal of the rest of this lecture.**

Recall: steepest descent analysis

- **Random matrix ensemble** $\frac{1}{\tilde{Z}_n} e^{-n \operatorname{Tr} V(M)} dM$
- **The eigenvalue correlation kernel is**

$$K_n(x,y) = \frac{\sqrt{e^{-nV(x)}}\sqrt{e^{-nV(y)}}}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} Y_+^{-1}(y) Y_+(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

where Y is the solution of the RH problem

RH-Y1 $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ **is analytic,**

RH-Y2 Y **has boundary values** $Y_\pm(x)$ **for** $x \in \mathbb{R}$ **and**

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & e^{-nV(x)} \\ 0 & 1 \end{pmatrix}, \qquad x \in \mathbb{R}$$

RH-Y3 $Y(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ **as** $z \to \infty$**.**

Equilibrium measure

- **Balance between mutual repulsion of eigenvalues and the confining potential V.**
- **To minimize**

$$-\iint \log|x-y| d\mu(x) d\mu(y) + \int V(x) d\mu(x)$$

is a well-studied equilibrium problem in logarithmic potential theory.

Mhaskar-Saff, Gonchar-Rakhmanov (1980s)

- **There is a unique minimizer $\mu = \mu_V$ which has a density**

$$d\mu_V(x) = \rho_V(x) dx$$

Equilibrium condition

- **There is a constant ℓ so that**

$$2\int \log\frac{1}{|x-y|}\rho_V(y)dy + V(x) = \ell \qquad \textbf{on}\ \ \mathrm{supp}(\rho_V)$$

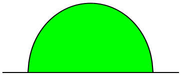
and

$$2\int \log\frac{1}{|x-y|}\rho_V(y)dy + V(x) \geq \ell \qquad \textbf{on}\ \mathbb{R}\backslash\mathrm{supp}(\rho_V)$$

Example: Semicircle law

- **In GUE case** $V(x) = x^2$ **the equilibrium density can be explicitly calculated**

$$\rho_V(x) = \frac{2}{\pi}\sqrt{2 - x^2},$$



$$-\sqrt{2} \le x \le \sqrt{2}$$

- **This is Wigner semi-circle law**

- **If V is real analytic then the support is a finite union of intervals**

$$\text{supp}(\rho_V) = \bigcup_{j=1}^{N} [a_j, b_j]$$

Global eigenvalue behavior

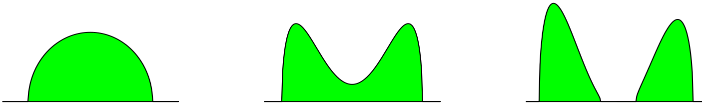
- **Global (or macroscopic) eigenvalue behavior is governed by the minimizer of the equilibrium problem**

$$\lim_{n\to\infty} \frac{1}{n} K_n(x,x) = \rho_V(x)$$

- **This is one of the outcomes of the steepest descent analysis, although it can be established by more elementary means as well.**

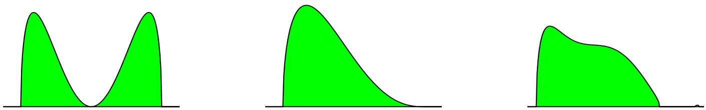
Regular cases

- $\rho_V > 0$ **in the interior of each interval,**
- ρ_V **vanishes like a square root at each endpoint,**
- $2\int \log \frac{1}{|x-y|}\rho_V(y)dy + V(x) > \ell$ **outside** $\text{supp}(\rho_V)$**.**



Singular cases

- **Singular case I:** ρ_V **vanishes at an interior point**
- **Singular case II:** ρ_V **vanishes to higher order at an endpoint.**
- **Singular case III: Equality in** $2\int \log \frac{1}{|x-y|}\rho_V(y) + V(x) \geq \ell$ **at exterior point.**



- **Different local eigenvalue behavior in singular cases near critical points.**

First transformation

- **We use the equilibrium measure in the first transformation of the RH problem**

  RH-Y1 $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ **is analytic,**

  RH-Y2 $Y_+(x) = Y_-(x) \begin{pmatrix} 1 & e^{-nV(x)} \\ 0 & 1 \end{pmatrix}$, **for** $x \in \mathbb{R}$,

  RH-Y3 $Y(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$, **as** $z \to \infty$.

- **We use** g**-function**

$$g(z) = \int \log(z - s)\rho_V(s)ds$$

First transformation

- **We define**

$$T(z) = \begin{pmatrix} e^{n\ell/2} & 0 \\ 0 & e^{-n\ell/2} \end{pmatrix} Y(z) \begin{pmatrix} e^{-n(g(z)+\ell/2)} & 0 \\ 0 & e^{n(g(z)+\ell/2)} \end{pmatrix}$$

- **Then** $T(z) = I + \mathcal{O}(1/z)$ **as** $z \to \infty$**.**

ϕ functions

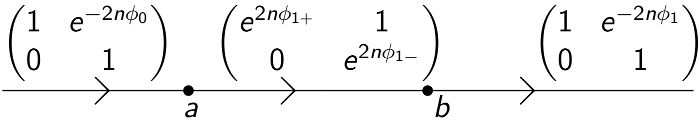
- **Jumps can all be expressed nicely in terms of analytic functions ϕ_k, $k = 0, \dots, N$.**

$$2\phi_k(x) = -g_+(x) - g_-(x) + V(x) - \ell, \qquad x \in (b_k, a_{k+1})$$

- **ϕ_k has analytic continuation which is such that**

$$g_+(x) - g_-(x) = -2\phi_{k+}(x) = 2\phi_{k-}(x)$$
$$\textbf{for } x \in (a_k, b_k) \cup (a_{k+1}, b_{k+1})$$

Jumps for T in one-interval case



- $\phi_1(x) > 0$ **for** $x > b$**,**
- $\phi_0(x) > 0$ **for** $x < a$**,**
- $\phi_{1+} = -\phi_{1-}$ **is purely imaginary on** (a, b) **and**

$$\frac{d}{dx}\phi_{1+}(x) = \pi i \rho_V(x) \qquad \textbf{with} \qquad \rho_V(x) > 0$$

Correlation kernel in terms of T

- **Recall that**

$$K_n(x,y) = \frac{\sqrt{e^{-nV(x)}}\sqrt{e^{-nV(y)}}}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} Y_+^{-1}(y) Y_+(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- **Assume** $x, y \in (a,b)$**.**
- **Transformation** $Y \mapsto T$ **gives**

$$K_n(x,y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & e^{-n\phi_{1+}(y)} \end{pmatrix} T_+^{-1}(y) T_+(x) \begin{pmatrix} e^{-n\phi_{1+}(x)} \\ 0 \end{pmatrix}.$$

  - **This is based on** $2g_+ - V + \ell = -2\phi_+$ **on** (a,b)**.**

Second transformation $T \mapsto S$

- **Factorization of jump matrix for T on (a, b),**

$$\begin{pmatrix} e^{2n\phi_{1+}} & 1 \\ 0 & e^{2n\phi_{1-}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{2n\phi_{1-}} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{2n\phi_{1+}} & 1 \end{pmatrix}.$$

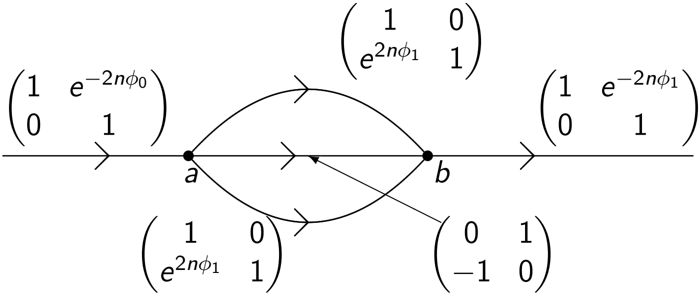
- **Open a lens around each $[a, b]$ and define**

$$S = T \begin{pmatrix} 1 & 0 \\ -e^{2n\phi_1} & 1 \end{pmatrix} \quad \text{in upper part of the lens}$$

$$S = T \begin{pmatrix} 1 & 0 \\ e^{2n\phi_1} & 1 \end{pmatrix} \quad \text{in lower part of the lens.}$$

and $S = T$ outside the lenses.

RH problem for S in one-interval case



- **We have** $\phi_1 > 0$ **on** (b, ∞) **and** $\phi_0 > 0$ **on** $(-\infty, a)$**.**
- **From Cauchy-Riemann equations:**

$$\operatorname{Re} \phi_1 < 0 \qquad \text{on the lips of the lens}$$

provided that $\rho_V(x) > 0$ **on** (a, b)

Correlation kernel in terms of S

- **We have for** $x, y \in (a, b)$**,**

$$K_n(x,y) = \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & e^{-n\phi_{1+}(y)} \end{pmatrix} T_+^{-1}(y) T_+(x) \begin{pmatrix} e^{-n\phi_{1+}(x)} \\ 0 \end{pmatrix}.$$

- **Transformation** $T \mapsto S$ **gives**

$$K_n(x,y) = \frac{1}{2\pi i(x-y)} \times$$
$$\begin{pmatrix} -e^{n\phi_{1+}(y)} & e^{-n\phi_{1+}(y)} \end{pmatrix} S_+^{-1}(y) S_+(x) \begin{pmatrix} e^{-n\phi_{1+}(x)} \\ e^{n\phi_{1+}(x)} \end{pmatrix}$$

Sine kernel in the bulk

- **The outcome of the steepest descent analysis will be that for** $x, y \in (a+\delta, b-\delta)$**,**

$$S_+^{-1}(y) S_+(x) = I + \mathcal{O}(x-y) \qquad \textbf{as } y \to x$$

- **Then for** x **and** y **close to** $x^* \in (a,b)$**,**

$$K_n(x,y) \approx \frac{1}{2\pi i(x-y)} \begin{pmatrix} -e^{n\phi_{1+}(y)} & e^{-n\phi_{1+}(y)} \end{pmatrix} \begin{pmatrix} e^{-n\phi_{1+}(x)} \\ e^{n\phi_{1+}(x)} \end{pmatrix}$$

- **Replacing** x, y **by** $x^* + \frac{x}{n\rho_V(x^*)}$ **and** $x^* + \frac{y}{n\rho_V(x^*)}$ **then we arrive in the limit** $n \to \infty$ **at the sine kernel**

$$\frac{\sin \pi(x-y)}{\pi(x-y)}.$$

Global parametrix in one-interval case

- **We keep only the jump matrix on $[a, b]$, and look for N satisfying**

  RH-N1 N **is analytic in** $\mathbb{C} \setminus [a, b]$.

  RH-N2 $N_+ = N_- \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ **on** (a, b).

  RH-N3 $N(z) = I + \mathcal{O}\left(\frac{1}{z}\right)$ **as** $z \to \infty$.

Solution in one-interval case

- **A solution in the one-interval case is**

$$N(z) = \begin{pmatrix} \frac{1}{2}\left(\beta(z) + \beta^{-1}(z)\right) & \frac{1}{2i}\left(\beta(z) - \beta^{-1}(z)\right) \\ -\frac{1}{2i}\left(\beta(z) - \beta^{-1}(z)\right) & \frac{1}{2}\left(\beta(z) + \beta^{-1}(z)\right) \end{pmatrix}$$

with $\beta(z) = \left(\frac{z-b}{z-a}\right)^{1/4}$

  - **This can be checked from the property** $\beta_+ = i\beta_-$ **on** (a, b)**.**

- **The global parametrix is more complicated in the multi-interval case.**

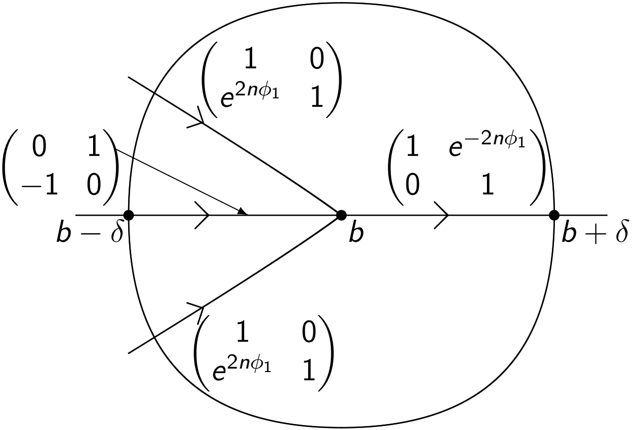
Local parametrix

- N is **unbounded** near endpoints $z = a$ and $z = b$.
  - Since S remains bounded near endpoints, N cannot be a good approximation to S near $z = a$ and $z = b$.
- We need **local parametrices** P in small neighborhoods

$$U_\delta(b) = \{z \in \mathbb{C} \mid |z - b| < \delta\}$$

$$U_\delta(a) = \{z \in \mathbb{C} \mid |z - a| < \delta\}$$

RH problem for P



- **Matching condition: Uniformly for $z \in \partial U_\delta(b)$,**

$$P(z) = \left(I + O\left(\frac{1}{n}\right)\right) N(z) \qquad \text{as } n \to \infty$$

Reduction to constant jumps

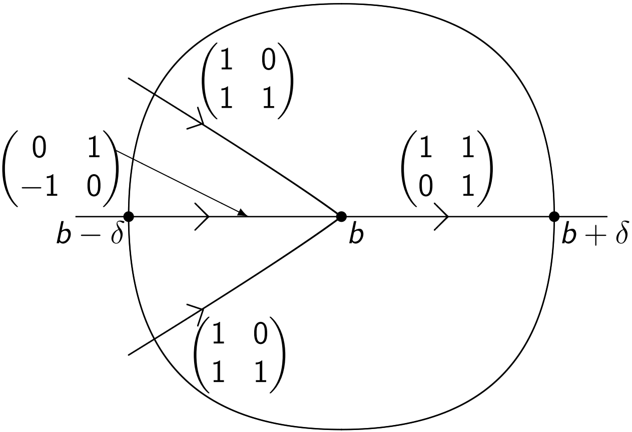
- **We write P in the form**

$$P = \widetilde{P} \begin{pmatrix} e^{n\phi_1} & 0 \\ 0 & e^{-n\phi_1} \end{pmatrix}$$

Then $\widetilde{P}$ should satisfy jumps

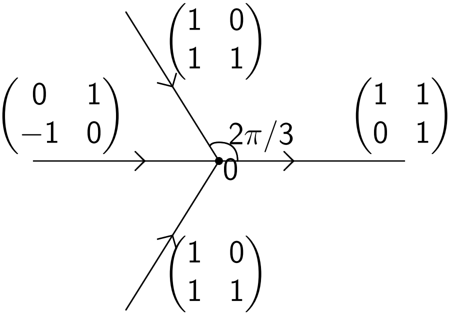
- $\widetilde{P}_+ = \widetilde{P}_- \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ **on** $(a, b) \cap U_\delta(b)$ **[Use $\phi_{1+} = -\phi_{1-}$]**
- $\widetilde{P}_+ = \widetilde{P}_- \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ **on the lips of the lens inside $U_\delta(b)$.**
- $\widetilde{P}_+ = \widetilde{P}_- \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ **on** $(b, \infty) \cap U_\delta(b)$

Jumps for $\widetilde{P}$



- **Jump matrices coincide with jump matrices in Airy RH problem.**
- **We solve the RH problem for $\widetilde{P}$ by mapping it to the Airy RH problem.**

Reminder: Airy RH problem



As $\zeta \to \infty$, we have

$$A(\zeta) = \left(I + \mathcal{O}\left(\frac{1}{\zeta}\right)\right) \begin{pmatrix} \zeta^{-1/4} & 0 \\ 0 & \zeta^{1/4} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} e^{-\frac{2}{3}\zeta^{3/2}} & 0 \\ 0 & e^{\frac{2}{3}\zeta^{3/2}} \end{pmatrix}$$

Conformal mapping

- **We take** $\widetilde{P}$ **in the form**

$$\widetilde{P}(z) = E_n(z) A(n^{2/3} f(z))$$

where

- $\zeta = f(z)$ **is a conformal map from** $U_\delta(b)$ **to a neighborhood of** 0 **in the** ζ**-plane,**
- $E_n(z)$ **is an analytic prefactor**

- **Then** $\widetilde{P}$ **satisfies the correct jumps.**

Matching

- **We use the freedom we have in choosing f and E_n to satisfy the matching condition as well**
- **We want for z on $\partial U_\delta(b)$**

$$E_n(z)A(n^{2/3}f(z)) = (I+\mathcal{O}(1/n))N(z)\begin{pmatrix} e^{-n\phi_1(z)} & 0 \\ 0 & e^{n\phi_1(z)} \end{pmatrix}$$

- **To match the exponential part we have to take**

$$f(z) = \left[\frac{3}{2}\phi_1(z)\right]^{2/3}$$

- **This is indeed a conformal map, but only in case equilibrium measure vanishes as square root at b.**

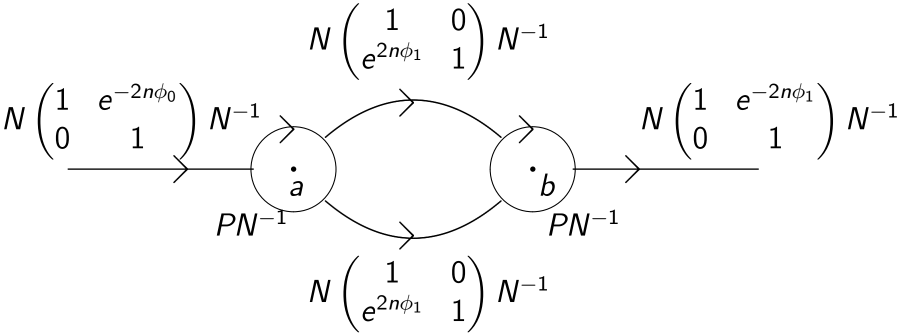
Third transformation $S \mapsto R$

- **Similar construction gives the local parametrix, which we also call P, in a neighborhood of a.**
- **Then define**

$$R(z) = S(z)N(z)^{-1}, \qquad \text{for } z \in \mathbb{C} \setminus (\Sigma_S \cup \overline{U}_\delta(a) \cup \overline{U}_\delta(b))$$
$$R(z) = S(z)P(z)^{-1}, \qquad \text{for } z \in (U_\delta(a) \cup U_\delta(b)) \setminus \Sigma_S.$$

- R **is analytic in** $\mathbb{C} \setminus (\Sigma_S \cup \partial U_\delta(a) \cup \partial U_\delta(b))$.
- **Since** S **and** N **have the same jump matrix on** (a, b), R **has analytic continuation across** $(a + \delta, b - \delta)$,
- **Similarly,** R **has analytic continuation across parts of** Σ_S **inside** $U_\delta(a)$ **and** $U_\delta(b)$.

Jumps in the RH problem for R



- **From matching conditions** $PN^{-1} = I + \mathcal{O}(1/n)$ **as** $n \to \infty$**, uniformly on** $\partial U_\delta(a) \cup \partial U_\delta(b)$**.**
- **The other jump matrices are** $I + \mathcal{O}(e^{-cn})$

Conclusion

- **We are in a good situation and we can conclude**

$$R(z) = I + \mathcal{O}\left(\frac{1}{n(|z|+1)}\right) \qquad \text{as } n \to \infty,$$

uniformly for $z \in \mathbb{C} \setminus \Sigma_R$**.**

Correlation kernel at the edge

- **For** x, y **in** $U_\delta(b)$, **we have**

$$K_n(x,y) = \frac{1}{2\pi i(x-y)}$$

$$\begin{pmatrix} -e^{n\phi_{1+}(y)} & e^{-n\phi_{1+}(y)} \end{pmatrix} S_+^{-1}(y) S_+(x) \begin{pmatrix} e^{-n\phi_{1+}(x)} \\ e^{n\phi_{1+}(x)} \end{pmatrix}$$

- **Now**

$$S_+(x) = R(x) E_n(x) A_+(n^{2/3} f(x)) \begin{pmatrix} e^{n\phi_{1+}(x)} & 0 \\ 0 & e^{-n\phi_{1+}(x)} \end{pmatrix}$$

and

$$E_n(y)^{-1} R(y)^{-1} R(x) E_n(x) \approx I$$

as $x \approx y$ **and** $n \to \infty$**.**

Airy kernel at the edge

- **Hence**

$$K_n(x,y) \approx \frac{1}{2\pi i(x-y)} \begin{pmatrix} -1 & 1 \end{pmatrix} A_+(n^{2/3}f(y))^{-1} A_+(n^{2/3}f(x)) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

- **For suitable $c > 0$ we have**

$$n^{2/3} f\left(b + \frac{x}{(cn)^{2/3}}\right) \to x, \qquad n^{2/3} f\left(b + \frac{y}{(cn)^{2/3}}\right) \to y.$$

- **Rescaled kernel tends to**

$$\frac{1}{2\pi i(x-y)} \begin{pmatrix} -1 & 1 \end{pmatrix} A_+(y)^{-1} A_+(x) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

which is the Airy kernel for $x, y < 0$.

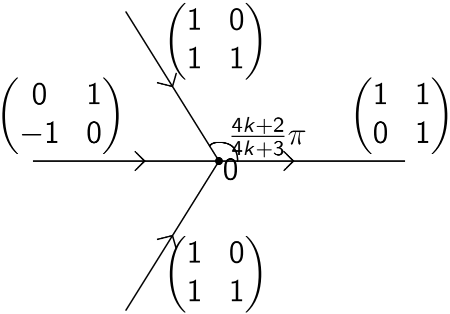
Singular cases

- **The steepest descent analysis does not work in singular cases.**
  - **Singular case I: ρ_V vanishes at an interior point x^***
  - **Singular case II: ρ_V vanishes to higher order at an endpoint.**

- **In singular case I we cannot open the lens near x^* and get good decay property of $e^{2n\phi(z)}$ on the lips of the lens.**
- **In singular case II the Airy parametrix does not work at the edge point. We cannot match it with the global parametrix.**

Singular case II

- **If ρ_V vanishes like $(b-x)^{2k+1/2}$ with $k \geq 1$, we would need the solution to the following RH problem for the construction of the local parametrix**



- **As $\zeta \to \infty$, we have the asymptotic condition**

$$\Psi(\zeta) = \left(I + \mathcal{O}\left(\frac{1}{\zeta}\right)\right) \begin{pmatrix} \zeta^{-1/4} & 0 \\ 0 & \zeta^{1/4} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} e^{-c_k \zeta^{\frac{4k+3}{2}}} & 0 \\ 0 & e^{c_k \zeta^{\frac{4k+3}{2}}} \end{pmatrix}$$

Ψ-kernel as scaling limit

- **RH problem cannot be solved with classical special functions.**
  - **Existence of solution can be proved with operator theoretic methods (Fredholm theory) and so-called vanishing lemma (Zhou (1989)).**
    Deift, Kriecherbauer, McLaughlin, Venakides, Zhou (1999)
- **The scaling limit of the OP kernel near the edge is now**

$$\frac{1}{2\pi i(x-y)} \begin{pmatrix} -1 & 1 \end{pmatrix} \Psi_+^{-1}(y)\Psi_+(x) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

  - **To prove this we can just follow the proof for the Airy kernel in the regular case.**
- **What can we say about Ψ ?**

Differential equation for Ψ

- Ψ **satisfies a differential equation.**
  - **The jump matrices for Ψ are constant on the four rays and therefore we find that $\frac{d}{d\zeta}\Psi$ satisfies the same jumps.**
  - **Then $\left(\frac{d}{d\zeta}\Psi\right)\Psi^{-1}$ is entire function, say it is $A = A(\zeta)$:**

$$\frac{d}{d\zeta}\Psi = A\Psi$$

  - **From asymptotic condition it follows that A is polynomial in ζ.**
    For $k = 1$ the degrees are

$$\deg A_{11} = 1, \qquad \deg A_{12} = 2,$$
$$\deg A_{21} = 3, \qquad A_{22} = -A_{11}.$$

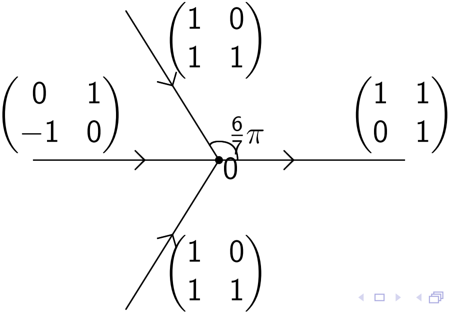
  - **We do not know coefficients of polynomials A_{ij}.**

Introduce extra parameter

- **Modify the RH problem by introducing parameter s in the asymptotic condition (written here for case $k = 1$)** $\Psi(\zeta) =$

$$\left(I + \mathcal{O}\left(\frac{1}{\zeta}\right)\right)\begin{pmatrix}\zeta^{-1/4} & 0\\ 0 & \zeta^{1/4}\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\begin{pmatrix}e^{-\left(\frac{1}{105}\zeta^{\frac{7}{2}}+s\zeta^{\frac{1}{2}}\right)} & 0\\ 0 & e^{\left(\frac{1}{105}\zeta^{\frac{7}{2}}+s\zeta^{\frac{1}{2}}\right)}\end{pmatrix}$$

- **Jump conditions remain the same.**



Lax pair

- **Solution also depends on s:** $\quad \Psi = \Psi(\zeta; s)$
- **Differential equation continues to hold**

$$\frac{\partial}{\partial \zeta}\Psi = A\Psi,$$

with $A = A(\zeta; s)$ polynomial in ζ of same degrees as before but with coefficients depending on s.

- **Since jumps do not depend on s, we also have a differential equation**

$$\frac{\partial}{\partial s}\Psi = B\Psi$$

  - **The two linear ODEs form a Lax pair.**

- **B is rather simple:** $\quad B = \begin{pmatrix} 0 & 1 \\ \zeta - 2u & 0 \end{pmatrix}$ **for some $u = u(s)$.**

Compatibility

- **The compatibility condition** $\frac{\partial^2}{\partial s \partial \zeta}\Psi = \frac{\partial^2}{\partial \zeta \partial s}\Psi$ **gives**

$$AB - BA = \frac{\partial B}{\partial \zeta} - \frac{\partial A}{\partial s}$$

- **Using this, we can express all entries of** A **in terms of** $u = u(s)$ **and its derivatives, for example**

$$A_{11} = -A_{22} = -\frac{1}{240}\left(4u_s\zeta + 12uu_s + u_{sss}\right)$$

- **It also follows that** u **must satisfy a nonlinear fourth order ODE**

$$\frac{1}{240}u_{ssss} + \frac{1}{24}\left(u_s^2 + 2uu_{ss}\right) + \frac{1}{6}u^3 + s = 0.$$

- **This is the second member of the Painlevé I hierarchy.**
  - **Painlevé I equation** $u_{ss} = 6u^2 + s$ **itself would be connected with vanishing of equilibrium measure with exponent** $3/2$ **(which cannot happen).**

Description of Ψ

- **To describe Ψ we first need to characterize the special solution of the second member of the Painlevé I hierarchy that is involved**
  - u **is characterized by its asymptotic behavior**

$$u(s) \sim \mp (6|s|)^{1/3} + \mathcal{O}(s^{-1}) \qquad \text{as } s \to \pm\infty.$$

  - **Show that this solution has no poles on the real line, and in particular not a pole at $s = 0$.**
- **Given u we can set up the differential equation**

$$\frac{\partial \Psi}{\partial \zeta} = A\Psi$$

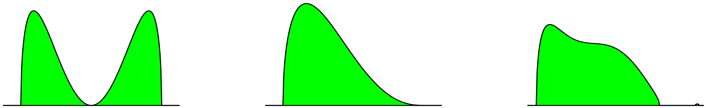
in particular for $s = 0$, since u has no pole at $s = 0$.

  - **Characterize the solution Ψ by its asymptotic behavior as $\zeta \to \infty$.**

Claeys-Vanlessen (2007)

Other singular cases

- Singular case I: ρ_V vanishes at an interior point
- Singular case II: ρ_V vanishes to higher order at an endpoint.
- Singular case III: equality at exterior point.



- Ψ functions for Painlevé II + hierarchy
  Bleher-Its (2003), Claeys-Kuijlaars (2006)
- Ψ functions for Painlevé I hierarchy
  Claeys-Vanlessen (2007)
- Finite size GUE (and generalizations)
  Claeys (2008), Mo (2008), Bertola-Lee (2009)