Integrability Property of Graph Invariants

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Reference

S. Chmutov, M. Kazarian, S. Lando,
Integrability property of graph invariants,
arXiv:1809.0434

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For ordinary graphs all the weights are taken to be 1,

$$\chi_G(c) = (-1)^{|V(G)|} W_G(c, -c, c, \dots).$$

Example

The weighted chromatic polynomials for the two connected graphs with three vertices are

$$\begin{aligned} W_{P_3}(q_1, q_2, \dots) &= q_1^3 + 2q_1q_2 + q_3, \\ W_{K_3}(q_1, q_2, \dots) &= q_1^3 + 3q_1q_2 + 2q_3. \end{aligned}$$

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Conjecture (R. Stanley)

The weighted chromatic polynomial distinguishes trees.

Generating functions for weighted chromatic polynomial

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\begin{aligned}
\mathcal{W}^\circ(q_1,q_2,\dots) &= \sum_G \frac{W_G(q_1,q_2,\dots)}{|Aut(G)|}\\
&= 1+\frac{1}{1!}q_1+\frac{1}{2!}\left(2q_1^2+q_2\right)+\frac{1}{3!}\left(8q_1^3+12q_1q_2+5q_3\right)\\
&\quad +\frac{1}{4!}\left(64q_1^4+192q_1^2q_2+48q_2^2+160q_1q_3+79q_4\right)+\dots\\
\mathcal{W}(q_1,q_2,\dots) &= \sum_{\substack{G \text{ connected}\\ \text{non-empty}}} \frac{W_G(q_1,q_2,\dots)}{|Aut(G)|}\\
&= \frac{1}{1!}q_1+\frac{1}{2!}\left(q_1^2+q_2\right)+\frac{1}{3!}\left(4q_1^3+9q_1q_2+5q_3\right)\\
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$$\mathcal{W}^\circ = \exp(\mathcal{W})$$

Integrability theorem

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Theorem

After an appropriate rescaling of the variables $q_i = c_i p_i$, $i = 1, 2, 3, \dots$, the function $\mathcal{W}$ becomes a solution to the KP (Kadomtsev–Petviashvili) hierarchy of partial differential equations, and $\mathcal{W}^\circ$ a τ-function of the KP hierarchy.

Kadomtsev–Petviashvili hierarchy

The KP hierarchy is an “integrable” infinite system of partial differential equations for a function in infinitely many variables, the first of which is

$$\frac{\partial^2 F}{\partial p_2^2} = \frac{\partial^2 F}{\partial p_1 \partial p_3} - \frac{1}{2}\left(\frac{\partial^2 F}{\partial p_1^2}\right)^2 - \frac{1}{12}\frac{\partial^4 F}{\partial p_1^4}.$$

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For functions independent of variables with even indices, the KP hierarchy degenerates into the KdV hierarchy.

Another solution to KP: Generating function for the numbers of rooted maps (I. Goulden, D. Jackson, 2008)

Define exponential generating functions in a variable w (recording the number of faces), a variable z (recording the number of edges), and infinitely many variables $p_1, p_2, \dots$ (recording the verticies' valencies):

$$R^\circ(w, z; p_1, p_2, \dots) = \sum_{m,n,\mu} \frac{r^\circ_{m,n;\mu}}{2n} p_{\mu_1} p_{\mu_2} \dots \frac{w^m}{m!} \frac{z^n}{n!};$$

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$$R(w, z; p_1, p_2, \ldots) = \sum_{m,n,\mu} \frac{r_{m,n;\mu}}{2n} p_{\mu_1} p_{\mu_2} \ldots \frac{w^m}{m!} \frac{z^n}{n!};$$

Here $\mu = (\mu_1, \mu_2, \ldots)$, $\mu_1 \geq \mu_2 \geq \ldots$ runs over *all* partitions $\mu \vdash 2n$, $r^\circ_{m,n;\mu}$ is the number of rooted maps, and $r_{m,n;\mu}$ is the number of connected rooted maps with m faces, n edges and partition of valencies of the vertices μ.

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The two are related by

$$R^\circ = \exp(R).$$

Hopf algebra of graphs (Rota, around 1970)

$\mathcal{G}_k$, $k = 0, 1, 2, \dots$, the vector space (say, over $\mathbb{C}$) spanned by simple graphs with k vertices. The vector space of graphs

$$\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1 \oplus \mathcal{G}_2 \oplus \dots .$$

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Comultiplication $\mathcal{G} \to \mathcal{G} \otimes \mathcal{G}$ — is defined on a graph as the sum over all splittings of the set of its vertices into two disjoint subset,

$$G \mapsto \sum_{V(G)=I \sqcup J} G|_I \otimes G|_J.$$

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Both operations are graded and together they make $\mathcal{G}$ into a connected commutative cocommutative Hopf algebra.

Umbral graph invariants

Definition

An *umbral graph invariant* is a graded Hopf algebra homomorphism $\mathcal{G} \to \mathbb{C}[q_1, q_2, \dots]$.

Comultiplication: $q_i \mapsto 1 \otimes q_i + q_i \otimes 1$, $i = 1, 2, \dots$.

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One more example: the *Abel polynomial* A_G of a graph G is defined as

$$A_G(q_1, q_2, \dots) = \sum_{\text{forests } F \subset E(G)} \prod_{\text{trees } T \text{ in } F} |V(T)| q_{|V(T)|}.$$

The coefficient of $q_1^{m_1} q_2^{m_2} \dots$ in A_G is the number of rooted forests in G having m_1 trees with 1 vertex, m_2 trees with 2 vertices, $\dots$.

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The Abel polynomial of complete graphs:

$$A_{K_n}(x, x, x, \ldots) = x(x+n)^{n-1} = A_n(x).$$

Generating function for umbral invariant

Let I be an umbral graph polynomial invariant. Define two generating functions by

$$\mathcal{I}^\circ(q_1, q_2, \dots) = \sum_{G} \frac{I_G(q_1, q_2, \dots)}{|Aut(G)|}$$

$$\mathcal{I}(q_1, q_2, \dots) = \sum_{\substack{G \text{ connected} \\ \text{non-empty}}} \frac{I_G(q_1, q_2, \dots)}{|Aut(G)|}$$

$$\mathcal{I}^\circ = \exp(\mathcal{I})$$

Define constants i_n, $n = 1, 2, \dots,$ by

$$i_n = [q_n] \sum_{G, |V(G)|=n} \frac{I_G(q_1, q_2, \dots)}{|Aut(G)|}.$$

Main theorem

Theorem

Suppose all the constants i_n, $n = 1, 2, 3, \ldots$, are nonzero. Then after an appropriate rescaling of the variables $q_n = c_n p_n$, $n = 1, 2, 3, \ldots$, the function $\mathcal{I}$ becomes a solution to the KP (Kadomtsev–Petviashvili) hierarchy of partial differential equations, and $\mathcal{I}^\circ$ a τ-function of the KP hierarchy. The solution and the τ-function are the same for all umbral graph invariants.

Rescaling

Theorem

After the rescaling of the variables $q_n = \frac{2^{n(n-1)/2}(n-1)!}{i_n} \cdot p_n$, *the generating function* $\mathcal{I}^\circ$ *becomes the following linear combination of one-part Schur polynomials:*

$$\mathcal{S}(p_1, p_2, \dots) = 1 + 2^0 s_1(p_1) + 2^1 s_2(p_1, p_2) + \dots + 2^{n(n-1)/2} s_n(p_1, \dots, p_n) + \dots.$$

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The *one-part Schur polynomials* are defined through the expansion

$$1 + s_1(p_1) + s_2(p_1, p_2) + s_3(p_1, p_2, p_3) + \dots = e^{\frac{p_1}{1} + \frac{p_2}{2} + \frac{p_3}{3} + \dots}$$

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It is known that any linear combination of one-part Schur polynomials with the free term 1 is a τ-function for the KP hierarchy.

Other Hopf algebras

The Hopf algebra of graphs is not unique. Other examples include

- weighted graphs;
- k-regular hypergraphs;
- binary delta-matroids;
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Theorem (E. Krasilnikov, 2019)

None of the above Hopf algebras possesses integrability property similar to that of the Hopf algebra of graphs.

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Exception: Hopf algebra of framed graphs (=simple graphs with loops allowed).

Thank you for your attention