Riemannian geometry without indices.

Joint work with Pierre Goussard.

Main objects

M	smooth manifold of dimension n,
V	(pseudo)Euclidean space of dimension n .
V	also a trivial bundle on M with the fibre V.
η	metric on V.
Vielbein	$\theta \in \Omega^1(M, \mathrm{V})$. $\theta : TM \to \mathrm{V}$ isomorphism.
Metric on M:	$g = \theta^* \eta$,
Connection on V	$\omega \in \Omega^1(M, o(\mathrm{V}))$.

Clifford algebra $Cl(\mathrm{V})$

$Cl(\mathrm{V}) = \{< V > | vv' + v'v = \eta(v, v')\}$.
$\Lambda(\mathrm{V}) = \{< V > | vv' + v'v = 0\}$. External algebra.
$\Lambda(\mathrm{V})$ acts on $Cl(\mathrm{V})$ by

$$v \wedge a \mapsto \frac{1}{2}(va + (-1)^{\deg a} av)$$

inducing an isomorphism $\Lambda(\mathrm{V}) \to Cl(\mathrm{V})$ as $\Lambda(\mathrm{V})$ modules.
Therefore $Cl(\mathrm{V}) = \oplus_i Cl^i(\mathrm{V})$.
str : $Cl(\mathrm{V}) \to \mathbb{C}$ a map defined by $\mathrm{str}(ab) = (-1)^{\deg a \deg b} str(ba)$ and str $Vol = 1$.

Ω^{pq} - algebra of Clifford forms.

$$\Omega^{pq} = \Omega^q(M, Cl^q(M)).$$

Vielbein	$\theta \in \Omega^{11}$
Connection form	$\omega \in \Omega^{21}$
Covariant derivative	$\nabla : \Omega^{pq} \to \Omega^{p,q+1}$
	$\nabla x = dx + \omega x - (-1)^q x\omega$ for any $x \in \Omega^{pq}$
Curvature	$R \in \Omega^{22}$. $R = d\omega + \omega^2$
Torsion	$t \in \Omega^{12}$. $t = \nabla\theta = d\theta + \omega\theta + \theta\omega$,
Gauge transformation	$\theta \to G^{-1}\theta G$, $\omega \to G^{-1}\omega G + G^{-1}dG$,
	where $G \in \exp(\Omega^{20})$.

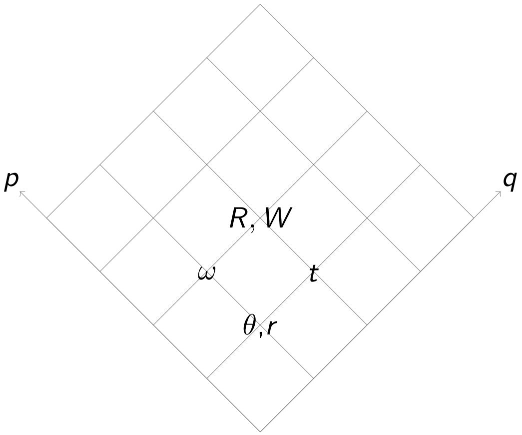
Hodge $*$

$$* : Cl^i \to Cl^{n-i}$$
$$*f(\xi_1, \ldots, \xi_n) = \int e^{\eta^{ij}\xi_i\zeta_j} f(\zeta_1, \ldots, \zeta_n) d\zeta_n \cdots d\zeta_1.$$
$$*_1 : \Omega^{pq} \to \Omega^{p,n-q}$$
$$*_2 : \Omega^{pq} \to \Omega^{n-p,q}$$

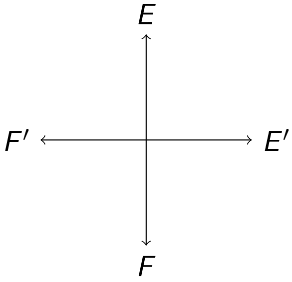
$\mathfrak{sl}(2) \times \mathfrak{sl}(2)$

$$E : \Omega^{pq} \to \Omega^{p+1,q+1},\ 2Ex = \theta x + (-1)^{p+q} x\theta$$
$$E' : \Omega^{pq} \to \Omega^{p+1,q-1}, E' = *_1^{-1} E *_1$$
$$F' : \Omega^{pq} \to \Omega^{p+1,q-1}, F' = *_2^{-1} E *_2$$
$$F : \Omega^{pq} \to \Omega^{p-1,q-1},\ F = *_1^{-1} *_2^{-1} E *_2 *_1.$$
$$H : \Omega^{pq} \to \Omega^{p,q},\ H = n - p - q,$$
$$H : \Omega^{pq} \to \Omega^{p,q},\ H = p - q.$$

Claim:
The operators E, F, H, E', F', H' generate the action of $\mathfrak{sl}(2) \times \mathfrak{sl}(2)$ on $\Omega^{\cdot\cdot}$.



p
q
R, W
ω
t
θ,r



E
F′
E′
F

Ex 1. Uniqueness of the torsion zero connection.

Proof: $t = d\theta + \omega\theta + \theta\omega = d\theta + 2E\theta$.
$E : \Omega^{21} \to \Omega^{12}$ is an isomorphism since $H_{\Omega^{12}} = -1$, $H_{\Omega^{21}} = 1$.
Therefore for a given θ there exists a unique ω such that the torsion vanishes.

Ex 2. Bianchi identity

The curvature R is symmetric considered as an element of
$Cl^2(\mathrm{V}) \otimes Cl^2(\mathrm{V})$
Proof:
$t = 0 \implies$
$\nabla t = dt + \omega t - t\omega = R\theta - \theta R = -2E'R = 0.$
Since $H'R = (2-2)R = 0$ we have also
$F'R = 0$
therefore $\exp(\frac{\pi}{2}(E' - F'))R = R.$

□

Ex. 3. Weil tensor

The Weyl tensor is a projection of the curvature R alng the image of E. It is conformaly invariant.
Proof:
Lemma: If (θ, ω) is torsion zero then
$(\tilde{\theta}, \tilde{\omega}) = (e^{\phi}\theta, \, = \omega + \theta\varepsilon - \varepsilon\,\theta)$, where $\phi \in \Omega^{01}$ and
$\varepsilon = \frac{1}{4}F' d\phi \in \Omega^{10}$ is torsion zero.

The corresponding curvature is
$\tilde{R} = R - \theta\rho - \rho\theta = R - 2E\rho$
where $\rho = d\varepsilon + \omega\varepsilon - \varepsilon\omega + \varepsilon\theta\varepsilon \in \Omega^{11}$.

□

Example of the computation:

$$\begin{aligned}\tilde{R} =& d\tilde{\omega} + \tilde{\omega}^2 = R + d(\theta\varepsilon - \varepsilon\theta) + \omega(\theta\varepsilon - \varepsilon\theta) + (\theta\varepsilon - \varepsilon\theta)\omega + (\theta\varepsilon - \varepsilon\theta)^2 = \\ =& -(\omega\theta + \theta\omega)\varepsilon + \varepsilon(\omega\theta + \theta\omega) - \theta d\varepsilon - d\varepsilon\theta + \omega(\theta\varepsilon - \varepsilon\theta) + (\theta\varepsilon - \varepsilon\theta)\omega + \\ &+ (\theta\varepsilon - \varepsilon\theta)^2 = R - \theta(d\varepsilon + \omega\varepsilon - \varepsilon\omega + \varepsilon\theta\varepsilon) - (d\varepsilon + \omega\varepsilon - \varepsilon\omega + \varepsilon\theta\varepsilon)\theta.\end{aligned}$$

Ex 4. Einstein equation.

Hilbert action: $S(\theta, \omega) = \operatorname{str} \int_M \theta^{n-2}(d\omega + \omega^2) = \operatorname{str} \int_M E^{n-2} R$.

$$\begin{aligned} \frac{\delta S}{\delta \theta} =& (-1)^{(n^2+n)/2}(n-2)E^{n-3}(d\omega + \omega^2) \\ \frac{\delta S}{\delta \omega} =& (-1)^{(n^2-n)/2}E^{n-3}(d\theta + \theta\omega + \omega\theta). \end{aligned}$$

$E^{n-3}t = 0 \implies t = 0$
Einstein equation: $E^{n-3}R = 0 \implies$ Ricci tensor $r = FR = 0$.
In dimension 4 $HR = 0$, therefore R is invariant under the whole $\mathfrak{sl}(2) \times \mathfrak{sl}(2)$.

Generalisation: Kähler case.

Two vielbeins θ and $\bar{\theta}$
Four copies of the algebra $\mathfrak{sl}(2)$.

Claim:
These $\mathfrak{sl}(2)$-s generate the affine group $\widehat{\mathfrak{sl}}(4)$