Demazure crystals for Kohnert polynomials

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References

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Demazure crystals for Kohnert polynomials
Consider the cohomology theory for the complete flag manifold of nested subspaces
\[ \mathcal{F} : \mathcal{F}^{(0)} \subset \mathcal{F}^{(1)} \subset \cdots \subset \mathcal{F}^{(n)} \quad \text{dim}(\mathcal{F}^{(i)}) = i \]

Let \( \mathcal{I}_n \) be the ideal generated by symmetric polynomials with no constant term.

**Theorem (Borel 1953)**

The cohomology ring \( H^*(\mathcal{F}) \) is \( \mathbb{C}[x_1, \ldots, x_n]/\mathcal{I}_n \) and has a linear basis of cosets \( [X_w] \) for each \( w \in S_n \), and so we may compute
\[
[X_u \cap X_v] = [X_u] \cdot [X_v] = \sum_w c_{u,v}^w [X_w].
\]

**Divided difference operators** \( \partial_i \) act by
\[
\partial_i f = \frac{f - s_i \cdot f}{x_i - x_{i+1}}
\]

For any reduced expression \( w = s_{i_k} \cdots s_{i_1} \)
\[
\partial_w = \partial_{i_k} \cdots \partial_{i_1}
\]

Schubert polynomials are a \( \mathbb{Z} \)-basis for the polynomial ring \( \mathbb{Z}[x_1, \ldots, x_n] \).

**Definition (Lascoux–Schützenberger 1982)**

The Schubert polynomial \( \mathcal{S}_w \) is
\[
\mathcal{S}_w = \partial_{w^{-1}w_0} \left( x_1^{n-1} x_2^{n-2} \cdots x_{n-1} \right)
\]
where \( w_0 = n \cdots 21 \) is the long element.

**Example (Schubert polynomials)**

- \( \mathcal{S}_{w_0} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1} \)
- \( \mathcal{S}_{v(\lambda,n)} = s_\lambda (x_1, \ldots, x_n) \)

**Theorem (Lascoux–Schützenberger 1982)**

\( \mathcal{S}_w \) represents the Schubert class \( [X_w] \), and
\[
\mathcal{S}_u \mathcal{S}_v = \sum_w c_{u,v}^w \mathcal{S}_w
\]
Finite dimensional irreducible representations of \( \mathfrak{g} \) decompose into weight spaces \( V^\lambda = \bigoplus_a V_a^\lambda \).

The Weyl group acts on extremal weight spaces \( \{ V_{w \cdot \lambda}^\lambda \mid w \in W \} \), which are all 1-dimensional.

**Definition (Demazure (1974))**

The Demazure module \( V_w^\lambda \) is the \( b \)-submodule of the irreducible \( \mathfrak{g} \)-representation \( V^\lambda \) generated by the extremal weight space \( V_{w \cdot \lambda}^\lambda \). Demazure characters are \( \text{char}(V_w^\lambda) = \kappa_{w \cdot \lambda} \).

**Example (Demazure modules)**

- \( V_{id}^\lambda = V_\lambda^\lambda \) is the 1-dim highest wt space
- \( V_{w_0}^\lambda = V_{\operatorname{rev}(\lambda)}^\lambda = V^\lambda \) is the full module

For \( \mathfrak{gl}_n \), index Demazure modules by

\[ (w, \lambda) \mapsto w \cdot \lambda \quad a \mapsto (w_a, \operatorname{sort}(a)) \]

where \( w_a \) is the shortest s.t. \( w_a \cdot a = \lambda \).

**Example (Demazure characters)**

- \( \kappa_\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n} \)
- \( \kappa_{\operatorname{rev}(\lambda)} = s_\lambda(x_1, \ldots, x_n) \)

Demazure characters are a \( \mathbb{Z} \)-basis for the polynomial ring \( \mathbb{Z}[x_1, \ldots, x_n] \) whose structure constants are not nonnegative.
A **diagram** is finitely many cells in $\mathbb{N} \times \mathbb{N}$.

The weight is $\text{wt}(D)_i = \#\text{cells in row } i$.

A **key diagram** is a left-justified diagram.

Axel Kohnert (1962-2013) was a student of Lascoux who devised a simple model for Demazure characters.

**Definition (Kohnert 1991)**

A **Kohnert move** on a diagram selects the rightmost cell $c$ of a row and moves $c$ to the first available position below, jumping over other cells in its way as needed.

Let $\text{KD}(D)$ be the set of **Kohnert diagrams** for $D$.

**Theorem (Kohnert 1991)**

The Demazure character $\kappa_a$ is given by

$$\kappa_a = \sum_{T \in \text{KD}(\text{key}(a))} x_1^{\text{wt}(T)_1} \cdots x_n^{\text{wt}(T)_n}$$
Kohnert bases

Use this to generalize Demazure characters and Schubert polynomials simultaneously.

**Definition (Assaf–Searles 2019)**

The *Kohnert polynomial* $\mathcal{K}_D$ of a diagram $D$ is

$$\mathcal{K}_D = \sum_{T \in \text{KD}(D)} x_1^{\text{wt}(T)}_1 \cdots x_n^{\text{wt}(T)}_n$$

**Example**

- Schur polynomials $\mathcal{K}_\lambda = s_\lambda(x_1 \cdots x_n)$
- Demazure characters $\mathcal{K}_{\text{key}}(a) = \kappa_a$
- Schubert polynomials $\mathcal{K}_{\text{Rothe}}(w) = \mathcal{S}_w$

Each Kohnert polynomial has a unique leading term minimal in lexicographic order.

**Definition (Assaf–Searles 2019)**

A basis $\{\mathcal{B}_a\}$ is a *Kohnert basis* if $\mathcal{B}_a = \mathcal{K}_D$ for some diagram $D$ with $\text{wt}(D) = a$.

For each $a$, choose *any* diagram $D$ with $\text{wt}(D) = a$ to construct a Kohnert basis.

**Definition (southwest diagrams)**

A diagram $D$ is *southwest* if $(r_2, c_1), (r_1, c_2) \in D$ s.t. $r_1 < r_2$ and $c_1 < c_2$ implies $(r_1, c_1) \in D$.

Partition diagrams, key diagrams and Rothe diagrams are all southwest diagrams.

**Conjecture (Assaf–Searles 2019)**

Kohnert polynomials of *southwest diagrams* are positive sums of Demazure characters.
Schur polynomials are also characters for finite connected normal $\mathfrak{gl}_n$ crystals.

Crystal basis $\mathcal{B}$, weight map $\text{wt} : \mathcal{B} \to \mathbb{Z}^n$
crystal lowering operators $f_i : \mathcal{B} \to \mathcal{B} \cup \{0\}$ such that $\text{wt}(b) - \text{wt}(f_i(b)) = e_i - e_{i+1}$.

The character of a crystal is

$$\text{char}(\mathcal{B}) = \sum_{b \in \mathcal{B}} x_1^{\text{wt}(b)_1} \cdots x_n^{\text{wt}(b)_n}$$

The standard $\mathfrak{gl}_n$ crystal has $\text{wt}([i]) = e_i$

The connected (finite, normal) $\mathfrak{gl}_n$ crystals are indexed by dominant weights (partitions).

For $\mathcal{B}(\lambda)$ is the crystal for the irrep $S_\lambda(\mathbb{C}^n)$

$$\text{char}(\mathcal{B}(\lambda)) = \text{char}(S_\lambda(\mathbb{C}^n)) = s_\lambda(x_1, \ldots, x_n)$$
Define **crystal operators** $e_i$ on $SSYT(\lambda)$ that change an $i + 1$ to an $i$ in $T$ by

**Definition (Pairing rule)**

Two cells $i$ and $i + 1$ are paired if in the same column or $i + 1$ left of $i$ and no unpaired cells $i$ or $i + 1$ between.

Compute $e_2$ by changing a 3 to a 2:

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**Definition (Crystal raising operators)**

For $T \in SSYT(\lambda)$ and $1 \leq i < n$, the crystal raising operator $e_i$ acts on $T$ by

- $e_i(T) = 0$ if $T$ has no unpaired $i + 1$
- change leftmost unpaired $i + 1$ to $i$
Define operators $\mathcal{D}_i$ on subsets $X \subseteq B$ by
$$\mathcal{D}_i X = \{ b \in B \mid e^k_i (b) \in X \}$$
For $w = s_{i_k} \cdots s_{i_1}$ reduced expression
$$B_w(\lambda) = \mathcal{D}_{i_k} \cdots \mathcal{D}_{i_1} \{ u_\lambda \}$$
where $u_\lambda$ is the highest weight of $B(\lambda)$.

**Theorem (Kashiwara 1993)**

The Demazure character $\kappa_a$ is given by
$$\kappa_a = \text{char} \left( V^\lambda_w \right) = \text{char} \left( B_w(\lambda) \right)$$

**Example (Compute $B_{2413}(2, 2, 1, 0)$)**

For $w = 2413$, we may take $w = s_1 s_3 s_2$

$$\kappa_{(1,2,0,2)} = x_1^2 x_2^2 x_3 + x_1^2 x_2 x_3^2$$
$$+ x_1^2 x_2 x_4 + x_1^2 x_2 x_3 x_4 + x_1^2 x_2 x_4^2$$
$$+ x_1 x_2^2 x_3^2 + x_1 x_2^2 x_3 x_4 + x_1 x_2 x_4^2$$
Define crystal operators $e_i$ on $\text{KD}(D)$ that moves a cell of $T$ in row $i$ down to row $i+1$.

**Definition (Pairing rule)**

Two cells in rows $i$ and $i+1$ are paired if in the same column or higher is right of lower with no unpaired cells between.

Compute $e_2$ by moving cell in row 3 to row 2:

$e_2$ changes $\kappa(0, 3, 2, 0) + \kappa(0, 3, 1, 1)$ to $\kappa(0, 3, 2, 0) + \kappa(0, 3, 1, 1)$.

**Definition (Crystal raising operators)**

For $T \in \text{KD}(D)$ and $1 \leq i < n$, the crystal raising operator $e_i$ acts on $T$ by

1. $e_i(T) = 0$ if no unpaired cell in row $i+1$
2. push rightmost unpaired cell in row $i+1$ down to row $i$
Kohnert crystal for southwest diagrams

Theorem (Assaf 2019$^+$)

For $D$ a southwest diagram, if $T \in \text{KD}(D)$ and $e_i(T) \neq 0$, then $e_i(T) \in \text{KD}(D)$.

Example (Closure of crystal operators)

In general, crystal operators do not act by Kohnert moves on diagrams.

If $D$ is not southwest, then either $e_i(D) \notin \text{KD}(D)$ or this is not a Demazure crystal.

Define rectification operators that map any diagram to one in $\text{KD}(\text{key}(a))$ then to $\text{SSYT}(\lambda)$ and which commute with crystal operators. This embeds $\text{KD}(D)$ into a normal crystal.

Theorem (Assaf 2019$^+$)

For $D$ a southwest diagram, the rectification map embeds $\text{KD}(D)$ as a Demazure subset of a normal crystal, and so the Kohnert polynomial $K_D$ is a positive sum of Demazure characters.
A connected crystal has a unique highest weight element $u$ characterized by $e_i(u) = 0$ for all $i$.

$$\text{char } (\mathcal{B}) = \sum_{u \in \mathcal{B} \text{ s.t. } e_i(u) = 0 \forall i} s_{\text{wt}(u)}(x_1, \ldots, x_n)$$

**Theorem (Assaf–Searles 2019)**

The Kohnert quasisymmetric functions are well-defined by $K_D(X) = \lim_{m \to \infty} \mathcal{R}_{0m \times D}$, and expand nonnegatively into Gessel’s fundamental basis for quasisymmetric functions.

In particular, for Demazure characters we have $\lim_{m \to \infty} \kappa_{0m \times a}(x_1, \ldots, x_m, 0, \ldots, 0) = s_{\text{sort}(a)}(X)$

**Corollary**

For $D$ a southwest diagram, the Kohnert quasisymmetric function $K_D$ is Schur positive.

**Example (The five highest weight elements of $K_D(D)$ for $D$ the left diagram)**

$K_D(X) = s_{(3,2,1,1)}(X) + s_{(3,2,2)}(X) + s_{(3,3,1)}(X) + s_{(4,1,1,1)}(X) + s_{(4,2,1)}(X)$
Demazure lowest weights

Demazure crystals have unique highest weights but $B_w(\lambda)$ has highest weight $\lambda$ for every $w$.

Example (The five highest weight elements of $KD(D)$ for $D$ the left diagram)

$$\lim_{m \to \infty} K_{0m} \times D = s(3,2,1,1) + s(3,2,2) + s(3,3,1) + s(4,1,1,1) + s(4,2,1).$$

We give an explicit algorithm for constructing Demazure lowest weight elements from highest weight elements, and so obtain an explicit formula for the Demazure character expansion.

Example (The five Demazure lowest weight elements of $KD(D)$ for $D$ the left diagram)

$$K_D = \kappa(0,1,3,0,1,2) + \kappa(0,2,3,0,0,2) + \kappa(0,3,3,0,0,1) + \kappa(0,1,4,0,1,1) + \kappa(0,2,4,0,0,1).$$

Open: Is there a simple rule on the diagram $D$ that generates Demazure lowest weights?
Stanley (1984) defined symmetric functions $S_w$ using reduced words for a permutation $w$.

**Theorem (Edelman–Greene 1987)**

\[
S_w = \sum_{\rho \in R(w)} \delta_{\text{Des}}(\rho) \quad \exists T \text{ increasing, row}(T) = \rho
\]

\[
S_{13625847} = s_{(3,2,1,1)} + s_{(3,2,2)} + s_{(3,3,1)} + s_{(4,1,1,1)} + s_{(4,2,1)}
\]

Macdonald (1991) proved $S_w$ is the stable limit of $G_w$, specifically $\lim_{m \to \infty} G_{1m} \times w = S_w$.


**Theorem (Assaf 2019+)**

\[
G_w = \sum_{\rho \in R(w)} \kappa_{\text{des}}(\text{lift}(\rho)) \quad \exists T \text{ increasing, row}(T) = \rho
\]

where lift($\rho$) $\in R(w)$ is explicit.

\[
G_{13625847} = \kappa_{(0,1,3,0,1,2)} + \kappa_{(0,2,3,0,0,2)} + \kappa_{(0,3,3,0,0,1)} + \kappa_{(0,1,4,0,1,1)} + \kappa_{(0,2,4,0,0,1)}
\]
The Rothe diagram of a permutation $w$ is $\mathbb{D}(w) = \{(i, w_j) \mid i < j \text{ and } w_i > w_j\}$.

**Theorem (Assaf 2019+)**

There is a bijection between increasing Young tableaux with reading words in $R(w)$ and highest weight elements of the Kohnert crystal on $KD(\mathbb{D}(w))$. In particular, $S_w = K_{\mathbb{D}(w)}$.

Kohnert asserted $S_w = K_{\mathbb{D}(w)}$. Winkel (1999, 2002) gave two published proofs of this.

**Example (Schur expansion bijection for 13625847)**

$\begin{array}{cccc}
6 & 5 & 3 & 7 \\
5 & 3 & 7 & 6 \\
3 & 7 & 6 & 2
\end{array}$

$\begin{array}{cccc}
6 & 7 & 5 & 3 \\
7 & 5 & 3 & 6 \\
5 & 3 & 6 & 2
\end{array}$

$\begin{array}{cccc}
6 & 3 & 5 & 7 \\
5 & 3 & 7 & 6 \\
3 & 7 & 6 & 2
\end{array}$

$\begin{array}{cccc}
6 & 5 & 3 & 7 \\
5 & 3 & 7 & 6 \\
3 & 7 & 6 & 2
\end{array}$

$\begin{array}{cccc}
6 & 3 & 5 & 7 \\
5 & 3 & 7 & 6 \\
3 & 7 & 6 & 2
\end{array}$

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**Theorem (Assaf 2019+)**

There is a bijection between Yamanouchi key tableaux with reading words in $R(w)$ and Demazure lowest weight elements of the Kohnert crystal on $KD(\mathbb{D}(w))$. In particular, Kohnert’s rule for Schubert polynomials holds: $\mathcal{S}_w = K_{\mathbb{D}(w)}$.

**Example (Demazure expansion bijection for 13625847)**

$\begin{array}{cccc}
6 & 7 & 5 & 3 \\
7 & 5 & 3 & 6 \\
5 & 3 & 6 & 2
\end{array}$

$\begin{array}{cccc}
6 & 7 & 5 & 3 \\
7 & 5 & 3 & 6 \\
5 & 3 & 6 & 2
\end{array}$

$\begin{array}{cccc}
6 & 3 & 5 & 7 \\
5 & 3 & 7 & 6 \\
3 & 7 & 6 & 2
\end{array}$

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Demazure crystals for Kohnert polynomials


Merci!