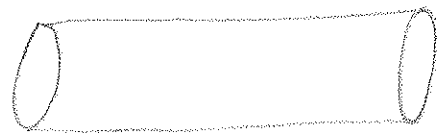
String interactions

So far we have considered the propagation of free bosonic strings, and the interpretation of the oscillation modes on the 2d worldsheet as quantum fields of given mass & spin in space-time.

Summary:

For closed strings: free propagation is described by a cylindrical worldsheet

$0 \leq \sigma < 2\pi, \quad -\infty < \tau < \infty$



supporting $D=26$ massless scalars X^μ

and the bc ghosts.

This can be mapped to the plane \ origin via $z = e^{w}$, $w = \tau + i\sigma$

* Quantization with respect to ∂_τ on the cylinder corresponds to radial quantization on the plane.

* Mode expansions of chiral primary fields on the cylinder are mapped to Taylor expansions on the plane:

$$\phi_{cyl}(w) = \sum_{n \in \mathbb{Z}} \phi_n \, e^{-n(\tau + i\sigma)}$$

$$\phi_{plane}(z) = \left(\frac{dz}{dw}\right)^{-\Delta} \phi_{cyl}(w) = \sum_{n \in \mathbb{Z}} \phi_n \, z^{-n-\Delta}$$

For quasi-primary fields, there may be corrections, eg

$$T_{plane}(z) = \sum_{n \in \mathbb{Z}} L_n \, z^{-n-2}$$

$$T_{cyl}(w) = \sum_{n \in \mathbb{Z}} \left(L_n - \frac{c}{24}\delta_{n,0}\right) e^{-n(\tau + i\sigma)}$$

Eg $$\partial X^\mu_{cyl}(w) = \frac{l_s}{\sqrt{2}} \sum_{n \in \mathbb{Z}} \alpha^\mu_n \, e^{-n(\tau + i\sigma)}$$

$$\partial X^\mu_{plane}(z) = \frac{l_s}{\sqrt{2}} \sum_{n \in \mathbb{Z}} \alpha^\mu_n \, z^{-n-1}$$

* Local operators on the plane are in 1-1 correspondence with states on the cylinder:

$$|\mathcal{O}\rangle = \lim_{z\to 0} \mathcal{O}(z,\bar z)\,|0\rangle$$

in such way that a transition amplitude

$$\langle \mathcal{O}_n | \prod_{i=2}^{n-1} \mathcal{O}_i(\tau_i,\sigma_i) \; | \mathcal{O}_1\rangle$$

↑ product of time ordered operators

is equal to the n-pt correlator

$$\lim_{z_n,\bar z_n\to\infty,\; z_1,\bar z_1\to 0} z_n^{2h_n}\bar z_n^{2\bar h_n} \langle \prod_{i=1}^{n} \mathcal{O}_i(z_i,\bar z_i)\rangle$$

* By construction, the state $|0\rangle$ is related to the identity operator $\mathbb{1}$ and annihilated by all $L_{n\geq -1}$

(since $T(z)|0\rangle$ regular at $z=0$)

In particular, it is invariant under $SL(2,\mathbb{C})$ (generated by $L_0, L_{\pm 1}, \tilde L_0, \tilde L_{\pm 1}$)

* Similarly

$$\alpha^\mu_{-m}|0\rangle = \frac{\sqrt2}{\ell_s}\oint \frac{dz}{2\pi} z^{-m}\,\partial X^\mu(z)|0\rangle$$

$$= \frac{i\sqrt2}{\ell_s}\frac{\partial^m X^\mu}{(m-1)!}(0)\,|0\rangle \qquad \text{if } m\geq 1\,,\ 0 \text{ if } m\leq 0$$

$$= \left|\frac{i\sqrt2}{\ell_s}\frac{\partial^m X^\mu}{(m-1)!}\right\rangle \qquad \text{if } m\geq 1$$

moreover $\alpha_0^\mu|0\rangle = 0$ so $|0\rangle$ is translationally invariant.

* The state associated to e^{ipX} is the ground state with momentum p:

$$\alpha^\mu_{-m}|p\rangle = \frac{\sqrt2}{\ell_s}\oint\frac{dz}{2\pi} z^{-m}\underbrace{\partial X^\mu(z)\, e^{ipX}(0)}_{\frac{ip\,e^{ipX}}{z} + \text{reg}}|0\rangle$$

$$= 0 \text{ if } m\leq -1,\ \frac{\sqrt2}{\ell_s}p^\mu|p\rangle \text{ if } m=0\ ,\left|\frac{i\sqrt2}{\ell_s}\frac{\partial^m X^\mu e^{ipX}}{(m-1)!}\right\rangle \text{ if } m\geq 1$$

* Similarly, for the b, c ghosts

$$b_{-m}|1\rangle \longleftrightarrow \frac{1}{(m-2)!}\partial^{m-2} b \qquad m \geq 2$$

$$\left(b_{-1}|1\rangle = b_{-1}^2 |\downarrow\rangle = 0 \right.$$
$$b_0 |1\rangle = -b_{-1} b_0 |\downarrow\rangle = 0$$

$$c_{-m}|1\rangle \longleftrightarrow \frac{1}{(m+1)!}\partial^{m+1} c \qquad m \geq -1$$

Recall $\begin{cases} c_0|\downarrow\rangle = |\uparrow\rangle \\ \langle 1| c_{-1} c_0 c_1 |1\rangle = 1 \end{cases}$

$$|1\rangle = b_{-1}|\downarrow\rangle$$

$$|\downarrow\rangle = (c_1 b_{-1} + \cancel{c_{-1} b_1})|\downarrow\rangle = c_1|1\rangle = |c\rangle$$

$$|\uparrow\rangle = c_0 c_1 |1\rangle$$

* Conserved charge map to contour integral. e.g. $P_\mu = \oint d\sigma\, \partial_\tau X^\mu = i\oint dz\, \partial X^\mu (+c.c.)$: $P_\mu |e^{ipx}\rangle = \oint \frac{dz}{z} ip\, e^{ipX}(0) = ip|e^{ipx}\rangle$

* In general,

$$L_n|\mathcal{O}\rangle = \oint \frac{dz}{2\pi i} z^{n+1} T(z)\, \mathcal{O}(0)\, |0\rangle$$

$$= \oint \frac{dz}{2\pi i} z^{n+1} \Big[\underbrace{\cdots}_{\text{vanishes for primary}} + \frac{\Delta \mathcal{O}}{z^2} + \frac{\partial \mathcal{O}}{z} + \cdots \Big] |0\rangle$$

so $L_{-1}|\mathcal{O}\rangle = |\partial \mathcal{O}\rangle$ always

$L_0|\mathcal{O}\rangle = \Delta |\mathcal{O}\rangle$ for quasi primaries

$L_n|\mathcal{O}\rangle = 0 \quad \forall n>0$ for primaries.

* Physical states, such that $Q_{BRST}|\mathcal{O}\rangle = 0$,

are mapped to vertex operators which commute with the BRST charge

$$Q_{BRST} = \oint \frac{dz}{2\pi i}\left(c T^X + \frac{1}{2} c T^{gh} \right),$$

ie those without 1st order pole with $j_{BRST} = cT^X + \frac{1}{2} c T^{gh}$

Thus $\mathcal{O} = c(z)\, \bar{c}(\bar{z}) \cdot \Phi$

$\hookrightarrow$ $\boxed{\text{Primary field of weight } (1,1)}$

Indeed $c(z)T^X(z) \cdot c(w)\Phi(w) \sim \partial c \cdot c(w)\,(z-w) \cdot \left[\frac{\Delta \phi}{(z-w)^2} + \cdots \right]$

$$\frac{1}{2} c(z) T^{gh}(z) \cdot c(w)\Phi(w) \sim \frac{1}{2}\left[c(w) + (z-w)\partial c \right]\left[\frac{-c(w)}{(z-w)^2} + \frac{\partial c(w)}{(z-w)^2} \right] \phi(w) \sim \frac{c\partial c\, \phi}{z-w}$$

<u>For open strings</u> the free propagation is instead described by the strip $0 \le \sigma \le \pi$, $-\infty < \tau < \infty$

The same map $z = e^{w}$ maps it to the upper half plane

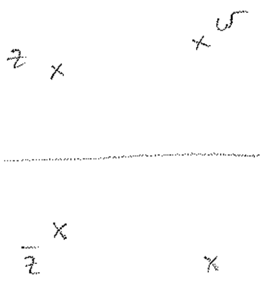
The condition that no momentum should flow across the boundary requires $T_{zz} = T_{\bar z \bar z}$ at $\mathrm{Im}\, z = 0$

The stress-tensor is still conserved and traceless: $T_{z\bar z} = 0$, $\partial_{\bar z} T_{zz} = 0$

It is sometimes convenient to extend $T \equiv T_{zz}$ to the whole complex plane, by requiring $T_{zz}(z) \equiv T_{\bar z \bar z}(\bar z)$ for $\mathrm{Im}\, z < 0$.
All integrals from $\sigma = 0, \pi$ at finite τ can then again be rewritten as contour integrals around $z = 0$.

Equivalently, the upper half plane can be obtained as the quotient of the full plane under the antiholomorphic involution $z \to \bar z$.

The propagator of free fields can then be obtained by method of images:



e.g. for a massless free boson:

$$\langle X(z,\bar z) X(w,\bar w) \rangle$$

$$= -\frac{\ell_s^2}{2} \left[\log|z-w|^2 \pm \log|z-\bar w|^2 \right]$$

+ : Neumann-Neumann

− : Dirichlet-Dirichlet

The $\ominus$ sign ensures that $X = 0$ when $w = \bar w$ or $z = \bar z$

$\oplus$ $\partial X = 0$ " "

For the b, c ghosts: require $\tilde b(\bar z) = b(z)$

$\tilde c(\bar z) = c(z)$

The state-operator correspondence proceeds as before, except that the local operators are on the boundary

Physical states are of the form $\mathcal{O} = c(z)\, \Phi$

where Φ is a primary boundary field of dimension 1.

Note that the conformal dimension of e^{ipX} is now $\ell_s^2 p^2$, since

$$\partial X^\mu(z) :e^{ipX(w)}: = -\frac{\ell_s^2}{2} \sum \frac{(ip)^n}{(n-1)!} \left(X(w,\bar w)\right)^{n-1} \left(\frac{1}{z-w} + \frac{1}{z-\bar w} \right)$$

$$= \frac{-i \ell_s^2 p^\mu}{2}\, e^{ipX(w)} \left(\frac{1}{z-w} + \frac{1}{z-\bar w} \right)$$

$$T(z) :e^{ipX}: \sim \frac{1}{\ell_s^2} \cdot \left(\frac{-i\ell_s^2 p^\mu}{2} \right)^2 \left(\frac{1}{z-w} + \frac{1}{z-\bar w} \right)^2 + \cdots$$

$$\sim \frac{\ell_s^2 (p^\mu)^2\, V_p}{(z-w)^2} \quad \text{at } w = \bar w$$

Thus, for the open string tachyon, $M^2 = -1/\ell_s^2$.

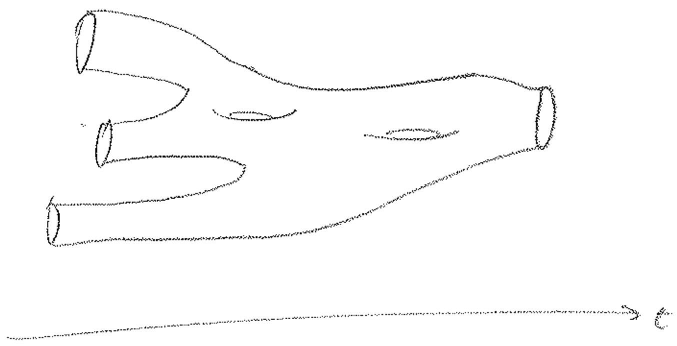
Rk the operator $\mathcal{O}$ is sometimes called the 'unintegrated' vertex operator associated to the state ϕ.
The operator Φ, having dimension 1, can be integrated in a conformally invariant way on the worldsheet (or on the boundary of the ws in the open string case) so is called 'integrated'

Φ is not BRST invariant, but $\delta_Q \Phi = i\varepsilon\, \partial_\alpha (c^\alpha \Phi)$ (Polchinski I, 5.4.5

so the integral $\int d^2z\, \Phi$ is BRST invariant.

Generalities on scattering amplitudes

In principle, we would like to compute transition amplitudes between arbitrary set of loops at initial / final time.



However, unlike in field theory, off-shell amplitudes are hard (if not impossible) to define. Not surprising since strings include gravity. Equivalently, there appears to exist no good notion of closed string field theory (this is alleviated for open strings, to a certain extent)

We'll focus on S matrix elements between asymptotic on-shell states

In that case, the external states correspond to infinite cylinders (or strips in open string case). By a conformal transformation, they can be represented by a vertex operator insertion.

One must integrate over all embeddings X^μ, metrics $\gamma_{\alpha\beta}$ and location of the vertices, and divide by the volume of the group of gauge symmetries: roughly,

$$S = \frac{\int DX^\mu D\gamma_{\alpha\beta} \prod_{i=1}^{n} dz_i d\bar{z}_i}{V_{\text{diff} \times \text{Weyl}}} \left\langle \prod \Phi_i(z_i, \bar{z}_i) \, e^{-S_P} \right\rangle$$

There are some complications however:

- The space of metrics is disconnected: components are labelled by the genus h, or the Euler number $\chi = 2-2h$.

  In principle, one could weight different components by a_h.

  To ensure the correct factorisation properties, a_h must be the exponential of a local functional of the metric.

  Since $\chi = \frac{1}{4\pi}\int d^2\sigma \sqrt{\gamma}\, R$ by the Gauss-Bonnet theorem, we take

$$a_h = g_s^{2h-2} = e^{-\int \frac{d^2\sigma}{4\pi}\sqrt{g}\, R\,\phi}$$

  where $g_s = e^{\phi}$ is the closed string coupling.

  It turns out that ϕ can be interpreted as the vev of the dilaton.

  There is also a factor of g_s from each puncture

- Locally, one can always go to conformal gauge $\gamma_{\alpha\beta} = \begin{pmatrix} -1 & \\ & +1 \end{pmatrix}$.

  and get rid of $\gamma_{\alpha\beta}$ entirely.

  Globally, there are in general obstructions, and one is left with an integral $\int_{\mathcal{M}_h} d^\mu t$ over a finite number of <u>complex structure moduli</u>

E.g on T^2: conformal structures are labelled by $\tau \in \mathbb{C}$, $\operatorname{Im}\tau \geq 0$

$$ds^2 = |dx - \tau dy|^2/\tau_2 \qquad x \equiv x+1, \quad y \equiv y+1$$

$$= dz\, d\bar{z}/\tau_2 \qquad z \equiv z+1 \equiv z+\tau$$

but τ and $\frac{a\tau+b}{c\tau+d}$ with a,b,c,d integer, $ad-bc=1$

give rise to isomorphic metrics.

In general, $\mu = 3h-3$ if $h>1$,
1 if $h=1$
0 if $h=0$

- Moreover, it can happen that the conformal structure is invariant under κ conformal transformations:

$h=0$: $\kappa = 3 = \dim SL(2,\mathbb{C})$

$h=1$: $\kappa = 1 = \dim \mathbb{R}^2$

$h>1$: $\kappa = 0$

If the number of insertions $n \geq \kappa$, we can fix this symmetry by fixing the position of κ of the vertex operators

- At the same time we must include the Jacobian for this gauge fixing, ie an integral over b, c ghosts. Remarkably, on a Riemann surface of genus h, # zero modes $(b) = \mu$!

zero modes $(c) = \kappa$

In order for the path integral to vanish, there should be at least μ insertions of b and κ insertions of c.

The final answer taking these facts into account is (Polchinski 5.3.9)

$$\left[S = \sum_{h=0}^{\infty} \int_{db_h} d^{\mu}t \int dX\, db\, dc \; \exp\left[-S_P - S_{gh} - \chi\phi\right] \left| \prod_{k=1}^{\mu} (b, \partial_{t_k}\hat{g}) \prod_{i=1}^{\kappa} c(\hat{z}_i) \prod_{i=\kappa+1}^{n} \int dz_i \right|^2 \prod_{i=1}^{n} \Phi_i(z_i) \right]$$

A similar results holds for open string amplitudes (with open string vertices integrated on the boundary) and for mixed open/closed amplitudes.

then, taking $z_1=0,\ z_2=1,\ z_3\to\infty,\ z_4=z$

$$S = g_s^2 \int dz_4 d\bar z_4\ z_3^2 \bar z_3^2 \cdot |z_3|^{l_s^2 p_1p_3} \ |z_4|^{l_s^2 p_1 p_4}$$
$$|z_3|^{l_s^2 p_2 p_3} \ |1-z_4|^{l_s^2 p_2 p_4} \cdot |z_3|^{l_s^2 p_3 p_4}$$

the power of z_3 is $2+(p_1p_3+p_2p_3+p_3p_4)l_s^2/_2 = 2-p_3^2 l_s^2 = 0$

leaving

$$S = g_s^2 \int dz d\bar z\ |z|^{-4-\frac{l_s^2 t}{2}}\ |1-z|^{-4-\frac{l_s^2 u}{2}}$$

Using $\int d^2z\ |z|^{2a-2}\ |1-z|^{2b-2} = \dfrac{2\pi\ \Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(1-a)\Gamma(1-b)\Gamma(1-c)}$

we arrive at the Virasoro Shapiro amplitude:

$$S = g_s^2\ (2\pi)^{26}\,\delta(\Sigma p_i) \cdot \frac{\Gamma(-1-\alpha's/4)\,\Gamma(-1-\alpha't/4)\,\Gamma(-\alpha'u/4)}{\Gamma(2+\alpha's/4)\,\Gamma(2+\alpha't/4)\,\Gamma(2+\frac{\alpha'u}{4})}$$

Since $\Gamma(x)$ has poles at $-x$ integer, we find poles at

$$\alpha's,\ \alpha't,\ \alpha'u = -4, 0, 4, 8, \dots$$

coming from the exchange of on-shell closed states.

In center of mass coordinates:

$$s = E^2$$
$$t = (4m^2-E^2)\sin^2\theta/_2$$
$$u = (4m^2-E^2)\cos^2\theta/_2$$

the hard scattering corresponds to $(s,t,u)\to\infty$ keeping ratio fixed

$$S \sim \exp\left[-\frac{l_s^2}{2}\ s\log s + t\log t + u\log u\right]$$

$$\sim \exp\left[-l_s^2 E^2 f(\theta)\right]$$: exponentially suppressed, unlike in field theory!

the Regge limit is $s\to\infty$, t fixed:

$$S \sim s^{2+\frac{l_s^2 t}{2}} \times \text{cte}$$

the size of the strings seems to grow as $l_s^2 \cdot E$

Tree level scattering of 3 closed string states:

$$S_{ijk} = \frac{1}{g_s^2} \cdot g_s^3 \left\langle |c(z_1)\, c(z_2)\, c(z_3)|^2 \; \phi_i(z_1,\bar z_1)\, \phi_j(z_2,\bar z_2)\, \phi_k(z_3,\bar z_3) \right\rangle$$

Recall $\langle \phi_i(z_1)\phi_j(z_2)\phi_k(z_3)\rangle = \dfrac{C_{ijk}}{z_{12}^{\Delta_1+\Delta_2-\Delta_3}\, z_{13}^{\Delta_1+\Delta_3-\Delta_2}\, z_{23}^{\Delta_2+\Delta_3-\Delta_1}} \times cc$

with $\Delta_i = 1$

while $\langle c(z_1)\, c(z_2)\, c(z_3)\rangle = z_{12}\; z_{13}\; z_{23}$

(due to zeromode contribution, $\langle c_{-1} c_0 c_1 \rangle = 1$)

thus $\boxed{S_{ijk} = g_s C_{ijk}}$

For 3 tachyons, $C_{ijk} = (2\pi)^{26}\delta^{26}(p_1+p_2+p_3)$

3 gravitons, one recovers the vertex of general relativity!

Consider now 4 closed string tachyons:

$$S = \frac{1}{g_s^2}\cdot g_s^4 \cdot \int dz_4 d\bar z_4 \left\langle |c(z_1)\, c(z_2)\, c(z_3)|^4 \prod_{i=1}^{4} e^{ip_iX}(z_i)\right\rangle$$

with $p_i^2 = 4/\ell_s^2$

Recall $\left\langle \prod_{i=1}^{n} e^{ip_iX}(z_i)\right\rangle = \prod_{i<j} |z_{ij}|^{\ell_s^2 p_ip_j} \cdot (2\pi)^{26}\delta^{26}(\Sigma p_i)$

Define Mandelstam variables

$s = -(p_1+p_2)^2 = -2p^2 - 2p_1p_2 = \frac{8}{\ell_s^2} - 2p_1p_2 \quad \Rightarrow p_1p_2 = \frac{4}{\ell_s^2} - s/2$

$t = -(p_2+p_3)^2$

$u = -(p_1+p_3)^2$

$s+t+u = -6p^2 - 2(p_1p_2+p_2p_3+p_1p_3) = -6p^2 - \left[(p_1+p_2+p_3)^2 - 3p^2\right] = -4p^2$

$= -16/\ell_s^2$

On the other hand, at low energy:

$$S_4 \sim g_s^2 \cdot (2\pi)^{26} \delta\left(\sum p_i\right) \cdot \left(\frac{64}{stu} + \left(\frac{1}{s} + \frac{1}{t} + \frac{1}{u}\right) + cte + \dots \right)$$

The divergences are due to exchange of massless gravitons etc.

The cte part is a contribution to T^4.

Similarly, 4-graviton scattering on the sphere at low energy has poles from exchange of gravitons + irreducible contribution to $h_{\mu\nu}^4$, consistent with Einstein's equations!

From these tree level scattering amplitudes, one can reconstruct an effective action for the low lying modes of the string:

$$S = \int d^{26}x \sqrt{-g}\, e^{-2\phi}$$
$$\times \left\{ (\partial T)^2 - \frac{4}{\alpha'} T^2 + \mathcal{O}(T^3) \right.$$
$$\left. + R + 4(\partial\phi)^2 - \frac{1}{12} H^2 + \dots \right\}$$

(Rk: an alternative way to find the effective action for the massless modes is to require that the non-linear σ model describing the string propagation in a curved background be conformally invariant.

$$S_P = \frac{1}{4\pi l_s^2} \int d^2\xi \sqrt{\gamma} \left(\gamma^{\alpha\beta} G_{\mu\nu}(X)\, \partial_\alpha X^\mu \partial_\beta X^\nu + \varepsilon^{\alpha\beta} B_{\mu\nu}(X)\, \partial_\alpha X^\mu \partial_\beta X^\nu \right.$$
$$\left. + l_s^2 R(\gamma)\, \phi(X) \right)$$

...

One - loop amplitudes

In field theory, the vacuum-to-vacuum transition amplitude for a free scalar field of mass m in D dimensions is

$$Z = \left[\det(-\partial^2 + m^2) \right]^{-\varepsilon/2} \qquad \text{with} \quad \begin{array}{l} \varepsilon = 1 \text{ for bosons} \\ \varepsilon = -1 \text{ for fermions} \end{array}$$

The connected vacuum amplitude is then

$$\log Z = -\frac{\varepsilon}{2} V_D \int \frac{d^D p}{(2\pi)^D} \log\left(p^2 - m^2 + i\varepsilon\right) \qquad \text{(Minkowskian)}$$

$$= -\frac{i\varepsilon}{2} V_D \int \frac{d^D p}{(2\pi)^D} \log\left(p^2 + m^2\right) \qquad \text{(Euclidean)}$$

It is useful to use the Schwinger time representation:

$$\int_0^\infty \frac{dt}{t^{1+s}} \exp(-\pi t M^2) = (\pi M^2)^s \, \Gamma(-s)$$

$$= -\frac{1}{s} - \left(\log \pi M^2 + \gamma \right) + O(s)$$

hence, up to an infinite additive constant,

$$\log Z = \frac{i\varepsilon}{2} V_D \int_0^\infty \frac{dt}{t} \frac{d^D p}{(2\pi)^D} \exp\left[-\pi t (p^2 + m^2)\right]$$

$$= \frac{i\varepsilon}{2} \frac{V_D}{(2\pi)^D} \int_0^\infty \frac{dt}{t^{1+D/2}} \exp\left[-\pi t m^2\right]$$

⚠ the UV divergence at $p \to \infty$ is seen as a divergence at $t = 0$

For a set of free fields of mass m_i, statistics ε_i

$$\log Z_{QFT} = \frac{i}{2} \frac{V_D}{(2\pi)^D} \int_0^\infty \frac{dt}{t^{1+D/2}} \sum_i \varepsilon_i \exp\left[-\pi t m_i^2\right]$$

[i includes a sum over all physical degrees of freedom,
e.g. $D-2$ for a gauge field
$(D-2)(D-1)/2 - 1$ for a graviton, etc]

RK This can be interpreted as an integral over all closed paths

$$\int_{X^\mu(0) = X^\mu(1)} DX^\mu(\tau) \exp\left[-\frac{1}{2} \int \left(\frac{(\partial X)^2}{e} + m^2 e \right) d\tau \right] \quad \text{with } e(\tau) \text{ gauged-fixed to } t = \int_0^1 e(\tau)\, d\tau$$

In bosonic closed string theory, the sum over physical d.o.f is most easily carried out in light cone gauge:

$$m_i^2 = \frac{2}{\ell_s^2}\left(L_0^{\perp} + \bar{L}_0^{\perp}\right) \quad , \quad \varepsilon_i = 1$$

where $$L_0^{\perp} = \sum_{m=1}^{\infty} \alpha^i_{-m}\, \alpha^i_m - \frac{D-2}{24}$$

$$\bar{L}_0^{\perp} = \sum_{m=1}^{\infty} \tilde{\alpha}^i_{-m}\, \tilde{\alpha}^i_m - \frac{D-2}{24}$$

subject to the level matching constraint $L_0^{\perp} - \bar{L}_0^{\perp} = 0$

Introducing a Lagrange multiplier θ for this constraint, the QFT result is

$$\log Z_{QFT} = \frac{i}{2}\frac{V_D}{(2\pi)^D}\int_0^{\infty}\frac{dt}{t^{1+D/2}}\int_{-1/2}^{1/2} d\theta$$

$$\times \operatorname{Tr}' \exp\left[-\frac{2\pi t}{\ell_s^2}\left(L_0^{\perp} + \bar{L}_0^{\perp}\right) + 2\pi i\theta\left(L_0^{\perp} - \bar{L}_0^{\perp}\right)\right]$$

let us define the 'complex Schwinger parameter'

$$\tau = \theta + \frac{it}{\ell_s^2} \equiv \tau_1 + i\tau_2 \quad , \quad q = e^{2\pi i\tau}$$

then

$$Z_{QFT} = \frac{i}{2}\frac{V_D}{(2\pi)^D}\iint_{\mathcal{F}_{QFT}}\frac{d\tau_1\, d\tau_2}{\tau_2^{1+D/2}} \operatorname{Tr}'\left[q^{L_0}\, \bar{q}^{\bar{L}_0}\right]$$

with $$\mathcal{F}_{QFT} = \left\{\tau_2 > 0 ,\ -\tfrac{1}{2} \le \tau_1 \le \tfrac{1}{2}\right\}$$

Moreover $$\operatorname{Tr}' q^{L_0} = q^{-\frac{D-2}{24}} \cdot \prod_{n=1}^{\infty}\left(1-q^n\right)^{-(D-2)}$$

$$= \eta^{-(D-2)}(\tau) \quad \text{where} \quad \eta(\tau) \equiv q^{\frac{1}{24}}\prod\left(1-q^n\right)$$

'Dedekind eta function'

This naive QFT result is UV divergent, due to the region $\tau_2 \to 0$.

The correct string theory result turns out to be given by the same integral, but with integration domain

$$\mathcal{F} = \mathcal{F}_{QFT} \cap \{ |\tau| > 1 \}$$

In particular, there are no UV divergences!

This can be justified as follows:

Recall the prescription

$$Z_h = \int_{\mathcal{M}_h} d^\mu t \int dX\, db\, dc \quad \exp[-S_p - S_{gh} - \chi\phi]$$

$$\left| \prod_{k=1}^{\mu} (b, \partial_{t_k} \hat{g}) \prod_{i=1}^{\kappa} c(\hat{z}_i) \prod_{i=\kappa+1}^{n} \int dz_i \right|^2 \prod_{i=1}^{n} \Phi_i(z_i)$$

for $n \geq \kappa$, zero otherwise

Here $h=1$, $\chi = 0$, $\mu = 1$, $\kappa = 1$

The moduli space of conformal structures on the torus is

$$\mathcal{M}_1 = \{ \tau \in \mathbb{C},\ \mathrm{Im}\,\tau > 0 \} / \underset{\uparrow}{SL(2,\mathbb{Z})}$$

modular group, acting by

$\tau \to \frac{a\tau + b}{c\tau + d}$ $\quad$ $ad - bc = 1$, $a,b,c,d \in \mathbb{Z}$

For a given τ, the metric (up to Weyl rescalings) is

$$\widehat{ds^2} = |dx - \tau dy|^2 = dz\, d\bar{z} \qquad z = x - \tau y$$

invariant up to scale under $\tau \to \frac{a\tau + b}{c\tau + d}$ $\quad \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$$(-y \quad x) \begin{pmatrix} \tau \\ 1 \end{pmatrix} \to (-y \quad x) \underbrace{\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \begin{pmatrix} \tau \\ 1 \end{pmatrix}$$

$\mathcal{F}$ defined above is a fundamental domain for the action of $SL(2,\mathbb{Z})$ on τ.

To find the measure is more tricky: up to numerical factors,

$$(b, \partial_{\bar\tau} g) = \int d^2 z \; b_{zz} \, \partial_{\bar\tau} g_{\bar z \bar z} = \frac{i}{\tau_2} \int d^2 z \; b_{zz}$$

$\to$ same as inserting $b(\hat z)$ at any fixed point

Similarly, rather than fixed the position of Φ_1 at $z_1 = \hat z_1$, over it, at the expense of dividing by the torus area:

$$Z_1 = \int_{\mathcal F} \frac{d\tau_1 d\tau_2}{\tau_2} \left\langle |b(\hat z) c(\hat z)|^2 \prod_{i=1}^{n} \int d^2 z_i \; \phi_i(z_i) \right\rangle_{b,c,X}$$

For $n=0$, this is to be interpreted as the dilaton tadpole at zero momentum (since the vacuum amplitude vanishes) : $\int \sqrt{-g}\, e^{-2\phi} \Lambda$ in effective action

The path integral for the coordinates X^μ can be written in operator language as a trace

$$\begin{aligned} Z_X(\tau) &= \mathrm{Tr} \left[\exp(-2\pi\tau_2 H + 2\pi i \tau_1 P) \right] \\ &= \mathrm{Tr} \left[\exp\left(-2\pi\tau_2 \left(L_0 + \tilde L_0 - \tfrac{1}{24}(c+\tilde c) \right) + 2\pi i \tau_1 (L_0 - \tilde L_0) \right) \right] \\ &= (q\bar q)^{-\frac{c}{24}} \, \mathrm{Tr} \, q^{L_0} \bar q^{\tilde L_0} \\ &= (\eta\bar\eta)^{-D} \cdot \underbrace{\frac{i V_d}{(4\pi^2 \alpha' \tau_2)^{D/2}}}_{\text{from zero modes}} \end{aligned}$$

For the b,c system:

$$Z_{b,c} \equiv \langle |b(\hat{z})\, c(\hat{z})|^2 \rangle$$

$$= \mathrm{Tr}\left((-1)^F\, b_0 c_0 \bar{b}_0 \bar{c}_0\, q^{L_0 - \frac{c}{24}}\, \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}\right)$$

$$= |\eta(\tau)|^4$$

where we recall that $c = -26$

and the ghost ground state $|\downarrow\rangle = c_1|0\rangle$

has $L_0 = -\frac{1}{2}$

In total,

$$Z_1 = \frac{iV_d}{(2\pi)^D} \int_{\mathcal{F}} \frac{d\tau_1\, d\tau_2}{\tau_2^{1+\frac{D}{2}}} \frac{(\eta\bar{\eta})^2}{(\eta\bar{\eta})^D}$$

in agreement with the field theory answer, up to the replacement $\mathcal{F}_{QFT} \longrightarrow \mathcal{F}$

Importantly, the integrand is invariant under modular transformations iff $\boxed{D=26}$ - using the modular properties of the Dedekind eta function:

$$\eta(\tau+1) = e^{\frac{i\pi}{12}}\, \eta(\tau)$$
$$\eta(-1/\tau) = (-i\tau)^{1/2}\, \eta(\tau)$$

$\longrightarrow$ Modular invariance uniquely selects the critical dimension.

<u>Rk</u> Due to the tachyon, the amplitude is IR divergent:

As $\tau_2 \to i\infty$, $\eta(\tau) \sim q^{\frac{1}{24}}(1 - q + O(q^2))$

$$Z_1 \sim V_{26} \int_{\mathcal{F}} \frac{d\tau_1\, d\tau_2}{\tau_2^{14}} \frac{(1+24q+\ldots)(1+24\bar{q}+\ldots)}{q\bar{q}}$$

$$\sim V_{26} \int \frac{d\tau_2}{\tau_2^{14}} \left(\exp[4\pi\tau_2] + \ldots\right) \quad \text{diverges as } \tau_2 \to \infty.$$

One loop amplitude in open bosonic string theory (w/ Neumann b.c.)

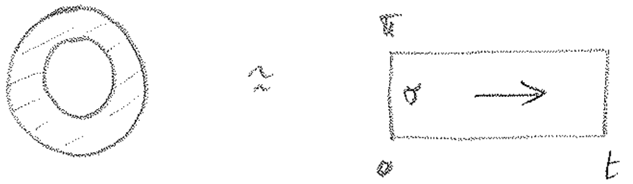
By the same reasoning, the QFT answer gives

$$Z_{QFT} = \frac{i}{2} V_D \cdot \int_0^\infty \frac{dt}{t^{1+D/2}} \sum_i \exp(-\pi t\, m_i^2)$$

$$= \frac{i}{2} V_D \int_0^\infty \frac{dt}{t^{1+D/2}} \operatorname{Tr} e^{-2\pi t L_0^\perp}$$

$$= \frac{i}{2} V_D \int_0^\infty \frac{dt}{t^{1+D/2}} \cdot \frac{1}{\eta^{D-2}(it)} \equiv Z_{annulus}$$

This is also the result of the path integral over the annulus:

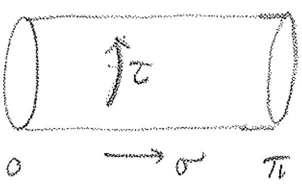


$$\mathcal{M}_{annulus} = \mathbb{R}^+_t$$

$$Z_X(t) = \left[t^{1/2} \eta(it) \right]^{-D}$$

$$Z_{bc}(t) = \left[\eta(it) \right]^2$$

Contrary to the closed string, there is no notion of modular invariance, and the one-loop amplitude is typically UV divergent.
However this UV divergence can be interpreted as an IR divergence in the 'closed string' channel; where σ is the closed string proper time



Using $\eta(it) = \frac{1}{\sqrt{t}}\, \eta\!\left(\frac{il}{\pi}\right)$ $\quad l = \frac{\pi}{t}$

$$Z_{annulus} = V_D \cdot \int_0^\infty dl\; \eta\left(\frac{il}{\pi}\right)^{-24}$$

$$\sim \int^\infty dl\, \left(e^{2l} + 24 + \dots \right)$$

closed string tachyon

graviton, dilaton, $B_{\mu\nu}$

$t \to 0$ is mapped to $l \to \infty$:

$\sum_m$ 

Let us now briefly consider open strings with Dirichlet boundary conditions:

$$X^i(\sigma=0)=0 \quad , \quad X^i(\sigma=\pi)=L^i$$

The mode expansion is now

$$X^i(\sigma,\tau) = x^i + \frac{L^i}{\pi}\sigma - \sqrt{2}\, l_s \sum_{n\in\mathbb{Z}} \alpha_n^i \sin(n\sigma)$$

$$H = \frac{(L^i)^2}{(2\pi l_s)^2} + \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n$$

Suppose we have Neumann bc for X^μ $\mu = 0 \dots p$

Dirichlet X^i $\mu = p+1 \dots D-1$

The annulus amplitude is then

$$Z_{annulus} = V_{p+1}\int\frac{d^{p+1}k}{(2\pi)^{p+1}}\int\frac{dt}{t}\exp\left[-2\pi l_s^2 k^2 t - t\frac{|L|^2}{2\pi l_s^2}\right]\cdot\frac{[\eta(it)]^2}{[\eta(it)]^{D/2}}$$

$$= V_{p+1}\int\frac{dt}{t^{1+\frac{p+1}{2}}}\exp\left[-t\frac{|L|^2}{2\pi l_s^2}\right]\frac{1}{\eta^{24}(it)}$$

$\underline{D=26}$

In the closed string channel,

$$V_{p+1}\int^{\infty}\frac{dl}{l}\cdot l^{-\frac{p+1}{2}}\, l^{12}\exp\left[-\frac{|L|^2}{2 l\, l_s^2}\right]\left[\eta\left(\frac{il}{\pi}\right)\right]^{-24}$$

$$\sim V_{p+1}\int\frac{dl}{l}\, l^{\frac{23-p}{2}}\exp\left[-\frac{|L|^2}{2 l\, l_s^2}\right]\cdot\left[e^{2l}+24+\dots\right]$$

by dimensional analysis: $|L|^{p-23}$

which is the Newton potential in $25-p$ transverse dimensions

$\rightsquigarrow$ the <u>D branes</u> at $X^i=0$ and $X^i=L^i$ are sources for the closed string fields, including the graviton. Their mass is of order $\frac{1}{g_s}$, such that $\frac{1}{g_s}\cdot g_s \cdot g_s \cdot \frac{1}{g_s} \sim 1$ $\square$